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Some corrected optimal quadrature formulas

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Abstract. The optimal 3-point quadrature formulae of closed type are derived and the estimations of error in terms of a variety on norms involving the second derivative are given. The corrected quadrature rules of the optimal quadrature formulae are considered. These results are obtained from an inequalities point of view.

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1. Introduction

The problem to construct the optimal quadratures formulas was studied by many authors. The first results were obtained by A. Sard, L.S. Meyers and S.M. Nikolski. In the last years a number of authors have obtained in many different ways the optimal quadrature formulas ([1], [5], [6], [10], [14], [15]). In this section we present the classical methods to construct this kind of quadrature formulas.

Let $\mathcal{H}$ be the class of sufficiently smooth functions $f : [a, b] \to \mathbb{R}$ and we consider the following quadrature formula with degree of exactness equal $n - 1$

$$\int_a^b f(x)dx = \sum_{i=0}^{m} \sum_{k=0}^{z_i-1} A_{ki} f^{(k)}(x_i) + \mathcal{R}_n[f], \tag{1.1}$$

where the nodes $a \le x_0 < x_1 < \cdots < x_m \le b$ have the multiplicities z_i, $1 \le z_i \le n$.

The quadrature formula (1.1) is called **optimal in the sense of Sard** in the space $\mathcal{H}$, if

$$\mathcal{E}_{m,n}(\mathcal{H}, A) = \sup_{f \in \mathcal{H}} |\mathcal{R}_n[f]|$$

attains the minimum value with regard to A, where $A = \{A_{ki}\}_{i=0}^{m}\,{}_{k=0}^{z_i-1}$ are the coefficients of quadrature formula.

The quadrature formula (1.1) is called **optimal in sense Nikolski** in the space $\mathcal{H}$, if

$$\mathcal{E}_{m,n}(\mathcal{H}, A, X) = \sup_{f\in\mathcal{H}} |\mathcal{R}_n[f]|$$

attains the minimum value with regard to A and X, where $A = \{A_{ki}\}_{i=0}^{m}\,{}_{k=0}^{z_i-1}$ are the coefficients and $X = (x_0, x_1, \ldots, x_m)$ are the nodes of quadrature formula.

We denote

$$W_p^n[a,b] := \left\{ f \in C^{n-1}[a,b], f^{(n-1)} \text{ absolutely continuous }, \left\| f^{(n)} \right\|_p < \infty \right\}$$

with

$$\|f\|_p := \left\{ \int_a^b |f(x)|^p \, dx \right\}^{\frac{1}{p}}, \quad \text{for } 1 \le p < \infty,$$
$$\|f\|_\infty := \sup_{x\in[a,b]} |f(x)| \, .$$

If $f \in W_p^n[a,b]$, by using Peano′s theorem, the remainder term can be written

$$\mathcal{R}_n[f] = \int_a^b K_n(t) f^{(n)}(t) dt,$$

where $K_n(t) = \mathcal{R}_n \left[\dfrac{(x-t)_+^{n-1}}{(n-1)!} \right]$.

For the remainder term we have the evaluation

$$|\mathcal{R}_n[f]| \le \left[\int_a^b \left| f^{(n)}(t) \right|^p dt \right]^{\frac{1}{p}} \left[\int_a^b |K_n(t)|^q \, dt \right]^{\frac{1}{q}}, \ \frac{1}{p} + \frac{1}{q} = 1, \tag{1.2}$$

with remark that in the cases $p = 1$ and $p = \infty$ this evaluation is

$$|\mathcal{R}_n[f]| \le \int_a^b \left| f^{(n)}(t) \right| dt \sup_{t\in[a,b]} |K_n(t)| \, , \tag{1.3}$$

$$|\mathcal{R}_n[f]| \le \sup_{t\in[a,b]} \left| f^{(n)}(t) \right| \int_a^b |K_n(t)| \, dt. \tag{1.4}$$

The φ-function method is a model of constructing the quadrature formulas and was given by D.V. Ionescu ([9]). Suppose that $f \in C^r[a,b]$ and for some given $n \in \mathbb{N}$ consider the nodes $a = x_0 < ... < x_n = b$. On each interval $[x_{k-1}, x_k]$, $k = 1, ..., n$, it is considered a function φ_k, $k = 1, ..., n$, with the property that

$$\varphi_k^{(r)} = 1, \ k = 1, ..., n. \tag{1.5}$$

One defines the function φ as follows

$$\varphi|_{[x_{k-1}, x_k]} = \varphi_k, \ k = 1, ..., n, \tag{1.6}$$

i.e., the restriction of the function φ to the interval $[x_{k-1}, x_k]$ is φ_k.

Using the integration by parts of the integral

$$S(f) := \int_a^b f(x)dx = \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} \varphi_k^{(r)}(x)f(x)dx,$$

one obtains the following quadrature formula

$$\int_a^b f(x)dx = \sum_{k=0}^{n}\sum_{j=0}^{r-1} A_{kj} f^{(j)}(x_k) + \mathcal{R}_n(f), \tag{1.7}$$

with

$$\mathcal{R}_n(f) = (-1)^r \int_a^b \varphi(x) f^{(r)}(x)dx \tag{1.8}$$

and

$$\begin{aligned} A_{0j} &= (-1)^{j+1}\varphi_1^{(r-j-1)}(x_0),\\ A_{kj} &= (-1)^j(\varphi_k - \varphi_{k+1})^{(r-j-1)}(x_k),\ k = 1,\dots,n-1,\\ A_{nj} &= (-1)^j\varphi_n^{(r-j-1)}(x_n),\ j = 0,1,\dots,r-1. \end{aligned} \tag{1.9}$$

In [1], T. Cătinaş and G. Coman studied the optimality in sense of Nikolski for a quadrature formula, using the method of φ-function.

In [16], N. Ujević and L. Mijić constructed a class of quadrature formulas of close type with 3 nodes. Let

$$K_2(\alpha,\beta,\gamma,\delta;t) = \begin{cases} \frac{1}{2}(t-\alpha)(t-\beta),\ t \in \left[a, \frac{a+b}{2}\right], \\ \\ \frac{1}{2}(t-\gamma)(t-\delta),\ t \in \left(\frac{a+b}{2}, b\right], \end{cases}$$

be a function which depends on the parameters $\alpha,\beta,\gamma,\delta \in \mathbb{R}$. Integrating by parts the integral $\int_a^b K_2(\alpha,\beta,\gamma,\delta;t)f''(t)dt$, and putting conditions that the coefficients of the first derivatives to be zero, N. Ujević and L. Mijić were constructed the following class of quadrature formulas of close type

$$\begin{aligned} \int_a^b f(t)dt &= A_0(\alpha,\beta,\gamma,\delta)f(a) + A_1(\alpha,\beta,\gamma,\delta)f\left(\frac{a+b}{2}\right) \\ &+ A_2(\alpha,\beta,\gamma,\delta)f(b) + \mathcal{R}[f], \end{aligned}$$

where

$$\mathcal{R}[f] = \int_a^b K_2(\alpha,\beta,\gamma,\delta)f''(t)dt.$$

The parameters $\alpha,\beta,\gamma,\delta$ are obtained putting conditions that the remainder term which is evaluated in sense of (1.4) to be minimal, namely $\int_a^b |K_2(\alpha,\beta,\gamma,\delta)|\,dt$ to attains the minimum value.

The main result obtained by N. Ujević and L. Mijić in the above described procedure is formulated bellow.

Theorem 1.1. [16] *Let $I \subset \mathbb{R}$ be an open interval such that $[0,1] \subset I$ and let $f : I \to \mathbb{R}$ be a twice differentiable function such that f'' is bounded and integrable. Then we have*

$$\left| \int_0^1 f(t)dt - \frac{\sqrt{2}}{8} f(0) - \left(1 - \frac{\sqrt{2}}{4}\right) f\left(\frac{1}{2}\right) - \frac{\sqrt{2}}{8} f(1) \right| \leq \frac{2-\sqrt{2}}{48} \|f''\|_\infty. \tag{1.10}$$

The main purpose of the section 2 is to derive a quadrature formula of close type with 3-points which is optimal in sense Nikolski, namely we calculate the coefficients A_i, $i = \overline{0,2}$ and the node $a_1 \in (a,b)$ such that the quadrature formula

$$\int_a^b f(t)dt = A_0 f(a) + A_1 f(a_1) + A_2 f(b) + \mathcal{R}_2[f],$$

to be optimal, considering that the remainder term is evaluated in sense of (1.2) in the cases $p = 1$, $p = 2$ and $p = \infty$.

For the simplicity, in this paper we choose $[a,b] = [0,1]$. The corresponding results in the arbitrary interval $[a,b]$ can be obtained using the following lemma.

Lemma 1.2. [11] *If $-\infty < \alpha < \beta < +\infty$ and w is a weight function on (α,β) and $\int_\alpha^\beta f(t)w(t)dt = \sum_{i=0}^m A_i f(x_i) + r_m[f]$, $f \in L_w^1(\alpha,\beta)$, then*

$$W(x) = w\left(\alpha + (\beta-\alpha)\frac{x-a}{b-a}\right), \ x \in (a,b), \ -\infty < a < b < +\infty,$$

is a weight function on (a,b) and

$$\int_a^b F(x)W(x)dx = \frac{b-a}{\beta-\alpha} \sum_{i=0}^m A_i F\left(a + (b-a)\frac{x_i - \alpha}{\beta-\alpha}\right) + \mathcal{R}_m[F],$$

where $F \in L_w^1(a,b)$ and $\mathcal{R}_m[F] = \frac{b-a}{\beta-\alpha} r_m[\tilde{F}]$, $\tilde{F}(t) = F\left(a + (b-a)\frac{t-\alpha}{\beta-\alpha}\right)$.

2. The optimal 3-point quadrature formula of closed type

Let

$$\int_0^1 f(x)dx = A_0 f(0) + A_1 f(a_1) + A_2 f(1) + \mathcal{R}_2[f] \tag{2.1}$$

be a quadrature formula with degree of exactness equal 1.

Since the quadrature formula has degree of exactness 1, the remainder term verifies the conditions $\mathcal{R}_2[e_i] = 0$, $e_i(x) = x^i$, $i = 0, 1$, namely

$$\begin{cases} A_0 + A_1 + A_2 = 1 \\ A_1 a_1 + A_2 = \frac{1}{2} \end{cases} \tag{2.2}$$

and using Peano's theorem the remainder term has the following integral representation

$$\mathcal{R}_2[f] = \int_0^1 K_2(t) f''(t)dt, \text{ where} \tag{2.3}$$

$$K_2(t) = \mathcal{R}_2\left[(x-t)_+\right] = \begin{cases} \dfrac{1}{2}t^2 - A_0 t, & 0 \le t \le a_1, \\ \dfrac{1}{2}(1-t)^2 - A_2(1-t), & a_1 < t \le 1. \end{cases} \tag{2.4}$$

Theorem 2.1. *For $f \in W_\infty^2[0,1]$, the quadrature formula of the form (2.1), optimal with regard to the error, is*

$$\int_0^1 f(x)dx = \frac{\sqrt{2}}{8} f(0) + \frac{4-\sqrt{2}}{4} f\left(\frac{1}{2}\right) + \frac{\sqrt{2}}{8} f(1) + \mathcal{R}_2^{[1]}[f], \tag{2.5}$$

with

$$\mathcal{R}_2^{[1]}[f] = \int_0^1 K_2^{[1]}(t) f''(t)dt, \; K_2^{[1]}(t) = \begin{cases} \dfrac{1}{2}t^2 - \dfrac{\sqrt{2}}{8}t, & 0 \le t \le \dfrac{1}{2}, \\ \dfrac{1}{2}(1-t)^2 - \dfrac{\sqrt{2}}{8}(1-t), & \dfrac{1}{2} < t \le 1, \end{cases} \tag{2.6}$$

$$\left|\mathcal{R}_2^{[1]}[f]\right| \le \frac{2-\sqrt{2}}{48} \|f''\|_\infty \approx 0.0122 \|f''\|_\infty.$$

Proof. The remainder term (2.3) can be evaluate in the following way

$$\left|\mathcal{R}_2^{[1]}[f]\right| \le \|f''\|_\infty \int_0^1 \left|K_2^{[1]}(t)\right| dt.$$

The quadrature formula is optimal with regard to the error if

$$\int_0^1 \left|K_2^{[1]}(t)\right| dt \to \text{minimum}.$$

We have

$$\begin{aligned} \mathcal{I}(A_0, A_2, a_1) &= \int_0^1 |K_2^{[1]}(t)|dt = \int_0^{a_1} |K_2^{[1]}(t)|dt + \int_{a_1}^1 |K_2^{[1]}(t)|dt \\ &= \int_0^{2A_0} \left(A_0 t - \frac{1}{2}t^2\right) dt + \int_{2A_0}^{a_1} \left(\frac{1}{2}t^2 - A_0 t\right) dt \\ &+ \int_{a_1}^{1-2A_2} \left[\frac{1}{2}(1-t)^2 - A_2(1-t)\right] dt + \int_{1-2A_2}^1 \left[A_2(1-t) - \frac{1}{2}(1-t)^2\right] dt \\ &= \frac{4}{3}A_0^3 - \frac{1}{2}a_1^2 A_0 + \frac{1}{6}a_1^3 + \frac{4}{3}A_2^3 - \frac{1}{2}(1-a_1)^2 A_2 + \frac{1}{6}(1-a_1)^3. \end{aligned}$$

Putting condition that the partial derivatives to be zero, namely

$$\begin{cases} \dfrac{\partial \mathcal{I}(A_0, A_2, a_1)}{\partial A_0} = 4A_0^2 - \dfrac{1}{2}a_1^2 = 0, \\[2ex] \dfrac{\partial \mathcal{I}(A_0, A_2, a_1)}{\partial A_2} = 4A_2^2 - \dfrac{1}{2}(1-a_1)^2 = 0, \\[2ex] \dfrac{\partial \mathcal{I}(A_0, A_2, a_1)}{\partial a_1} = -a_1 A_0 + \dfrac{a_1^2}{2} - \dfrac{1}{2}(1-a_1)^2 + A_2(1-a_1) = 0, \end{cases}$$

we find the following values for the coefficients and the node of optimal quadrature formula

$$A_0 = A_2 = \frac{\sqrt{2}}{8},\ A_1 = 1 - \frac{\sqrt{2}}{4},\ a_1 = \frac{1}{2},\ \text{and} \int_0^1 |K_2^{[1]}(t)|dt = \frac{2-\sqrt{2}}{48}. \qquad \square$$

Remark 2.2. The optimal quadrature (2.5) coincides with the quadrature formula (1.10) obtained by N. Ujević and L. Mijić in [16], but this quadrature formula was obtained in different way than in [16]. This result motivated us to seek the quadrature formulas of type (2.1) such that the estimation of its error to be best possible in p-norm for $p = 2$ and $p = 1$.

Remark 2.3. For the remainder term of quadrature formula (2.5) can be established the following two estimations

$$\begin{aligned} \left|\mathcal{R}_2^{[1]}[f]\right| &\le \left[\int_0^1 \left(K_2^{[1]}(t)\right)^2 dt\right]^{\frac{1}{2}} \|f''\|_2 = \frac{1}{16}\sqrt{\frac{22-15\sqrt{2}}{15}}\|f''\|_2 \\ &\approx 0.0143\|f''\|_2,\ f \in W_2^2[0,1], \\ \left|\mathcal{R}_2^{[1]}[f]\right| &\le \sup_{t\in[0,1]} |K_2^{[1]}(t)|\cdot\|f''\|_1 = \frac{2-\sqrt{2}}{16}\|f''\|_1 \approx 0.0366\|f''\|_1,\ f \in W_1^2[0,1]. \end{aligned}$$

Theorem 2.4. *For $f \in W_2^2[0,1]$, the quadrature formula of the form (2.1), optimal with regard to the error, is*

$$\int_0^1 f(x)dx = \frac{3}{16}f(0) + \frac{5}{8}f\left(\frac{1}{2}\right) + \frac{3}{16}f(1) + \mathcal{R}_2^{[2]}[f], \tag{2.7}$$

with

$$\mathcal{R}_2^{[2]}[f] = \int_0^1 K_2^{[2]}(t)f''(t)dt, \quad K_2^{[2]}(t) = \begin{cases} \dfrac{1}{2}t^2 - \dfrac{3}{16}t, & 0 \le t \le \dfrac{1}{2}, \\[2ex] \dfrac{1}{2}(1-t)^2 - \dfrac{3}{16}(1-t), & \dfrac{1}{2} < t \le 1, \end{cases} \tag{2.8}$$

and

$$\left|\mathcal{R}_2^{[2]}[f]\right| \le \frac{\sqrt{5}}{160}\|f''\|_2 \approx 0.0140\|f''\|_2.$$

Proof. The remainder term (2.3) can be evaluated in the following way

$$\left|\mathcal{R}_2^{[2]}[f]\right| \leq \left[\int_0^1 \left(K_2^{[2]}(t)\right)^2 dt\right]^{\frac{1}{2}} \|f''\|_2.$$

The quadrature formula is optimal with regard to the error if

$$\int_0^1 \left(K_2^{[2]}(t)\right)^2 dt \to \text{minimum}.$$

We have

$$\begin{aligned}\mathcal{I}(A_0, A_2, a_1) &= \int_0^1 \left(K_2^{[2]}(t)\right)^2 dt = \int_0^{a_1} \left(\frac{1}{2}t^2 - A_0 t\right)^2 dt \\ &+ \int_{a_1}^1 \left[\frac{1}{2}(1-t)^2 - A_2(1-t)\right]^2 dt \\ &= \frac{1}{20}a_1^5 - \frac{A_0}{4}a_1^4 + \frac{A_0^2}{3}a_1^3 + \frac{(1-a_1)^5}{20} - A_2\frac{(1-a_1)^4}{4} + A_2^2\frac{(1-a_1)^3}{3}.\end{aligned}$$

Putting condition that the partial derivatives to be zero, namely

$$\begin{cases} \dfrac{\partial \mathcal{I}(A_0, A_2, a_1)}{\partial A_0} = -\dfrac{a_1^4}{4} + \dfrac{2}{3}A_0 a_1^3 = 0, \\[2ex] \dfrac{\partial \mathcal{I}(A_0, A_2, a_1)}{\partial A_2} = -\dfrac{(1-a_1)^4}{4} + \dfrac{2}{3}A_2(1-a_1)^3 = 0, \\[2ex] \dfrac{\partial \mathcal{I}(A_0, A_2, a_1)}{\partial a_1} = \dfrac{a_1^4}{4} - A_0 a_1^3 + A_0^2 a_1^2 - \dfrac{(1-a_1)^4}{4} + A_2(1-a_1)^3 - A_2^2(1-a_1)^2 = 0, \end{cases}$$

we find the following values for the coefficients and the node of optimal quadrature formula

$$A_0 = A_2 = \frac{3}{16},\ A_1 = \frac{5}{8},\ a_1 = \frac{1}{2},\ \text{and} \int_0^1 \left(K_2^{[2]}(t)\right)^2 dt = \frac{1}{2^{10}\cdot 5}.$$ □

Remark 2.5. For the remainder term of quadrature formula (2.7) can be established the following two estimations

$$\left|\mathcal{R}_2^{[2]}[f]\right| \leq \int_0^1 |K_2^{[2]}(t)| dt \cdot \|f''\|_\infty = \frac{19}{1536}\|f''\|_\infty \approx 0.0124\|f''\|_\infty, \text{ for } f \in W_\infty^2[0,1],$$

$$\left|\mathcal{R}_2^{[2]}[f]\right| \leq \sup_{t\in[0,1]} |K_2^{[2]}(t)| \cdot \|f''\|_1 = \frac{1}{32}\|f''\|_1 \approx 0.0313\|f''\|_1, \text{ for } f \in W_1^2[0,1].$$

Theorem 2.6. *For $f \in W_1^2[0,1]$, the quadrature formula of the form (2.1), optimal with regard to the error, is*

$$\int_0^1 f(x)dx = \frac{\sqrt{2}-1}{2}f(0) + (2-\sqrt{2})f\left(\frac{1}{2}\right) + \frac{\sqrt{2}-1}{2}f(1) + \mathcal{R}_2^{[3]}[f], \tag{2.9}$$

with

$$\mathcal{R}_2^{[3]}[f]=\int_0^1 K_2^{[3]}(t)f''(t)dt,\ K_2^{[3]}(t)=\begin{cases}\dfrac{1}{2}t^2-\dfrac{\sqrt{2}-1}{2}t,\ 0\le t\le\dfrac{1}{2},\\[2ex] \dfrac{1}{2}(1-t)^2-\dfrac{\sqrt{2}-1}{2}(1-t),\ \dfrac{1}{2}<t\le 1,\end{cases} \tag{2.10}$$

and

$$\left|\mathcal{R}_2^{[3]}[f]\right| \le \frac{3-2\sqrt{2}}{8}\|f''\|_1 \approx 0.0214\|f''\|_1.$$

Proof. The remainder term (2.3) can be evaluated in the following way

$$\left|\mathcal{R}_2^{[3]}[f]\right| \le \|f''\|_1 \cdot \sup_{0\le t\le 1}|K_2^{[3]}(t)|.$$

The quadrature formula is optimal with regard to the error if

$$\sup_{0\le t\le 1}|K_2^{[3]}(t)| \to \text{minimum}.$$

We have

$$\sup_{t\in[0,a_1]}|K_2^{[3]}(t)| = \max\left\{|K_2^{[3]}(A_0)|,|K_2^{[3]}(a_1)|\right\} = \max\left\{\frac{1}{2}A_0^2,\left|\frac{1}{2}a_1^2-A_0a_1\right|\right\},$$

$$\sup_{t\in[a_1,1]}|K_2^{[3]}(t)|=\max\left\{|K_2^{[3]}(a_1)|,|K_2^{[3]}(1-A_2)|\right\}=\max\left\{\left|\frac{1}{2}a_1^2-A_0a_1\right|,\frac{1}{2}A_2^2\right\},$$

therefore

$$\sup_{t\in[0,1]}|K_2^{[3]}(t)| = \max\left\{\frac{1}{2}A_0^2,\frac{1}{2}A_2^2,\frac{1}{2}a_1^2-A_0a_1\right\}. \tag{2.11}$$

Putting condition that $\sup_{t\in[0,1]}|K_2^{[3]}(t)|$ to attains the minimum value, which in our case is equivalent with $K_2^{[3]}(a_1)=-K_2^{[3]}(t_{\min})$, where $t_{\min}\in\{A_0,1-A_2\}$, we obtain $A_0=(\sqrt{2}-1)a_1$, respectively $A_2=(\sqrt{2}-1)(1-a_1)$. Since $K\in C[0,1]$, namely $K_2^{[3]}(a_1-0)=K_2^{[3]}(a_1+0)$, we can find the following relation

$$\frac{1}{2}a_1^2-(\sqrt{2}-1)a_1^2=\frac{1}{2}(1-a_1)^2-(\sqrt{2}-1)(1-a_1)^2.$$

From the above equality we obtain $a_1=\dfrac{1}{2}$ and the values for the coefficients of the optimal quadrature formula are $A_0=A_2=\dfrac{\sqrt{2}-1}{2}$, $A_1=2-\sqrt{2}$. From (2.11) it follows $\sup_{t\in[0,1]}|K_2^{[3]}(t)|=\dfrac{1}{2}A_0^2=\dfrac{3-2\sqrt{2}}{8}$. □

Remark 2.7. For the remainder term of quadrature formula (2.9) can be established the following two estimations

$$\begin{aligned}
\left|\mathcal{R}_2^{[3]}[f]\right| \le \int_0^1 |K_2^{[3]}(t)|dt \cdot \|f''\|_\infty &= \frac{37\sqrt{2}-52}{24}\|f''\|_\infty \\
&\approx 0.0136\|f''\|_\infty,\ f \in W_\infty^2[0,1], \\
\left|\mathcal{R}_2^{[3]}[f]\right| \le \left[\int_0^1 \left(K_2^{[3]}(t)\right)^2 dt\right]^{\frac{1}{2}} \|f''\|_2 &= \frac{1}{8}\sqrt{\frac{78-55\sqrt{2}}{15}}\|f''\|_2 \\
&\approx 0.0151\|f''\|_2,\ f \in W_2^2[0,1].
\end{aligned}$$

Remark 2.8. If we denote by $C_p^{[i]}$ the constants which appear in estimations of the following type

$$\left|\mathcal{R}_2^{[i]}[f]\right| \le C_p^{[i]} \left\|f''\right\|_p ,$$

where $i = 1, 2, 3$, $p = \infty, 2$, respectively 1, and $f \in W_p^2[0,1]$, from the above results the inequalities $C_\infty^{[1]} \le C_\infty^{[2]} \le C_\infty^{[3]}$, $C_2^{[2]} \le C_2^{[1]} \le C_2^{[3]}$ and $C_1^{[3]} \le C_1^{[2]} \le C_1^{[1]}$ are true. Therefore, we can assert that our results are better than Ujević and Mijić's result, if we consider 2-norm, respectively 1-norm.

3. The corrected quadrature formulae

In recent years some authors have considered so called perturbed (corrected) quadrature rules (see [2], [3], [4], [7], [8], [17]). By corrected quadrature rule we mean the formula which involves the values of the first derivative in end points of the interval not only the values of the function in certain points. These formulae have a higher degree of exactness than the original rule. The estimate of the error in corrected rule is better then in the original rule, in generally.

The main purpose of this section is to derive corrected rule of the optimal quadrature formulae obtained in previous section. Here we will show that the corrected formula improves the original formula. We mention that the corrected formula of (2.5) was considered by N. Ujević and L. Mijić in [16].

Let

$$\int_0^1 f(x)dx = A_0 f(0) + A_1 f\left(\frac{1}{2}\right) + A_2 f(1) + A\left[f'(1) - f'(0)\right] + \tilde{\mathcal{R}}_2[f], \tag{3.1}$$

where

$$\tilde{\mathcal{R}}_2[e_i] = 0,\ i = 0, 1, \text{ and } A = \int_0^1 K_2(t)dt$$

be the corrected quadrature formula of the rule (2.1).

Since the remainder term has degree of exactness 1 we can write

$$\tilde{\mathcal{R}}_2[f] = \int_0^1 \tilde{K}_2(t) f''(t)dt, \text{ where} \tag{3.2}$$

$$\tilde{K}_2(t) = \tilde{\mathcal{R}}_2\left[(x-t)_+\right] = K_2(t) - A. \tag{3.3}$$

From the relation (3.3) we remark that $\int_0^1 \tilde{K}_2(t)dt = 0$. Now we will calculate the coefficient A from corrected optimal quadrature formulas obtained in previous section. If we consider $f(x) = \frac{x^2}{2}$ in (3.1) we find

$$A = \frac{1}{6} - \frac{1}{2}A_1 - \frac{1}{2}A_2. \tag{3.4}$$

Using relations (3.3) and (3.4) we construct the following corrected quadrature formula of (2.5), (2.7), respectively (2.9):

$$\begin{aligned} \int_0^1 f(x)dx &= \frac{\sqrt{2}}{8}f(0) + \frac{4-\sqrt{2}}{4}f\left(\frac{1}{2}\right) + \frac{\sqrt{2}}{8}f(1) \\ &+ \frac{4-3\sqrt{2}}{96}\left[f'(1) - f'(0)\right] + \tilde{\mathcal{R}}_2^{[1]}[f], \end{aligned} \tag{3.5}$$

where

$$\tilde{\mathcal{R}}_2^{[1]}[f] = \int_0^1 \tilde{K}_2^{[1]}(t)f''(t)dt, \tag{3.6}$$

$$\tilde{K}_2^{[1]}(t) = \begin{cases} \frac{1}{2}t^2 - \frac{\sqrt{2}}{8}t - \frac{4-3\sqrt{2}}{96}, & 0 \le t \le \frac{1}{2} \\ \frac{1}{2}(1-t)^2 - \frac{\sqrt{2}}{8}(1-t) - \frac{4-3\sqrt{2}}{96}, & \frac{1}{2} < t \le 1, \end{cases}$$

$$\int_0^1 f(x)dx = \frac{3}{16}f(0) + \frac{5}{8}f\left(\frac{1}{2}\right) + \frac{3}{16}f(1) - \frac{1}{192}\left[f'(1) - f'(0)\right] + \tilde{\mathcal{R}}_2^{[2]}[f], \tag{3.7}$$

where

$$\tilde{\mathcal{R}}_2^{[2]}[f] = \int_0^1 \tilde{K}_2^{[2]}(t)f''(t)dt, \tag{3.8}$$

$$\tilde{K}_2^{[2]}(t) = \begin{cases} \frac{1}{2}t^2 - \frac{3}{16}t + \frac{1}{192}, & 0 \le t \le \frac{1}{2} \\ \frac{1}{2}(1-t)^2 - \frac{3}{16}(1-t) + \frac{1}{192}, & \frac{1}{2} < t \le 1, \end{cases}$$

respectively

$$\begin{aligned} \int_0^1 f(x)dx = \frac{\sqrt{2}-1}{2}f(0) + (2-\sqrt{2})f\left(\frac{1}{2}\right) + \frac{\sqrt{2}-1}{2}f(1) \\ + \frac{4-3\sqrt{2}}{24}\left[f'(1) - f'(0)\right] + \tilde{\mathcal{R}}_2^{[3]}[f], \end{aligned} \tag{3.9}$$

where

$$\tilde{\mathcal{R}}_2^{[3]}[f] = \int_0^1 \tilde{K}_2^{[3]}(t)f''(t)dt, \tag{3.10}$$

$$\tilde{K}_2^{[3]}(t)=\begin{cases}\dfrac{1}{2}t^2-\dfrac{\sqrt{2}-1}{2}t-\dfrac{4-3\sqrt{2}}{24}, & 0\le t\le \dfrac{1}{2}\\[2ex] \dfrac{1}{2}(1-t)^2-\dfrac{\sqrt{2}-1}{2}(1-t)-\dfrac{4-3\sqrt{2}}{24}, & \dfrac{1}{2}<t\le 1,\end{cases}$$

Denote by $\tilde{C}_p^{[i]}$ the constant which appear in estimations of the remainder term of corrected quadrature formulas, namely

$$\left|\tilde{\mathcal{R}}_2^{[i]}[f]\right|\le \tilde{C}_p^{[i]}\left\|f''\right\|_p,$$

where $i=1,2,3$, $p=\infty,2,$ respectively 1, and $f\in W_p^2[0,1]$. The constants $\tilde{C}_p^{[i]}$ can be calculated in a similar way with the constants $C_p^{[i]}$ defined in Remark 2.8. From the bellow table follows that for $p=\infty$ and $p=2$ the corrected formula improves the original formula.

$\diagdown i$	1	2	3
$C_\infty^{[i]}$	$\dfrac{2-\sqrt{2}}{48}\approx 0.0122$	$\dfrac{19}{1536}\approx 0.0124$	$\dfrac{37\sqrt{2}-52}{24}\approx 0.0136$
$\tilde{C}_\infty^{[i]}$	$\dfrac{5}{96}\sqrt{6}-\dfrac{29}{432}\sqrt{3}\approx 0.0113$	$\dfrac{19}{13824}\sqrt{57}\approx 0.0104$	$\dfrac{\sqrt{3(13-9\sqrt{2})^3}}{27}\approx 0.0091$
$C_2^{[i]}$	$\dfrac{1}{16}\sqrt{\dfrac{22-15\sqrt{2}}{15}}\approx 0.0143$	$\dfrac{\sqrt{5}}{160}\approx 0.0140$	$\dfrac{1}{8}\sqrt{\dfrac{78-55\sqrt{2}}{15}}\approx 0.0151$
$\tilde{C}_2^{[i]}$	$\dfrac{\sqrt{470-300\sqrt{2}}}{480}\approx 0.0141$	$\dfrac{\sqrt{155}}{960}\approx 0.0130$	$\left(\dfrac{1}{90}-\dfrac{1}{128}\sqrt{2}\right)\approx 0.0112$
$C_1^{[i]}$	$\dfrac{2-\sqrt{2}}{16}\approx 0.0366$	$\dfrac{1}{32}\approx 0.0313$	$\dfrac{3-2\sqrt{2}}{8}\approx 0.0214$
$\tilde{C}_1^{[i]}$	$\left(\dfrac{1}{12}-\dfrac{1}{32}\sqrt{2}\right)\approx 0.0391$	$\dfrac{7}{192}\approx 0.0365$	$\left(\dfrac{5}{24}-\dfrac{1}{8}\sqrt{2}\right)\approx 0.0316$

Theorem 3.1. *Let $f:[0,1]\to\mathbb{R}$ be an absolutely continuous function such that $f''\in L[0,1]$ and there exist real number $m[f],M[f]$ such that $m[f]\le f''(t)\le M[f]$, $t\in[0,1]$. Then*

$$\left|\tilde{\mathcal{R}}_2^{[1]}[f]\right|\le\frac{M[f]-m[f]}{2}\left(\frac{5\sqrt{6}}{96}-\frac{29\sqrt{3}}{432}\right)\approx 11306\times 10^{-6}\cdot\frac{M[f]-m[f]}{2}, \tag{3.11}$$

$$\left|\tilde{\mathcal{R}}_2^{[2]}[f]\right|\le\frac{M[f]-m[f]}{2}\cdot\frac{19\sqrt{57}}{13824}\approx 10377\times 10^{-6}\cdot\frac{M[f]-m[f]}{2}, \tag{3.12}$$

$$\left|\tilde{\mathcal{R}}_2^{[3]}[f]\right|\le\frac{M[f]-m[f]}{2}\cdot\frac{\sqrt{3(13-9\sqrt{2})^3}}{27}\approx 9104\times 10^{-6}\cdot\frac{M[f]-m[f]}{2}. \tag{3.13}$$

If there exist a real number $m[f]$ such that $m[f]\le f''(t)$, $t\in[0,1]$, then

$$\begin{aligned}\left|\tilde{\mathcal{R}}_2^{[1]}[f]\right| &\le \frac{1}{4}\left(\frac{1}{3}-\frac{\sqrt{2}}{8}\right)(f'(1)-f'(0)-m[f])\\ &\approx 39139\times 10^{-6}\,(f'(1)-f'(0)-m[f]),\end{aligned} \tag{3.14}$$

$$\begin{aligned}\left|\tilde{\mathcal{R}}_2^{[2]}[f]\right| &\le \frac{7}{192}\left(f'(1)-f'(0)-m[f]\right)\\ &\approx 36458\times 10^{-6}\left(f'(1)-f'(0)-m[f]\right),\end{aligned} \tag{3.15}$$

$$\begin{aligned}\left|\tilde{\mathcal{R}}_2^{[3]}[f]\right| &\le \frac{5-3\sqrt{2}}{24}\cdot\left(f'(1)-f'(0)-m[f]\right)\\ &\approx 31557\times 10^{-6}\left(f'(1)-f'(0)-m[f]\right).\end{aligned} \tag{3.16}$$

If there exist a real number $M[f]$ such that $f''(t)\le M[f]$, $t\in[0,1]$, then

$$\begin{aligned}\left|\tilde{\mathcal{R}}_2^{[1]}[f]\right| &\le \frac{1}{4}\left(\frac{1}{3}-\frac{\sqrt{2}}{8}\right)\left[M[f]-\left(f'(1)-f'(0)\right)\right]\\ &\approx 39139\times 10^{-6}\left[M[f]-\left(f'(1)-f'(0)\right)\right],\\ \left|\tilde{\mathcal{R}}_2^{[2]}[f]\right| &\le \frac{7}{192}\left[M[f]-\left(f'(1)-f'(0)\right)\right]\\ &\approx 36458\times 10^{-6}\left[M[f]-\left(f'(1)-f'(0)\right)\right],\\ \left|\tilde{\mathcal{R}}_2^{[3]}[f]\right| &\le \frac{5-3\sqrt{2}}{24}\left[M[f]-\left(f'(1)-f'(0)\right)\right]\\ &\approx 31557\times 10^{-6}\left[M[f]-\left(f'(1)-f'(0)\right)\right].\end{aligned}$$

Proof. Since $\displaystyle\int_0^1 \tilde{K}_2(t)dt=0$, the remainder term (3.2) can be written in the following way

$$\tilde{\mathcal{R}}_2[f]=\int_0^1 \tilde{K}_2(t)\left(f''(t)-\frac{M[f]+m[f]}{2}\right)dt.$$

Therefore

$$\left|\tilde{\mathcal{R}}_2[f]\right|\le\left\|f''-\frac{M[f]+m[f]}{2}\right\|_\infty\cdot\|\tilde{K}_2\|_1\le\frac{M[f]-m[f]}{2}\cdot\|\tilde{K}_2\|_1.$$

Calculating the norm of the kernel $\tilde{K}_2$ from the integral representation of the remainder term (3.6), (3.8) and (3.10), respectively, the first part of the theorem is proved.

To prove the relations (3.14), (3.15) and (3.16) respectively, we consider the following estimation of remainder term

$$\begin{aligned}\left|\tilde{\mathcal{R}}_2[f]\right| &= \left|\int_0^1 \tilde{K}_2(t)\left(f''(t)-m\right)dt\right|\le\sup_{t\in[0,1]}|\tilde{K}_2(t)|\cdot\int_0^1\left(f''(t)-m\right)dt\\ &=\left\|\tilde{K}_2\right\|_\infty\cdot\left(f'(1)-f'(0)-m\right).\end{aligned}$$

The last part of the theorem can be proved using the following estimation of remainder term

$$\begin{aligned}\left|\tilde{\mathcal{R}}_2[f]\right| &= \left|\int_0^1 \tilde{K}_2(t)\left(f''(t)-M\right)dt\right|\le\sup_{t\in[0,1]}|\tilde{K}_2(t)|\cdot\int_0^1\left(M-f''(t)\right)dt\\ &=\left\|\tilde{K}_2\right\|_\infty\cdot\left[M-\left(f'(1)-f'(0)\right)\right].\end{aligned}$$

□

Let $f, g : [a, b] \to \mathbb{R}$ be integrable functions on $[a, b]$. The functional

$$T(f,g) := \frac{1}{b-a}\int_a^b f(t)g(t)dt - \frac{1}{b-a}\int_a^b f(t)dt \cdot \frac{1}{b-a}\int_a^b g(t)dt, \tag{3.17}$$

is well known in the literature as the Čebyšev functional. It was proved that $T(f,f) \geq 0$ and the inequality $|T(f,g)| \leq \sqrt{T(f,f)} \cdot \sqrt{T(g,g)}$ holds. Denote by $\sigma(f, a, b) = \sqrt{T(f,f)}$.

Theorem 3.2. *Let $f : [0,1] \to \mathbb{R}$ be an absolutely continuous function such that $f'' \in L_2[0,1]$. Then*

$$\left|\tilde{\mathcal{R}}_2^{[1]}[f]\right| \leq \sqrt{\frac{47}{23040} - \frac{\sqrt{2}}{768}} \cdot \sigma(f''; 0, 1) \approx 14089 \times 10^{-6}\sigma(f''; 0, 1), \tag{3.18}$$

$$\left|\tilde{\mathcal{R}}_2^{[2]}[f]\right| \leq \frac{\sqrt{155}}{960} \cdot \sigma(f''; 0, 1) \approx 12969 \times 10^{-6}\sigma(f''; 0, 1), \tag{3.19}$$

$$\left|\tilde{\mathcal{R}}_2^{[3]}[f]\right| \leq \frac{1}{120}\sqrt{320 - 225\sqrt{2}} \cdot \sigma(f''; 0, 1) \approx 11186 \times 10^{-6}\sigma(f''; 0, 1). \tag{3.20}$$

Proof. The remainder term of the corrected quadrature formula (3.2) can be written in such way

$$\tilde{\mathcal{R}}_2[f] = \int_0^1 \tilde{K}_2(t)f''(t)dt = \int_0^1 \left[K_2(t) - \int_0^1 K_2(t)dt\right] f''(t)dt$$
$$= \int_0^1 K_2(t)f''(t)dt - \int_0^1 K_2(t)dt \cdot \int_0^1 f''(t)dt = T(K_2, f'').$$

From the above relation we obtain

$$\left|\tilde{\mathcal{R}}_2[f]\right| = |T(K_2, f'')| \leq \sqrt{T(K_2, K_2)}\sqrt{T(f'', f'')} = \sigma(K_2; 0, 1) \cdot \sigma(f''; 0, 1).$$

Calculating $\sigma(K_2; 0, 1)$ for the kernel defined in (2.6), (2.8) and (2.10), respectively, the theorem is proved. □

Remark 3.3. The inequalities (3.18), (3.19) and (3.20), respectively, are sharp in the sense that the constants $\sqrt{\frac{47}{23040} - \frac{\sqrt{2}}{768}}$, $\frac{\sqrt{155}}{960}$ and $\frac{1}{120}\sqrt{320 - 225\sqrt{2}}$ respectively, cannot be replaced by a smaller ones. To prove that we define the functions

$$F^{[i]}(x) = \int_0^x \left(\int_0^t K_2^{[i]}(u)du\right) dt, \quad i = 1, 2, 3. \tag{3.21}$$

For the function (3.21) the right-hand side of the inequalities (3.18), (3.19) and (3.20), respectively are equal with $T(K_2^{[i]}, K_2^{[i]})$, $i = 1, 2, 3$, respectively, and the left-hand side becomes

$$\left|\tilde{\mathcal{R}}_2^{[i]}[f]\right| = \left|\int_0^1 \tilde{K}_2^{[i]}(t) \cdot K_2^{[i]}(t)dt\right| =$$
$$\left|\int_0^1 \left[K_2^{[i]}(t) - \int_0^1 K_2^{[i]}(t)dt\right] K_2^{[i]}(t)dt\right| = T(K_2^{[i]}, K_2^{[i]}), \ i = 1, 2, 3.$$

Remark 3.4. Denote by Z_i, $i = 1, 2, 3$, the constants which appear in one of the following types of estimations obtained in Theorem 3.1 and Theorem 3.2, namely

$$\left|\tilde{\mathcal{R}}_2^{[i]}[f]\right| \leq Z_i \cdot \frac{M[f]-m[f]}{2},$$

$$\left|\tilde{\mathcal{R}}_2^{[i]}[f]\right| \leq Z_i \cdot (f'(1) - f'(0) - m[f]),$$

$$\left|\tilde{\mathcal{R}}_2^{[i]}[f]\right| \leq Z_i \cdot (M[f] - [f'(1) - f'(0)])$$

or

$$\left|\tilde{\mathcal{R}}_2^{[i]}[f]\right| \leq Z_i \cdot \sigma(f''; 0, 1).$$

Since for every $i = 1, 2, 3$ we have $Z_3 \leq Z_2 \leq Z_1$, for the corrected quadrature formulas, our results are better than Ujević and Mijić's results obtained in [16].

The corrected quadrature formulas (3.5), (3.7), and (3.9), respectively have degree of exactness 3, which is higher than the original rule, namely for $j = \overline{1,3}$, $\tilde{R}_2^{[j]}[e_i] = 0$ and $\tilde{R}_2^{[j]}[e_4] \neq 0$, where $e_i(x) = x^i$, $i = \overline{0,4}$. Using Peano's Theorem, the remainder term can be written

$$\mathcal{R}_4[f] = \int_0^1 K_4(t) f^{(4)}(t), \;\; K_4(t) = \mathcal{R}_4\left[\frac{(x-t)_+^3}{3!}\right], \tag{3.22}$$

where by $\mathcal{R}_4$ we denote the new integral representation of the remainder term of these quadrature formulas.

In the next part of this paper, using relation (3.22), we will give new estimations of the remainder term in quadrature formulas (3.5), (3.7), and (3.9), respectively.

Theorem 3.5. *If $f \in C^4[0,1]$, then the remainder term of quadrature formula (3.5)has the integral representation*

$$\mathcal{R}_4^{[1]}[f] = \int_0^1 K_4^{[1]}(t) f^{(4)}(t) dt, \;\; \textit{where}$$

$$K_4^{[1]}(t) = \begin{cases} \dfrac{1}{24} t^2 \left(t^2 - \dfrac{\sqrt{2}}{2} t - \dfrac{4-3\sqrt{2}}{8}\right), \; 0 \leq t \leq \dfrac{1}{2}, \\[2ex] \dfrac{1}{24}(1-t)^2 \left((1-t)^2 - \dfrac{\sqrt{2}}{2}(1-t) - \dfrac{4-3\sqrt{2}}{8}\right), \; \dfrac{1}{2} < t \leq 1, \end{cases}$$

and the following estimations hold

$$\begin{aligned}
\left|\mathcal{R}_4^{[1]}[f]\right| &\leq \sqrt{\int_0^1 \left(K_4^{[1]}(t)\right)^2 dt}\sqrt{\int_0^1 \left(f^{(4)}(t)\right)^2 dt} \\
&= \frac{\sqrt{23170 - 15645\sqrt{2}}}{80640}\|f^{(4)}\|_2 \approx 4.008 \times 10^{-4}\|f^{(4)}\|_2,
\end{aligned}$$

$$\begin{aligned}
\left|\mathcal{R}_4^{[1]}[f]\right| &\leq \int_0^1 \left|K_4^{[1]}(t)\right| dt \cdot \sup_{t\in[0,1]} |f^{(4)}(t)| \\
&= \frac{200-171\sqrt{2}-(90-68\sqrt{2})\sqrt{5-3\sqrt{2}}+2(15-8\sqrt{2})\sqrt{43-30\sqrt{2}}}{11520}\|f^{(4)}\|_\infty \\
&\approx 2.946 \times 10^{-4} \cdot \|f^{(4)}\|_\infty,
\end{aligned}$$

$$\begin{aligned}
\left|\mathcal{R}_4^{[1]}[f]\right| &\leq \sup_{t\in[0,1]} |K_4^{[1]}(t)| \cdot \int_0^1 |f^{(4)}(t)| dt \\
&= \frac{2-\sqrt{2}}{768} \cdot \|f^{(4)}\|_1 \approx 7.627 \times 10^{-4} \cdot \|f^{(4)}\|_1.
\end{aligned}$$

Theorem 3.6. *If $f \in C^4[0,1]$, then the remainder term of quadrature formula (3.7)has the integral representation*

$$\mathcal{R}_4^{[2]}[f] = \int_0^1 K_4^{[2]}(t) f^{(4)}(t) dt, \ \textit{where}$$

$$K_4^{[2]}(t) = \begin{cases} \frac{1}{24}t^2\left(t^2 - \frac{3}{4}t + \frac{1}{16}\right), \ 0 \leq t \leq \frac{1}{2}, \\ \frac{1}{24}(1-t)^2\left((1-t)^2 - \frac{3}{4}(1-t) + \frac{1}{16}\right), \ \frac{1}{2} < t \leq 1, \end{cases}$$

and the following estimations hold

$$\begin{aligned}
\left|\mathcal{R}_4^{[2]}[f]\right| &\leq \sqrt{\int_0^1 \left(K_4^{[2]}(t)\right)^2 dt}\sqrt{\int_0^1 \left(f^{(4)}(t)\right)^2 dt} \\
&= \frac{\sqrt{2905}}{161280}\|f^{(4)}\|_2 \approx 3.342 \times 10^{-4}\|f^{(4)}\|_2,
\end{aligned}$$

$$\begin{aligned}
\left|\mathcal{R}_4^{[2]}[f]\right| &\leq \int_0^1 \left|K_4^{[2]}(t)\right| dt \cdot \sup_{t\in[0,1]} |f^{(4)}(t)| \\
&= \frac{125\sqrt{5}-103}{737280} \cdot \|f^{(4)}\|_\infty \approx 2.394 \times 10^{-4} \cdot \|f^{(4)}\|_\infty,
\end{aligned}$$

$$\begin{aligned}\left|\mathcal{R}_4^{[2]}[f]\right| &\leq \sup_{t\in[0,1]}|K_4^{[2]}(t)|\cdot\int_0^1|f^{(4)}(t)|dt\\ &= \frac{1}{1536}\cdot\|f^{(4)}\|_1\approx 6.51\times10^{-4}\cdot\|f^{(4)}\|_1.\end{aligned}$$

Theorem 3.7. *If $f\in C^4[0,1]$, then the remainder term of quadrature formula (3.9)has the integral representation*

$$\mathcal{R}_4^{[3]}[f]=\int_0^1 K_4^{[3]}(t)f^{(4)}(t)dt,\ \textit{where}$$

$$K_4^{[3]}(t)=\begin{cases}\dfrac{1}{24}t^2\left(t^2-2(\sqrt{2}-1)t-\dfrac{4-3\sqrt{2}}{2}\right),\ 0\leq t\leq\dfrac{1}{2},\\[2ex] \dfrac{1}{24}(1-t)^2\left((1-t)^2-2(\sqrt{2}-1)(1-t)-\dfrac{4-3\sqrt{2}}{2}\right),\ \dfrac{1}{2}<t\leq1,\end{cases}$$

and the following estimations hold

$$\begin{aligned}\left|\mathcal{R}_4^{[3]}[f]\right| &\leq \sqrt{\int_0^1\left(K_4^{[3]}(t)\right)^2dt}\sqrt{\int_0^1\left(f^{(4)}(t)\right)^2dt}\\ &= \frac{\sqrt{68530-48405\sqrt{2}}}{40320}\|f^{(4)}\|_2\approx 2.148\times10^{-4}\|f^{(4)}\|_2,\end{aligned}$$

$$\begin{aligned}\left|\mathcal{R}_4^{[3]}[f]\right| &\leq \int_0^1\left|K_4^{[3]}(t)\right|dt\cdot\sup_{t\in[0,1]}|f^{(4)}(t)|\\ &= \frac{78470-55487\sqrt{2}-32(550-389\sqrt{2})\sqrt{10-7\sqrt{2}}}{5760}\cdot\|f^{(4)}\|_\infty\\ &\approx 1.461\times10^{-4}\cdot\|f^{(4)}\|_\infty,\end{aligned}$$

$$\begin{aligned}\left|\mathcal{R}_4^{[3]}[f]\right| &\leq \sup_{t\in[0,1]}|K_4^{[3]}(t)|\cdot\int_0^1|f^{(4)}(t)|dt\\ &= \frac{3-2\sqrt{2}}{384}\cdot\|f^{(4)}\|_1\approx 4.468\times10^{-4}\cdot\|f^{(4)}\|_1.\end{aligned}$$

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