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Two remarks on harmonic Bergman spaces in $\mathbb{B}^n$ and $\mathbb{R}^{n+1}_+$

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Abstract. Sharp estimates on distances in spaces of harmonic functions in the unit ball and the upper half space are obtained.

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1. Introduction and preliminaries

In this note we obtain distance estimates in spaces of harmonic functions on the unit ball and on the upper half space. This line of investigation can be considered as a continuation of papers [1], [4] and [5]. These results are contained in the second section of the paper. The first section is devoted to preliminaries and main definitions which are needed for formulations of main results. Almost all objects we define and definitions can be found in [2] and in [6].

Let $\mathbb{B}$ be the open unit ball in $\mathbb{R}^n$, $\mathbb{S} = \partial\mathbb{B}$ is the unit sphere in $\mathbb{R}^n$, for $x \in \mathbb{R}^n$ we have $x = rx'$, where $r = |x| = \sqrt{\sum_{j=1}^n x_j^2}$ and $x' \in \mathbb{S}$. Normalized Lebesgue measure on $\mathbb{B}$ is denoted by $dx = dx_1 \dots dx_n = r^{n-1} dr dx'$ so that $\int_{\mathbb{B}} dx = 1$. We denote the space of all harmonic functions in an open set Ω by $h(\Omega)$. In this paper letter C designates a positive constant which can change its value even in the same chain of inequalities.

For $0 < p < \infty$, $0 \le r < 1$ and $f \in h(\mathbb{B})$ we set

$$M_p(f, r) = \left(\int_{\mathbb{S}} |f(rx')|^p dx' \right)^{1/p},$$

with the usual modification to cover the case $p = \infty$.

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For $0 < p < \infty$ and $\alpha > -1$ we consider weighted harmonic Bergman spaces $A^p_\alpha = A^p_\alpha(\mathbb{B})$ defined by

$$A^p_\alpha = \left\{ f \in h(\mathbb{B}) : \|f\|^p_{A^p_\alpha} = \int_{\mathbb{B}} |f(x)|^p (1-|x|^2)^\alpha dx < \infty \right\}.$$

For $p = \infty$ this definition is modified in a standard manner:

$$A^\infty_\alpha = A^\infty_\alpha(\mathbb{B}) = \left\{ f \in h(\mathbb{B}) : \|f\|_{A^\infty_\alpha} = \sup_{x\in\mathbb{B}} |f(x)|(1-|x|^2)^\alpha < \infty \right\}, \quad \alpha > -1.$$

These spaces are complete metric spaces for $0 < p \leq \infty$, they are Banach spaces for $p \geq 1$.

Next we need certain facts on spherical harmonics and the Poisson kernel, see [2] for a detailed exposition. Let $Y^{(k)}_j$ be the spherical harmonics of order k, $1 \leq j \leq d_k$, on $\mathbb{S}$. Next,

$$Z^{(k)}_{x'}(y') = \sum_{j=1}^{d_k} Y^{(k)}_j(x') \overline{Y^{(k)}_j(y')}$$

are zonal harmonics of order k. Note that the spherical harmonics $Y^{(k)}_j$, ($k \geq 0$, $1 \leq j \leq d_k$) form an orthonormal basis of $L^2(\mathbb{S}, dx')$. Every $f \in h(\mathbb{B})$ has an expansion

$$f(x) = f(rx') = \sum_{k=0}^\infty r^k b_k \cdot Y^k(x'),$$

where $b_k = (b^1_k, \ldots, b^{d_k}_k)$, $Y^k = (Y^{(k)}_1, \ldots, Y^{(k)}_{d_k})$ and $b_k \cdot Y^k$ is interpreted in the scalar product sense: $b_k \cdot Y^k = \sum_{j=1}^{d_k} b^j_k Y^{(k)}_j$.

We denote the Poisson kernel for the unit ball by $P(x, y')$, it is given by

$$\begin{aligned} P(x,y') = P_{y'}(x) &= \sum_{k=0}^\infty r^k \sum_{j=1}^{d_k} Y^{(k)}_j(y') Y^{(k)}_j(x') \\ &= \frac{1}{n\omega_n} \frac{1-|x|^2}{|x-y'|^n}, \ x = rx' \in \mathbb{B}, \quad y' \in \mathbb{S}, \end{aligned}$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$. We are going to use also a Bergman kernel for A^p_β spaces, this is the following function

$$Q_\beta(x,y) = 2\sum_{k=0}^\infty \frac{\Gamma(\beta+1+k+n/2)}{\Gamma(\beta+1)\Gamma(k+n/2)} r^k \rho^k Z^{(k)}_{x'}(y'), \ x = rx', \ y = \rho y' \in \mathbb{B}. \tag{1.1}$$

For details on this kernel we refer to [2], where the following theorem can be found.

Theorem 1.1. [2] *Let $p \geq 1$ and $\beta \geq 0$. Then for every $f \in A^p_\beta$ and $x \in \mathbb{B}$ we have*

$$f(x) = \int_0^1 \int_{\mathbb{S}^{n-1}} Q_\beta(x,y) f(\rho y')(1-\rho^2)^\beta \rho^{n-1} d\rho dy', \ y = \rho y'.$$

This theorem is a cornerstone for our approach to distance problems in the case of the unit ball. The following lemma gives estimates for this kernel, see [2], [3].

Lemma 1.2. *1. Let $\beta > 0$. Then, for $x = rx', y = \rho y' \in \mathbb{B}$ we have*

$$|Q_\beta(x,y)| \leq \frac{C}{|\rho x - y'|^{n+\beta}}.$$

2. Let $\beta > -1$. Then

$$\int_{\mathbb{S}^{n-1}} |Q_\beta(rx', y)| dx' \leq \frac{C}{(1-r\rho)^{1+\beta}}, \ |y| = \rho, \quad 0 \leq r < 1.$$

3. Let $\beta > n-1$, , $0 \leq r < 1$ and $y' \in \mathbb{S}^{n-1}$. Then

$$\int_{\mathbb{S}^{n-1}} \frac{dx'}{|rx' - y'|^\beta} \leq \frac{C}{(1-r)^{\beta-n+1}}.$$

Lemma 1.3. [2] *Let $\alpha > -1$ and $\lambda > \alpha + 1$. Then*

$$\int_0^1 \frac{(1-r)^\alpha}{(1-r\rho)^\lambda} dr \leq C(1-\rho)^{\alpha+1-\lambda}, \ 0 \leq \rho < 1.$$

Lemma 1.4. *For $\delta > -1$, $\gamma > n + \delta$ and $\beta > 0$ we have*

$$\int_{\mathbb{B}} |Q_\beta(x,y)|^{\frac{\gamma}{n+\beta}} (1-|y|)^\delta dy \leq C(1-|x|)^{\delta-\gamma+n}, \ x \in \mathbb{B}.$$

Proof. Using Lemma 1.2 and Lemma 1.3 we obtain:

$$\begin{aligned}\int_{\mathbb{B}} |Q_\beta(x,y)|^{\frac{\gamma}{n+\beta}} (1-|y|)^\delta dy &\leq C \int_{\mathbb{B}} \frac{(1-|y|)^\delta}{|\rho r x' - y'|^\gamma} dy \\ &\leq C \int_0^1 (1-\rho)^\delta \int_{\mathbb{S}} \frac{dy'}{|\rho r x' - y'|^\gamma} dy' d\rho \\ &\leq C \int_0^1 (1-\rho)^\delta (1-r\rho)^{n-\gamma-1} d\rho \leq C(1-r)^{n+\delta-\gamma}.\end{aligned}$$

□

We set $\mathbb{R}^{n+1}_+ = \{(x,t) : x \in \mathbb{R}^n, t > 0\} \subset \mathbb{R}^{n+1}$. We usually denote points in $\mathbb{R}^{n+1}_+$ by $z = (x,t)$ or $w = (y,s)$ where $x, y \in \mathbb{R}^n$ and $s, t > 0$.

For $0 < p < \infty$ and $\alpha > -1$ we consider spaces

$$\tilde{A}^p_\alpha(\mathbb{R}^{n+1}_+) = \tilde{A}^p_\alpha = \left\{ f \in h(\mathbb{R}^{n+1}_+) : \int_{\mathbb{R}^{n+1}_+} |f(x,t)|^p t^\alpha dx dt < \infty \right\}.$$

Also, for $p = \infty$ and $\alpha > 0$, we set

$$\tilde{A}^\infty_\alpha(\mathbb{R}^{n+1}_+) = \tilde{A}^\infty_\alpha = \left\{ f \in h(\mathbb{R}^{n+1}_+) : \sup_{(x,t) \in \mathbb{R}^{n+1}_+} |f(x,t)| t^\alpha < \infty \right\}.$$

These spaces have natural (quasi)-norms, for $1 \leq p \leq \infty$ they are Banach spaces and for $0 < p \leq 1$ they are complete metric spaces.

We denote the Poisson kernel for $\mathbb{R}^{n+1}_+$ by $P(x,t)$, i.e.

$$P(x,t) = c_n \frac{t}{(|x|^2 + t^2)^{\frac{n+1}{2}}}, \ x \in \mathbb{R}^n, t > 0.$$

For an integer $m \geq 0$ we introduce a Bergman kernel $Q_m(z,w)$, where $z = (x,t) \in \mathbb{R}^{n+1}_+$ and $w = (y,s) \in \mathbb{R}^{n+1}_+$, by

$$Q_m(z,w) = \frac{(-2)^{m+1}}{m!} \frac{\partial^{m+1}}{\partial t^{m+1}} P(x-y, t+s).$$

The terminology is justified by the following result from [2].

Theorem 1.5. *Let $0 < p < \infty$ and $\alpha > -1$. If $0 < p \leq 1$ and $m \geq \frac{\alpha+n+1}{p} - (n+1)$ or $1 \leq p < \infty$ and $m > \frac{\alpha+1}{p} - 1$, then*

$$f(z) = \int_{\mathbb{R}^{n+1}_+} f(w) Q_m(z,w) s^m dyds, \ f \in \tilde{A}^p_\alpha, \quad z \in \mathbb{R}^{n+1}_+. \tag{1.2}$$

The following elementary estimate of this kernel is contained in [2]:

$$|Q_m(z,w)| \leq C \left[|x-y|^2 + (s+t)^2 \right]^{-\frac{n+m+1}{2}}, \quad z = (x,t), w = (y,s) \in \mathbb{R}^{n+1}_+. \tag{1.3}$$

2. Estimates for distances in harmonic function spaces in the unit ball and related problems in $\mathbb{R}^{n+1}_+$

In this section we investigate distance problems both in the case of the unit ball and in the case of the upper half space. The method we use here originated in [7], see [1], [4], [5] for various modification of this method.

Lemma 2.1. *Let $0 < p < \infty$ and $\alpha > -1$. Then there is a $C = C_{p,\alpha,n}$ such that for every $f \in A^p_\alpha(\mathbb{B})$ we have*

$$|f(x)| \leq C(1-|x|)^{-\frac{\alpha+n}{p}} \|f\|_{A^p_\alpha}, \ x \in \mathbb{B}.$$

Proof. We use subharmonic behavior of $|f|^p$ to obtain

$$|f(x)|^p \leq \frac{C}{(1-|x|)^n} \int_{B(x,\frac{1-|x|}{2})} |f(y)|^p dy$$

$$\leq C \frac{(1-|x|)^{-\alpha}}{(1-|x|)^n} \int_{B(x,\frac{1-|x|}{2})} |f(y)|^p (1-|y|)^\alpha dy \leq C(1-|x|)^{-\alpha-n} \|f\|^p_{A^p_\alpha}. \qquad \square$$

This lemma shows that A^p_α is continuously embedded in $A^\infty_{\frac{\alpha+n}{p}}$ and motivates the distance problem that is investigated in Theorem 2.3.

Lemma 2.2. *Let $0 < p < \infty$ and $\alpha > -1$. Then there is $C = C_{p,\alpha,n}$ such that for every $f \in \tilde{A}^p_\alpha$ and every $(x,t) \in \mathbb{R}^{n+1}_+$ we have*

$$|f(x,t)| \leq C y^{-\frac{\alpha+n+1}{p}} \|f\|_{\tilde{A}^p_\alpha}. \tag{2.1}$$

The above lemma states that $\tilde{A}^p_\alpha$ is continuously embedded in $\tilde{A}^\infty_{\frac{\alpha+n+1}{p}}$, its proof is analogous to that of Lemma 2.1.

For $\epsilon > 0$, $t > 0$ and $f \in h(\mathbb{B})$ we set

$$U_{\epsilon,t}(f) = U_{\epsilon,t} = \{x \in \mathbb{B} : |f(x)|(1-|x|)^t \geq \epsilon\}.$$

Theorem 2.3. *Let $p > 1$, $\alpha > -1$, $t = \frac{\alpha+n}{p}$ and $\beta > \max(\frac{\alpha+n}{p} - 1, \frac{\alpha}{p})$. Set, for $f \in A^\infty_{\frac{\alpha+n}{p}}(\mathbb{B})$:*

$$t_1(f) = \operatorname{dist}_{A^\infty_{\frac{\alpha+n}{p}}}(f, A^p_\alpha),$$

$$t_2(f) = \inf\left\{\epsilon > 0 : \int_{\mathbb{B}} \left(\int_{U_{\epsilon,t}} |Q_\beta(x,y)|(1-|y|)^{\beta-t}dy\right)^p (1-|x|)^\alpha dx < \infty\right\}.$$

Then $t_1(f) \asymp t_2(f)$.

Proof. We begin with inequality $t_1(f) \geq t_2(f)$. Assume $t_1(f) < t_2(f)$. Then there are $0 < \epsilon_1 < \epsilon$ and $f_1 \in A^p_\alpha$ such that $\|f - f_1\|_{A^\infty_t} \leq \epsilon_1$ and

$$\int_{\mathbb{B}} \left(\int_{U_{\epsilon,t}(f)} |Q_\beta(x,y)|(1-|y|)^{\beta-t}dy\right)^p (1-|x|)^\alpha dx = +\infty.$$

Since $(1-|x|)^t|f_1(x)| \geq (1-|x|)^t|f(x)| - (1-|x|)^t|f(x) - f_1(x)|$ for every $x \in \mathbb{B}$ we conclude that $(1-|x|)^t|f_1(x)| \geq (1-|x|)^t|f(x)| - \epsilon_1$ and therefore

$$(\epsilon - \epsilon_1)\chi_{U_{\epsilon,t}(f)}(x)(1-|x|)^{-t} \leq |f_1(x)|, \; x \in \mathbb{B}.$$

Hence

$$\begin{aligned}
+\infty &= \int_{\mathbb{B}} \left(\int_{U_{\epsilon,t}(f)} |Q_\beta(x,y)|(1-|y|)^{\beta-t}dy\right)^p (1-|x|)^\alpha dx \\
&= \int_{\mathbb{B}} \left(\int_{\mathbb{B}} \frac{\chi_{U_{\epsilon,t}(f)}(y)}{(1-|y|)^t}|Q_\beta(x,y)|(1-|y|)^\beta dy\right)^p (1-|x|)^\alpha dx \\
&\leq C_{\epsilon,\epsilon_1} \int_{\mathbb{B}} \left(\int_{\mathbb{B}} |f_1(y)||Q_\beta(x,y)|(1-|y|)^\beta dy\right)^p (1-|x|)^\alpha dx = M,
\end{aligned}$$

and we are going to prove that M is finite, arriving at a contradiction. Let q be the exponent conjugate to p. We have, using Lemma 1.4,

$$\begin{aligned}
I(x) &= \left(\int_{\mathbb{B}} |f_1(y)|(1-|y|)^\beta |Q_\beta(x,y)|dy\right)^p \\
&= \left(\int_{\mathbb{B}} |f_1(y)|(1-|y|)^\beta |Q_\beta(x,y)|^{\frac{1}{n+\beta}(\frac{n}{p}+\beta-\epsilon)} |Q_\beta(x,y)|^{\frac{1}{n+\beta}(\frac{n}{q}+\epsilon)}dy\right)^p \\
&\leq \int_{\mathbb{B}} |f_1(y)|^p(1-|y|)^{p\beta}|Q_\beta(x,y)|^{\frac{n+p\beta-p\epsilon}{n+\beta}}dy \left(\int_{\mathbb{B}} |Q_\beta(x,y)|^{\frac{n+q\epsilon}{n+\beta}}dy\right)^{p/q} \\
&\leq C(1-|x|)^{-p\epsilon} \int_{\mathbb{B}} |f_1(y)|^p(1-|y|)^{p\beta}|Q_\beta(x,y)|^{\frac{n+p\beta-p\epsilon}{n+\beta}}dy
\end{aligned}$$

for every $\epsilon > 0$. Choosing $\epsilon > 0$ such that $\alpha - p\epsilon > -1$ we have, by Fubini's theorem and Lemma 1.4:

$$\begin{aligned}
M &\leq C \int_{\mathbb{B}} |f_1(y)|^p(1-|y|)^{p\beta} \int_{\mathbb{B}} (1-|x|)^{\alpha-p\epsilon}|Q_\beta(x,y)|^{\frac{n+p\beta-p\epsilon}{n+\beta}}dxdy \\
&\leq C \int_{\mathbb{B}} |f_1(y)|^p(1-|y|)^\alpha dy < \infty.
\end{aligned}$$

In order to prove the remaining estimate $t_1(f) \leq Ct_2(f)$ we fix $\epsilon > 0$ such that the integral appearing in the definition of $t_2(f)$ is finite and use Theorem 1.1, with $\beta > \max(t-1, 0)$:

$$\begin{aligned} f(x) &= \int_{\mathbb{B}\setminus U_{\epsilon,t}(f)} Q_\beta(x,y) f(y)(1-|y|^2)^\beta dy + \int_{U_{\epsilon,t}(f)} Q_\beta(x,y) f(y)(1-|y|^2)^\beta dy \\ &= f_1(x) + f_2(x). \end{aligned}$$

Since, by Lemma 1.4, $|f_1(x)| \leq 2^\beta \int_{\mathbb{B}} |Q_\beta(x,y)|(1-|w|)^{\beta-t} dy \leq C(1-|x|)^{-t}$ we have $\|f_1\|_{A_t^\infty} \leq C\epsilon$. Thus it remains to show that $f_2 \in A_\alpha^p$ and this follows from

$$\|f_2\|_{A_\alpha^p}^p \leq \|f\|_{A_t^\infty}^p \int_{\mathbb{B}} \left(\int_{U_{\epsilon,t}(f)} |Q_\beta(x,y)|(1-|y|^2)^{\beta-t} dy \right)^p (1-|x|)^\alpha dx < \infty. \qquad \square$$

The above theorem has a counterpart in the $\mathbb{R}_+^{n+1}$ setting. As a preparation for this result we need the following analogue of Lemma 1.4.

Lemma 2.4. *For $\delta > -1$, $\gamma > n+1+\delta$ and $m \in \mathbb{N}_0$ we have*

$$\int_{\mathbb{R}_+^{n+1}} |Q_m(z,w)|^{\frac{\gamma}{n+m+1}} s^\delta dy ds \leq Ct^{\delta-\gamma+n+1}, \ t > 0.$$

Proof. Using Fubini's theorem and estimate (1.3) we obtain

$$\begin{aligned} I(t) = \int_{\mathbb{R}_+^{n+1}} |Q_m(z,w)|^{\frac{\gamma}{n+m+1}} s^\delta dy ds &\leq C \int_0^\infty s^\delta \left(\int_{\mathbb{R}^n} \frac{dy}{[|y|^2 + (s+t)^2]^\gamma} \right) ds \\ &= C \int_0^\infty s^\delta (s+t)^{n-\gamma} ds = Ct^{\delta-\gamma+n+1}. \end{aligned} \qquad \square$$

For $\epsilon > 0$, $\lambda > 0$ and $f \in h(\mathbb{R}_+^{n+1})$ we set:

$$V_{\epsilon,\lambda}(f) = \{(x,t) \in \mathbb{R}_+^{n+1} : |f(x,t)| t^\lambda \geq \epsilon\}.$$

Theorem 2.5. *Let $p > 1$, $\alpha > -1$, $\lambda = \frac{\alpha+n+1}{p}$, $m \in \mathbb{N}_0$ and $m > \max(\frac{\alpha+n+1}{p} - 1, \frac{\alpha}{p})$. Set, for $f \in \tilde{A}^\infty_{\frac{\alpha+n+1}{p}}(\mathbb{R}_+^{n+1})$:*

$$s_1(f) = \operatorname{dist}_{\tilde{A}^\infty_{\frac{\alpha+n+1}{p}}}(f, \tilde{A}_\alpha^p),$$

$$s_2(f) = \inf \left\{ \epsilon > 0 : \int_{\mathbb{R}_+^{n+1}} \left(\int_{V_{\epsilon,\lambda}} Q_m(z,w) s^{m-\lambda} dy ds \right)^p t^\alpha dx dt < \infty \right\}.$$

Then $s_1(f) \asymp s_2(f)$.

The proof of this theorem closely parallels the proof of the previous one, in fact, the role of Lemma 1.4 is taken by Lemma 2.4 and the role of Theorem 1.1 is taken by Theorem 1.5. We leave details to the reader.

Remark. Results of this note very recently were extended by the authors to all values of positive p. Proofs of these assertions are heavily based on the well-known so-called Whitney decomposition of the upper halfspace of $\mathbb{R}^{n+1}$ and the unit ball B and some nice properties and estimates of the related Whitney cubes and harmonic functions on them, which partially can be found in [6].

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