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An asymptotic formula for Jain's operators

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Abstract. We investigate a class of linear positive operators of discrete type depending on a real parameter. By additional conditions imposed on this parameter, the considered sequence turns into an approximation process.

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1. Introduction

In 1970 G.C. Jain has introduced in [1] a new class of positive linear operators based on a Poisson-type distribution. In 1984 starting from Jain's operator, S.Umar and Q. Razi introduced in [6] a class of modified Szász-Mirakjan operators and studied their approximation properties. Later on, in 1995 L. Rempulska approached in [5] a Voronovskaja type result for some operators of Szász-Mirakjan type.

Present paper aims to prove a Voronovskaja type result for a class of linear positive operators of discrete type depending on a real parameter. In first section of this paper, we collect some basic results concerning Jain's operator, $P_n^{[\beta]}$, and we also compute other similar relations starting from those who are already proved by Jain.

In Section 2 will be highlighted the main results obtained and Section 3 will host the proofs of the stated results.

First of all, we recall the form of a Poisson-type distribution.

Lemma 1.1. ([1]) *For* $0 < \alpha < \infty,\quad |\beta| < 1,$ *let*

$$\omega_\beta(k,\alpha) = \alpha(\alpha + k\beta)^{k-1} e^{-(\alpha+k\beta)}/k!\ ;\ \ k \in \mathbb{N}_0. \tag{1.1}$$

then

$$\sum_{k=0}^{\infty} \omega_\beta(k,\alpha) = 1. \tag{1.2}$$

Lemma 1.2. ([1]) *Let*

$$S(r,\alpha,\beta) = \sum_{k=0}^{\infty} (\alpha + \beta k)^{k+r-1} e^{-(\alpha+\beta k)}/k!,\ \ r = 0, 1, 2, \ldots \tag{1.3}$$

and

$$\alpha S(0,\alpha,\beta)=1. \tag{1.4}$$

Then

$$S(r,\alpha,\beta)=\sum_{k=0}^{\infty}\beta^k(\alpha+k\beta)S(r-1,\alpha+k\beta,\beta). \tag{1.5}$$

The functions $S(r,\alpha,\beta)$ satisfy the recurrence formula

$$S(r,\alpha,\beta)=\alpha S(r-1,\alpha,\beta)+\beta S(r,\alpha+\beta,\beta). \tag{1.6}$$

The above formula implies

$$S(1,\alpha,\beta)=\sum_{k=0}^{\infty}\beta^k=\frac{1}{1-\beta} \tag{1.7}$$

and

$$S(2,\alpha,\beta)=\sum_{k=0}^{\infty}\frac{\beta^k(\alpha+k\beta)}{1-\beta}=\frac{\alpha}{(1-\beta)^2}+\frac{\beta^2}{(1-\beta)^3}. \tag{1.8}$$

We easily get

Lemma 1.3. *Let S be the function defined in Lemma* 1.2. *Then, one has*

(i) $S(3,\alpha,\beta)=\dfrac{\alpha^3}{(1-\beta)^3}+\dfrac{3\alpha\beta^2}{(1-\beta)^4}+\dfrac{\beta^3+2\beta^4}{(1-\beta)^5}$,

(ii) $S(4,\alpha,\beta)=\dfrac{\alpha^3}{(1-\beta)^4}+\dfrac{6\alpha^2\beta^2}{(1-\beta)^5}+\dfrac{\alpha\beta^3(11\beta+4)}{(1-\beta)^6}+\dfrac{6\beta^6+8\beta^5+\beta^4}{(1-\beta)^7}$.

The operator defined by Jain is given by

$$(P_n^{[\beta]}f)(x)=\sum_{k=0}^{\infty}\omega_\beta(k,nx)\cdot f\left(\frac{k}{n}\right),\quad f\in C[0,\infty), \tag{1.9}$$

where $0\le\beta<1$ and $\omega_\beta(k,\alpha)$ has been defined in (1.1).

Remark 1.4. If we take $\beta=0$ in (1.9) we obtain Szász -Mirakjan operator [3], [4].

$$(P_n^{[0]}f)(x)\equiv(S_nf)(x)=e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}\cdot\frac{k}{n},\quad x\ge 0. \tag{1.10}$$

We denote by $e_j(t)$ the monomial of degree j, $e_j(t)=t^j$.
Taking in view Lemma 1.2, in [1] has been established the following identities.

$$(P_n^{[\beta]}e_0)(x)=1. \tag{1.11}$$

$$(P_n^{[\beta]}e_1)(x)=xS(1,nx+\beta,\beta)=\frac{x}{1-\beta}. \tag{1.12}$$

$$\begin{aligned}(P_n^{[\beta]}e_2)(x)&=\frac{x}{n}\Big[S(2,nx+2\beta,\beta)+S(1,nx+\beta,\beta)\Big]\\&=\frac{x^2}{(1-\beta)^2}+\frac{x}{n(1-\beta)^3}.\end{aligned} \tag{1.13}$$

2. Main results

In what follows, $C_2[0,\infty)$ represents the space of all continuous functions having the second derivative continuous.

In this section we first define the function

$$\varphi_x \in C_2[0,\infty),\ \varphi_x(t) = t - x. \tag{2.1}$$

We also compute the values of $P_n^{[\beta]}$ on φ_x^3 and φ_x^4.

In order to present our main theorem, we need the following lemmas.

Lemma 2.1. *The operators defined by* (1.9) *verify the following identities.*

(i) $$(P_n^{[\beta]}e_3)(x) = \frac{x^3}{(1-\beta)^3} + \frac{3x^2}{n(1-\beta)^4} - \frac{x(6\beta^4 - 6\beta^3 - 2\beta - 1)}{n^2(1-\beta)^5}.$$

(ii) $$(P_n^{[\beta]}e_4)(x) = \frac{x^4}{(1-\beta)^4} + \frac{6x^3}{n(1-\beta)^5} - \frac{x^2(36\beta^4 - 72\beta^3 + 36\beta^2 - 8\beta - 7)}{n^2(1-\beta)^6} + \frac{x(105\beta^5 - 14\beta^4 - 2\beta^3 + 12\beta^2 + 8\beta + 1)}{n^3(1-\beta)^7}.$$

Remark 2.2. Examining the relations (i) and (ii) in Lemma 2.1, based on Korovkin theorem [2] and Theorem 2.1 in [1], we may observe that $(P_n^{[\beta]})_{n\geq 1}$ does not form an approximation process. In order to transform it into an approximation process, we replace the constant β by a number $\beta_n \in [0,1)$.

If

$$\lim_{n\to\infty} \beta_n = 0, \tag{2.2}$$

then Lemma 2.1 ensures us that $\lim_{n\to\infty}(P_n^{[\beta_n]}e_j)(x) = x^j$, $j = \overline{0,2}$ uniformly in $C([0,\infty))$.

On the basis of relations (1.12), (1.13) and Lemma 1.2 we deduce the following identities.

$$\begin{aligned}(P_n^{[\beta_n]}\varphi_x)(x) &= \sum_{k=0}^{\infty}(nx + k\beta_n)^{k-1}\cdot e^{-(nx+k\beta_n)}\frac{1}{k!}\cdot\frac{k-x}{n}\\ &= (P_n^{[\beta_n]}e_1)(x) - x(P_n^{[\beta_n]}e_0)(x)\\ &= \frac{x}{1-\beta_n} - x.\end{aligned} \tag{2.3}$$

$$\begin{aligned}(P_n^{[\beta_n]}\varphi_x^2)(x) &= \sum_{k=0}^{\infty}(nx + k\beta_n)^{k-1}\cdot e^{-(nx+k\beta_n)}\frac{1}{k!}\cdot\frac{(k-x)^2}{n^2}\\ &= (P_n^{[\beta_n]}e_2)(x) - 2x(P_n^{[\beta_n]}e_1)(x) + x^2(P_n^{[\beta_n]}e_0)(x)\\ &= \frac{x^2}{(1-\beta_n)^2} - \frac{2x^2}{1-\beta_n} + x^2 + \frac{x}{n(1-\beta_n)^3}.\end{aligned} \tag{2.4}$$

where φ_x is defined by (2.1).

Lemma 2.3. *Let the operator* $P_n^{[\beta_n]}$ *be defined by relation* (1.9) *and let* φ_x *be given by* (2.1). *Then*

(i) $$(P_n^{[\beta_n]}\varphi_x^3)(x) = \frac{x^3}{(1-\beta_n)^3} - \frac{3x^3}{(1-\beta_n)^2} + \frac{3x^3}{1-\beta_n} - x^3 + \frac{3x^2}{n(1-\beta_n)^4} - \frac{3x^2}{n(1-\beta_n)^3} - \frac{x(6\beta_n^4 - 6\beta_n^3 - 2\beta_n - 1)}{n^2(1-\beta_n)^5}.$$

(ii) $$(P_n^{[\beta_n]}\varphi_x^4)(x) = \frac{x^4}{(1-\beta_n)^4} - \frac{4x^4}{(1-\beta_n)^3} + \frac{6x^4}{(1-\beta_n)^2} - \frac{4x^4}{1-\beta_n} + x^4 + \frac{6x^3}{n(1-\beta_n)^5} - \frac{12x^3}{n(1-\beta_n)^4} + \frac{6x^3}{n(1-\beta_n)^3} - \frac{x^2(36\beta_n^4 - 72\beta_n^3 + 36\beta_n^2 - 8\beta_n - 7)}{n^2(1-\beta_n)^6} + \frac{4x^2(6\beta_n^4 - 6\beta_n^3 - 2\beta_n - 1)}{n^2(1-\beta_n)^5} + \frac{x(105\beta_n^5 - 14\beta_n^4 - 2\beta_n^3 + 12\beta_n^2 + 8\beta_n + 1)}{n^3(1-\beta_n)^7}.$$

Lemma 2.4. *Let* $P_n^{[\beta_n]}$ *be the Jain operator and let* φ_x *be defined in* (2.1). *In addition, if (2.2) holds, then*

$$P_n^{[\beta_n]}\varphi_x^4 \le \frac{12x^3}{n(1-\beta_n)^5} + \frac{24x^2}{n^2(1-\beta_n)^5} + \frac{106x}{n^3(1-\beta_n)^7}.$$

We may now present the main result.

Theorem 2.5. *Let* $f \in C_2([0,\infty))$ *and let the operator* $P_n^{[\beta_n]}$ *be defined as in* (1.9). *If* (2.2) *holds, then*

$$\lim_{n\to\infty} n\left(P_n^{[\beta_n]}(f;x) - f(x)\right) = \frac{x}{2}f''(x), \quad \forall x > 0.$$

3. Proofs

Proof of Lemma 1.3.

(i) $$\begin{aligned} S(3,\alpha,\beta) &= \sum_{k=0}^{\infty}\beta^k(\alpha+k\beta)S(2,\alpha+k\beta,\beta) \\ &= \sum_{k=0}^{\infty}\beta^k(\alpha+k\beta)\left(\frac{\alpha+k\beta}{(1-\beta)^2} + \frac{\beta^2}{(1-\beta)^3}\right) \\ &= \frac{1}{(1-\beta)^2}\sum_{k=0}^{\infty}\beta^k(\alpha+k\beta)(\alpha+k\beta) + \frac{\beta^2}{(1-\beta)^3}\sum_{k=0}^{\infty}\beta^k(\alpha+k\beta) \\ &= \frac{1}{(1-\beta)^2}\left(\frac{\alpha^2}{1-\beta} + \frac{2\alpha\beta^2}{(1-\beta)^2} + \frac{\beta^3(1+\beta)}{(1-\beta)^3}\right) \\ &+ \frac{\beta^2}{(1-\beta)^3}\left(\frac{\alpha}{1-\beta} + \frac{\beta^2}{(1-\beta)^2}\right) = \frac{\alpha^3}{(1-\beta)^3} + \frac{3\alpha\beta^2}{(1-\beta)^4} + \frac{\beta^3+2\beta^4}{(1-\beta)^5}. \end{aligned}$$

(ii)
$$
\begin{aligned}
S(4,\alpha,\beta) &= \sum_{k=0}^{\infty}\beta^k(\alpha+k\beta)S(3,\alpha+k\beta,\beta)\\
&= \sum_{k=0}^{\infty}\beta^k(\alpha+k\beta)\left[\frac{(\alpha+k\beta)^2}{(1-\beta)^3}+\frac{3(\alpha+k\beta)\beta^2}{(1-\beta)^4}+\frac{\beta^3+2\beta^4}{(1-\beta)^5}\right]\\
&= \frac{1}{(1-\beta)^3}\sum_{k=0}^{\infty}\beta^k(\alpha+k\beta)^3+\frac{3\beta^2}{(1-\beta)^4}\sum_{k=0}^{\infty}\beta^k(\alpha+k\beta)^2\\
&\quad+\frac{\beta^3+2\beta^4}{(1-\beta)^5}\sum_{k=0}^{\infty}\beta^k(\alpha+k\beta)\\
&= \frac{\alpha^3}{(1-\beta)^4}+\frac{6\alpha^2\beta^2}{(1-\beta)^5}+\frac{\alpha\beta^3(11\beta+4)}{(1-\beta)^6}+\frac{6\beta^6+8\beta^5+\beta^4}{(1-\beta)^7}. \qquad \square
\end{aligned}
$$

Proof of Lemma 2.1.

(i)
$$
\begin{aligned}
P_n^{[\beta]}(e_3;x) &= xn\sum_{k=0}^{\infty}(nx+k\beta)^{k-1}\cdot e^{-(nx+k\beta)}\frac{1}{k!}\cdot\frac{k^3}{n^3}\\
&= \frac{x}{n^2}\Big[S(3,nx+3\beta,\beta)+3S(2,nx+2\beta,\beta)+S(1,nx+\beta,\beta)\Big]\\
&= \frac{x^3}{(1-\beta)^3}+\frac{3x^2}{n(1-\beta)^4}-\frac{x(6\beta^4-6\beta^3-2\beta-1)}{n^2(1-\beta)^5}.
\end{aligned}
$$

(ii)
$$
\begin{aligned}
(P_n^{[\beta]}e_4)(x) &= xn\sum_{k=0}^{\infty}(nx+k\beta)^{k-1}\cdot e^{-(nx+k\beta)}\frac{1}{k!}\cdot\frac{k^4}{n^4}\\
&= \frac{x}{n^3}\Big[S(4,nx+4\beta,\beta)+6S(3,nx+3\beta,\beta)\\
&\quad+7S(2,nx+2\beta,\beta)+S(1,nx+\beta,\beta)\Big]\\
&= \frac{x^4}{(1-\beta)^4}+\frac{6x^3}{n(1-\beta)^5}-\frac{x^2(36\beta^4-72\beta^3+36\beta^2-8\beta-7)}{n^2(1-\beta)^6}\\
&\quad+\frac{x(105\beta^5-14\beta^4-2\beta^3+12\beta^2+8\beta+1)}{n^3(1-\beta)^7}. \qquad \square
\end{aligned}
$$

Proof of Lemma 2.3.

(i)
$$
\begin{aligned}
(P_n^{[\beta_n]}\varphi_x^3)(x) &= (P_n^{[\beta_n]}e_3)(x)-3x(P_n^{[\beta_n]}e_2)(x)+3x^2(P_n^{[\beta_n]}e_1)(x)-x^3(P_n^{[\beta_n]}e_0)(x)\\
&= \frac{x^3}{(1-\beta_n)^3}-\frac{3x^3}{(1-\beta_n)^2}+\frac{3x^3}{1-\beta_n}-x^3+\frac{3x^2}{n(1-\beta_n)^4}\\
&\quad-\frac{3x^2}{n(1-\beta_n)^3}-\frac{x(6\beta_n^4-6\beta_n^3-2\beta_n-1)}{n^2(1-\beta_n)^5}.
\end{aligned}
$$

(ii)
$$
\begin{aligned}
(P_n^{[\beta_n]}\varphi_x^4)(x) &= (P_n^{[\beta_n]}e_4)(x) - 4x(P_n^{[\beta_n]}e_3)(x) + 6x^2(P_n^{[\beta_n]}e_2)(x)\\
&\quad -4x^3(P_n^{[\beta_n]}e_1)(x) + x^4(P_n^{[\beta_n]}e_0)(x)\\
&= \frac{x^4}{(1-\beta_n)^4} - \frac{4x^4}{(1-\beta_n)^3} + \frac{6x^4}{(1-\beta_n)^2} - \frac{4x^4}{1-\beta_n} + x^4\\
&\quad + \frac{6x^3}{n(1-\beta_n)^5} - \frac{12x^3}{n(1-\beta_n)^4} + \frac{6x^3}{n(1-\beta_n)^3}\\
&\quad - \frac{x^2(36\beta_n^4 - 72\beta_n^3 + 36\beta_n^2 - 8\beta_n - 7)}{n^2(1-\beta_n)^6}\\
&\quad + \frac{4x^2(6\beta_n^4 - 6\beta_n^3 - 2\beta_n - 1)}{n^2(1-\beta_n)^5}\\
&\quad + \frac{x(105\beta_n^5 - 14\beta_n^4 - 2\beta_n^3 + 12\beta_n^2 + 8\beta_n + 1)}{n^3(1-\beta_n)^7}. \qquad \square
\end{aligned}
$$

Proof of Lemma 2.4. Starting from relation (*ii*) in Lemma 2.3, the entire proof of Lemma 2.4 is based on the following simple increases:

$$\frac{6x^3}{n(1-\beta_n)^3} \le \frac{6x^3}{n(1-\beta_n)^5},$$

$$6\beta_n^4 - 6\beta_n^3 - 2\beta_n - 1 \le 6, \quad 105\beta_n^5 - 14\beta_n^4 - 2\beta_n^3 + 12\beta_n^2 + 8\beta_n + 1 \le 106. \qquad \square$$

Proof of Theorem 2.5. Let $f, f', f'' \in C_2([0,\infty))$ and $x \in [0,\infty)$ be fixed. By the Taylor formula we have

$$f(t) = f(x) + f'(x)(t-x) + \frac{1}{2}f''(x)(t-x)^2 + r(t;x)(t-x)^2, \tag{3.1}$$

where $r(t;x)$ is the Peano form of the remainder, $r(\cdot;x) \in C_2([0,\infty))$ and

$$\lim_{t\to x} r(t;x) = 0.$$

Let φ_x be given by (2.1). We apply $P_n^{[\beta_n]}$ to (3.1) and we get

$$
\begin{aligned}
(P_n^{[\beta_n]}f)(x) - f(x) &= (P_n^{[\beta_n]}\varphi_x)(x)\cdot f'(x) + \frac{1}{2}(P_n^{[\beta_n]}\varphi_x^2)(x)\cdot f''(x)\\
&\quad + (P_n^{[\beta_n]}\varphi_x^2 \cdot r(\cdot;x))(x).
\end{aligned} \tag{3.2}
$$

Using the relations (2.3) and (2.4) one obtains

$$
\begin{aligned}
(P_n^{[\beta_n]}f)(x) - f(x) &= \left(\frac{x}{1-\beta_n} - x\right)f'(x)\\
&\quad + \frac{1}{2}\left[\frac{x^2}{(1-\beta_n)^2} - \frac{2x^2}{1-\beta_n} + x^2 + \frac{x}{n(1-\beta_n)^3}\right]f''(x)\\
&\quad + (P_n^{[\beta_n]}\varphi_x^2 \cdot r(\cdot;x))(x)
\end{aligned} \tag{3.3}
$$

For the last term, by applying the Cauchy-Schwartz inequality, we get

$$0 \le |(P_n^{[\beta_n]}\varphi_x^2 \cdot r(\cdot;x))(x)| \le \sqrt{(P_n^{[\beta_n]}\varphi_x^4)(x)} \cdot \sqrt{(P_n^{[\beta_n]}r^2(\cdot;x))(x)} \tag{3.4}$$

We have marked that $\lim_{t\to x} r(t,x) = 0$. In harmony with Remark 2.2 we have

$$\lim_{n\to\infty} P_n^{[\beta_n]}(r^2(x,x);x) = 0. \tag{3.5}$$

On the basis of (2.2) , (3.4), (3.5) and Lemma 2.4 , we get that

$$\lim_{n\to\infty} n\left(P_n^{[\beta_n]}(f;x) - f(x)\right) = \frac{x}{2}f''(x). \qquad \square$$

Remark 3.1. Considering Jain's operator P_n^β and taking $\beta = \beta_n$, with β_n satisfying (2.2) we have rediscovered the genuine Voronovskaja result for Szász operators (1.10). The same genuine Voronovskaja result was found once again in in [5, *Eq. (20)*] while studying some operators of Szász-Mirakjan type.

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