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The pairs of linear positive operators according to a general method of construction

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Abstract. We provide an estimate between discrete operators and their associated integral operators. A lot of examples are given.

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1. Introduction

Using a well-known method of construction for the pairs of linear positive operators, we give an estimate of the difference between the terms of these pairs. Let $(L_n)_{n\geq 1}$ be a sequence of linear positive operators, $L_n : \mathcal{L} \longrightarrow \mathcal{F}(I)$ having the form

$$L_n(f;x) = \sum_{k\geq 0} h_{n,k}(x)\nu_{n,k}(f),\ x \in I,\ f \in \mathcal{L}, \tag{1.1}$$

where $\mathcal{L}$ is the space of all real measurable bounded functions on I, for which $L_n f$ is well defined and $\mathcal{F}(I)$ is the space of all real valued functions defined on I.

Also the functions $f \in \mathcal{L}$ are $\mu_{n,k}$- integrable on I, $\mu_{n,k}$ being probability Borel measures on I and so that, the linear positive functional

$$\nu_{n,k}(f) = \int_I f(u)d\mu_{n,k}(u),\ f \in \mathcal{L} \tag{1.2}$$

is well defined for each $n \geq 1$ and $k \geq 0$. We assume that, $x_{n,k} \in I$ is the barycenter of the probability Borel measure $\mu_{n,k}$, i.e.

$$x_{n,k} = \nu_{n,k}(e_1) = \int_I u d\mu_{n,k}(u). \tag{1.3}$$

As usual, $e_i(x) = x^i$, $x \in I$, $i = 0, 1, 2, \ldots$ denote the test functions. In (1.1) we consider the positive functions $h_{n,k} \in \mathbf{C}_B(I)$ so that, $\sum_{k\geq 0} h_{n,k}(x) = 1$. With these remarks, the linear positive operators (1.1) preserve the constant functions. It is well known (see [3], [4], [8], [9], [22], [25]) that the sequence of positive linear operators (1.1) is associated with the next sequence of linear positive operators

$$P_n(f;x) = \sum_{k\geq 0} h_{n,k}(x) f\left(x_{n,k}\right), \; n \geq 1, \; k \geq 0, \; x \in I, \; f \in \mathcal{L} \tag{1.4}$$

where we consider that $\mathcal{L}$ is the common set of all real functions f on I for which $L_n f$, $\nu_{n,k}(f)$, $P_n f$ are well defined. We remark that

$$L_n\left(e_1;x\right) = P_n\left(e_1;x\right) = \sum_{k\geq 0} h_{n,k}(x) x_{n,k}.$$

In the next section, we present an estimate on the difference between the terms of the pair of operators (L_n, P_n).

2. An estimate on the difference $|L_n f - P_n f|$

The basic result for the next theorem is the barycenter inequality of ν a probability Radon measure on I

$$\nu(h) \geq h(b), \; h \in \mathbf{C}_B(I) \text{ convex},$$

with $b = \nu(e_1)$ the barycenter of probability Radon measure ν.

Indeed, if $h = \dfrac{\|f''\|}{2} e_2 \pm f$, $f \in \mathbf{C}_B^2(I)$, then the barycenter inequality becomes

$$|\nu(f) - f(b)| \leq \frac{\|f''\|}{2}\left[\nu(e_2) - b^2\right],$$

where $\|\cdot\|$ is the uniform norm.

Theorem 2.1. *If $(L_n)_{n\geq 1}$, $(P_n)_{n\geq 1}$, are two sequences of linear positive operators defined as (1.1) respectively (1.4) for $f \in \mathbf{C}_B^2(I) \subset \mathcal{L}$, then for $x \in I$ we shall have the estimation*

$$|L_n(f;x) - P_n(f;x)| \leq \frac{\|f''\|}{2} \sum_{k\geq 0} h_{n,k}(x)\left[\nu_{n,k}(e_2) - \left(\nu_{n,k}(e_1)\right)^2\right]. \tag{2.1}$$

3. Some examples

The main purpose of the present paper is to establish results of type (2.1) for a number of well-known pairs operators (L_n, P_n) used in approximation theory. In our examples, the functions $h_{n,k}(x)$ are the next discrete probability functions:

- **(i).** $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$, $x \in [0,1]$, $0 \leq k \leq n$, $n \geq 1$ (the binomial probability)
- **(ii).** $s_{n,k}(x) = e^{-nx} \dfrac{(nx)^k}{k!}$, $x \geq 0$, $k \geq 0$, $n \geq 1$ (the Poisson probability)
- **(iii).** $\pi_{n,k}(x) = \binom{n+k-1}{k} \dfrac{x^k}{(1+x)^{n+k}}$, $x \geq 0$, $k \geq 0$, $n \geq 1$ (the negative binomial probability or the Pascal probability).

Also, we consider the next probability density functions:

(iv). $\gamma_n(x) = \begin{cases} 0 & , x < 0 \\ \dfrac{1}{\Gamma(n)} x^{n-1} e^{-x} & , x \geq 0,\ n > 0 \end{cases}$ (the Gamma probability density function),

$\Gamma(a) = \int\limits_0^\infty e^{-x} x^{a-1} dx,\ a > 0$ (the Gamma function)

(v). $\beta_{k,n}(x) = \begin{cases} 0 & , x \notin [0,1] \\ \dfrac{1}{B(k,n)} x^{k-1}(1-x)^{n-1} & , x \in [0,1],\ k > 0,\ n > 0 \end{cases}$ (the Beta probability density function),

$B(k,n) = \int\limits_0^1 x^{k-1}(1-x)^{n-1} dx,\ k > 0,\ n > 0$ (the Beta function of the first kind)

(vi). $b_{k,n}(x) = \begin{cases} 0 & , x < 0 \\ \dfrac{1}{B(k,n)} \cdot \dfrac{x^{k-1}}{(1+x)^{n+k}} & , x \geq 0,\ k > 0,\ n > 0 \end{cases}$ (the Inverse-Beta probability density function),

$B(k,n) = \int\limits_0^\infty \dfrac{x^{k-1}}{(1+x)^{n+k}} dx,\ k > 0,\ n > 0$ (the Beta function of the second kind)

(vii). $\omega(x) = \begin{cases} 0 & , x \notin [a,b] \\ \dfrac{1}{b-a} & , x \in [a,b],\ a < b \end{cases}$ (the uniform continuous probability density function on $[a,b]$).

It is easy to see that, between the discrete probability functions and the probability density functions there are a link in the next sense:

$$\begin{cases} p_{n,k}(x) & = \dfrac{1}{n+1} \beta_{k+1,n-k+1}(x) \\ & = \beta_{k+1,n-k+1}(x) \int\limits_0^1 p_{n,k}(x) dx,\ 0 \leq k \leq n,\ n \geq 1,\ x \in [0,1], \\ s_{n,k}(x) & = \gamma_{k+1}(nx),\ k \geq 0,\ n \geq 1,\ x \geq 0, \\ \pi_{n,k}(x) & = \dfrac{1}{n-1} b_{k+1,n-1}(x),\ k \geq 0,\ n > 1,\ x \geq 0. \end{cases}$$

Using a general method of construction for the pairs of linear positive operators, we give the next examples.

A. Let $h_{n,k}(x) = p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$, $n \geq 1$, $k = \overline{0,n}$, $I = [0,1]$ be, the binomial probability.

A1. Taking

$$\nu_{n,k}(f) = \begin{cases} \int_0^1 f(u)\beta_{k,n-k}(u)du & , 1 \le k \le n-1,\ n \ge 2 \\ f(0) & , k=0 \\ f(1) & , k=n \end{cases}$$

$$= \begin{cases} (n-1)\int_0^1 f(u)p_{n-2,k-1}(u)du & , 1 \le k \le n-1,\ n \ge 2 \\ f(0) & , k=0 \\ f(1) & , k=n \end{cases}$$

and

$$x_{n,k} = \nu_{n,k}(e_1) = \begin{cases} \int_0^1 u\beta_{k,n-k}(u)du = \frac{k}{n} & , 1 \le k \le n-1,\ n \ge 2 \\ 0 & , k=0 \\ 1 & , k=n \end{cases}$$

we obtain the pair of operators $(DB_n(f;x), B_n(f;x))$ where

$$DB_n(f;x) = p_{n,0}(x)f(0) + (n-1)\sum_{k=1}^{n-1} p_{n,k}(x)\int_0^1 f(u)p_{n-2,k-1}(u)du$$
$$+p_{n,n}(x)f(1)$$

is the genuine Bernstein - Durrmeyer operator, defined and investigated by Goodman, T.N.T., Sharma, A. [15], [16] and $B_n(f;x) = \sum_{k=0}^{n} p_{n,k}(x)f\left(\frac{k}{n}\right)$ is the classical Bernstein operator.

We have with (2.1), for $f \in \mathbf{C}^2[0,1]$ the next estimation

$$|DB_n(f;x) - B_n(f;x)| \le \frac{\|f''\|}{2} \cdot \frac{n-1}{n+1} \cdot \frac{x(1-x)}{n}.$$

A2. If we have

$$\nu^*_{n,k}(f) = \int_0^1 f(u)\beta_{k+1,n-k+1}(u)du = (n+1)\int_0^1 f(u)p_{n,k}(u)du$$

and

$$x_{n,k} := \nu^*_{n,k}(e_1) = \frac{B(k+2,n-k+1)}{B(k+1,n-k+1)} = \frac{k+1}{n+2},\ 0 \le k \le n,$$

then, we get the pair of operators $(DB_n^*(f,x), B_n^*(f,x))$ with

$$\begin{aligned} DB_n^*(f;x) &= \sum_{k=0}^{n} p_{n,k}(x) \int_0^1 f(u)\beta_{k+1,n-k+1}(u)du \\ &= (n+1)\sum_{k=0}^{n} p_{n,k}(x) \int_0^1 f(u)p_{n,k}(u)du \end{aligned}$$

the classical Bernstein-Durrmeyer operator defined by Durrmeyer J.L.[12] and extensively studied by Derriennic M. M. [10], Ditzian Z., Ivanov K. [11], Gonska H.H., Zhou X. [14] and $B_n^*(f;x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k+1}{n+2}\right)$, the Bernstein-Stancu operator [27]. For they, if $f \in \mathbf{C}^2[0,1]$ then

$$|DB_n^*(f;x) - B_n^*(f;x)| \leq \frac{\|f''\|}{2} \cdot \frac{n(n-1)x(1-x)+n+1}{(n+2)^2(n+3)}.$$

A3. For the functional $\lambda_{n,k}(f) = (n+1) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(u)du$ associated with the uniform continuous probability density function

$$\omega_{n,k}(u) = \begin{cases} 0 & , u \notin \left[\frac{k}{n+1}, \frac{k+1}{n+1}\right] \\ n+1 & , u \in \left[\frac{k}{n+1}, \frac{k+1}{n+1}\right] \end{cases}$$

we have

$$x_{n,k} := \lambda_{n,k}(e_1) = (n+1) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} u du = \frac{2k+1}{2(n+1)},\ 0 \leq k \leq n,\ n \geq 1.$$

So, the pair of operators becomes $(KB_n(f;x), B_n^{**}(f;x))$ with

$$KB_n(f;x) = (n+1)\sum_{k=0}^{n} p_{n,k}(x) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(u)du \tag{3.1}$$

the Bernstein-Kantorovich [18] operator and

$$B_n^{**}(f;x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{2k+1}{2(n+1)}\right)$$

the Bernstein-Stancu [27] operator.

Using the Theorem 2.1 we have for $f \in \mathbf{C}^2[0,1]$ the estimate

$$|KB_n^*(f;x) - B_n^{**}(f;x)| \leq \frac{\|f''\|}{24(n+1)^2}.$$

A4. Consider now, the linear positive functional

$$\lambda_{n,k}^{*}(f;a_n,b_n)=\frac{n+1}{b_n-a_n}\int\limits_{\frac{k+a_n}{n+1}}^{\frac{k+b_n}{n+1}} f(u)du,\ 0\le a_n<b_n\le 1,$$

which can be associated with the uniform continuous probability density function

$$\omega_{n,k}^{*}(u;a_n,b_n)=\begin{cases}0 & ,\ u\notin\left[\frac{k+a_n}{n+1},\frac{k+b_n}{n+1}\right]\\[2ex] \frac{n+1}{b_n-a_n} & ,\ u\in\left[\frac{k+a_n}{n+1},\frac{k+b_n}{n+1}\right]\end{cases},\ 0\le a_n<b_n\le 1$$

and

$$x_{n,k}:=\lambda_{n,k}^{*}(e_1;a_n,b_n)=\frac{2k+a_n+b_n}{2(n+1)},\ 0\le k\le n,\ n\ge 1.$$

We obtain the pair of operators $(AL_n(f;x),BS_n(f;x))$ with

$$AL_n(f;x)=\sum_{k=0}^{n}p_{n,k}(x)\left(\frac{n+1}{b_n-a_n}\int\limits_{\frac{k+a_n}{n+1}}^{\frac{k+b_n}{n+1}} f(u)du\right)$$

$0\le a_n<b_n\le 1$, $f\in\mathbf{C}[0,1]$, $x\in[0,1]$, a generalization of Bernstein-Kantorovich operators (3.1) which was given by Altomare F., Leonessa V. [2] and its associated operators is the Bernstein-Stancu type operator [27]

$$BS_n(f;x)=\sum_{k=0}^{n}p_{n,k}(x)f\left(\frac{2k+a_n+b_n}{2(n+1)}\right),\ (a_n\to 0,\ b_n\to 1,\ n\to\infty).$$

If $f\in\mathbf{C}^2[0,1]$ we obtain with (2.1) the next estimate

$$|AL_n(f;x)-BS_n(f;x)|\le\|f''\|\,\frac{(b_n-a_n)^2}{24(n+1)^2}.$$

A5. Taking for $a,b>-1$, $\alpha\ge 0$, $c:=c_n=[n^{\alpha}]$ the positive linear beta functional

$$T_{k,n}^{a,b,c}(f)=\frac{1}{B\left(ck+a+1,c(n-k)+b+1\right)}\int\limits_0^1 f(u)u^{ck+a}(1-u)^{c(n-k)+b}du$$

and its associated linear positive beta operator

$$T_n^{a,b,c}(f;x)$$
$$=\frac{1}{B\left(cnx+a+1,cn(1-x)+b+1\right)}\int\limits_0^1 f(u)u^{cnx+a}(1-u)^{cn(1-x)+b}du$$

$x \in [0,1]$, we have with (1.1) a linear positive operator defined and investigated by Mache D. H. [19], [20], which represents a link between the Durrmeyer operator with Jacobi weights (for $\alpha = 0$) and the classical Bernstein operator

$$DM_n(f;x) = \sum_{k=0}^{n} p_{n,k}(x) T_{n,k}^{a,b,c}(f) = B_n\left(T_n^{a,b,c}\right)(f;x).$$

Because $T_{n,k}^{a,b,c}(e_1) = \dfrac{ck+a+1}{cn+a+b+2}$, Rasa I. [25] using (1.4) defined and investigated a new linear positive operator associated with $DM_n(f)$,

$$R_n(f;x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{ck+a+1}{cn+a+b+2}\right).$$

If $f \in \mathbf{C}^2[0,1]$ then for the pair of operators $(DM_n f, R_n f)$ with Theorem 2.1 we have the estimate

$$\begin{aligned}&|DM_n(f;x) - R_n(f;x)| \\ &\leq \frac{\|f''\|}{2} \cdot \frac{c^2 n(n-1)x(1-x) + cn(b-a)x + cn(a+1) + (a+1)(b+1)}{2(cn+a+b+2)^2(cn+a+b+3)}.\end{aligned}$$

B. Let $h_{n,k} := s_{n,k}(x) = e^{-nx}\dfrac{(nx)^k}{k!}$, $x \geq 0$, $k \geq 0$, $n \geq 1$ be, the Poisson probability function.

B1. If

$$\tau_{n,k}(f) = \begin{cases} n\int_0^\infty f(u)\gamma_k(nu)du = n\int_0^\infty f(u)s_{n,k-1}(u)du \\ \qquad = \dfrac{\int_0^\infty f(u)s_{n,k-1}(u)du}{\int_0^\infty s_{n,k-1}(u)du} & , k \geq 1 \\ 0 & , k = 0 \end{cases}$$

is a linear positive functional and

$$x_{n,k} := \tau_{n,k}(e_1) = \begin{cases} n\int_0^\infty u\gamma_k(nu)du = n\int_0^\infty s_{n,k-1}(u)du = \dfrac{k}{n} & , k \geq 1 \\ 0 & , k = 0 \end{cases}$$

then we have with (1.4) the classical Szasz-Mirakjan operator

$$S_n(f;x) = \sum_{k=0}^{\infty} s_{n,k}(x) f\left(\frac{k}{n}\right). \tag{3.2}$$

and with (1.1) the Phillips operator [23] or the genuine Szasz-Durrmeyer operator

$$\begin{aligned} DS_n(f,x) &= s_{n,0}(x)f(0) + \sum_{k=1}^{\infty} s_{n,k}(x) \int_0^{\infty} f(u)s_{n,k-1}(u)du \\ &= s_{n,0}(x)f(0) + \sum_{k=1}^{\infty} s_{n,k}(x) \int_0^{\infty} f(u)\gamma_k(nu)du \end{aligned}$$

For these two operators, with the Theorem 2.1, we obtain the estimation

$$\begin{aligned} |DS_n(f;x) - S_n(f;x)| &\le \frac{\|f''\|}{2} \sum_{k\ge 0} s_{n,k}(x) \left[\tau_{n,k}(e_2) - (\tau_{n,k}(e_1))^2\right] \\ &= \frac{\|f''\|}{2} \cdot \frac{x}{n}, \end{aligned}$$

$x \ge 0$, $f \in \mathbf{C}_B^2[0,\infty)$.

B2. We consider the linear positive functional

$$\tau^*_{n,k}(f) = n\int_0^{\infty} f(u)s_{n,k}(u)du = n\int_0^{\infty} f(u)\gamma_{k+1}(nu)du$$

and

$$x_{n,k} := \tau^*_{n,k}(e_1) = n\int_0^{\infty} u\gamma_{k+1}(nu)du = \frac{k+1}{n},\ k \ge 0.$$

So, using (1.4) we get a modification of Szasz-Mirakjan operator

$$S_n^*(f;x) = \sum_{k=0}^{\infty} s_{n,k}(x) f\left(\frac{k+1}{n}\right)$$

and with (1.1) the Szasz-Durrmeyer type operator, which was defined and studied by Mazhar, Totik [21]

$$\begin{aligned} DS_n^*(f,x) &= n\sum_{k=0}^{\infty} s_{n,k}(x) \int_0^{\infty} f(u)s_{n,k}(u)du \\ &= n\sum_{k=0}^{\infty} s_{n,k}(x) \int_0^{\infty} f(u)\gamma_{k+1}(nu)du. \end{aligned}$$

If $f \in \mathbf{C}_B^2[0,\infty)$ then

$$\begin{aligned} |DS_n^*(f;x) - S_n^*(f;x)| &\le \frac{\|f''\|}{2} \sum_{k\ge 0} s_{n,k}(x) \left[\tau^*_{n,k}(e_2) - \left(\tau^*_{n,k}(e_1)\right)^2\right] \\ &\le \frac{\|f''\|}{2} \left(\frac{x}{n} + \frac{1}{n^2}\right). \end{aligned}$$

B3. Taking the linear positive functional

$$\varphi_{n,k}(f)=\begin{cases} f(0) & ,\ k=0\\ \int\limits_0^\infty f(u)b_{k,n+1}(u)du & ,\ k>0,\end{cases}$$

with $b_{k,n+1}(u)=\begin{cases}0 & ,\ u\le 0 \text{ or } k=0\\ \dfrac{1}{B(k,n+1)}\cdot\dfrac{u^k}{(1+u)^{n+k+1}} & ,\ u>0,\ k>0\end{cases}$ the Inverse-Beta probability density function we obtain the knots

$$x_{n,k}:=\varphi_{n,k}(e_1)=\begin{cases}0 & ,\ k=0\\ \int\limits_0^\infty ub_{k,n+1}(u)du=\dfrac{B(k+1,n)}{B(k,n+1)}=\dfrac{k}{n} & ,\ k>0.\end{cases}$$

According to (1.1) we have the Szasz-Inverse Beta operator, defined by Govil N.K., Gupta, V., Noor M. A., [17] and studied by Finta Z., Govil N. K., Gupta V., [13], Cismaiu C., [5], [6], [7]

$$SA_n(f,x) \;=\; f(0)s_{n,0}(x)+\sum_{k=1}^{\infty}s_{n,k}(x)\int\limits_0^\infty f(u)b_{k,n+1}(u)du$$

and according to (1.4) we have the Szasz-Mirakjan S_nf operator (3.2). For the pair of operators (SA_nf,S_nf), if $f\in \mathbf{C}_B^2[0,\infty)$ we get the next estimate

$$\begin{aligned}|SA_n(f;x)-S_n(f;x)| &\le \frac{\|f''\|}{2}\sum_{k=1}^{\infty}s_{n,k}(x)\left[\varphi_{n,k}(e_2)-(\varphi_{n,k}(e_1))^2\right]\\ &\le \frac{\|f''\|}{2}\cdot\frac{1}{n-1}\left(x(x+1)+\frac{x}{n}\right),\ n>1.\end{aligned}$$

B4. For the linear functional

$$\phi_{n,k}(f)=\int\limits_{\frac{k}{n}}^{\frac{k+1}{n}}f(u)\rho_{n,k}(u)du=n\int\limits_{\frac{k}{n}}^{\frac{k+1}{n}}f(u)du,\ n\ge 1,\ k\ge 0$$

with

$$\rho_{n,k}(u)=\begin{cases}0 & ,\ u\notin\left[\dfrac{k}{n+1},\dfrac{k+1}{n+1}\right]\\ n & ,\ u\in\left[\dfrac{k}{n+1},\dfrac{k+1}{n+1}\right],\ n\ge 1,\ k\ge 0\end{cases}$$

the uniform continuous probability density function, we have

$$x_{n,k}:=\phi_{n,k}(e_1)=\int\limits_{\frac{k}{n}}^{\frac{k+1}{n}}u\rho_{n,k}(u)du=\frac{2k+1}{2n},\ n\ge 1,\ k\ge 0$$

and with (1.4) we obtain a modification of Szasz-Mirakjan operator

$$S_n^{**}(f;x) = \sum_{k=0}^{\infty} s_{n,k}(x) f\left(\frac{2k+1}{2n}\right).$$

The associated operator with (1.1) is the classical Szasz-Kantorovich operator [18] $K_n S_n^{**} : L_1([0,\infty)) \longrightarrow B([0,\infty))$,

$$L_1([0,\infty)) = \left\{ f : [0,\infty) \longrightarrow \mathbb{R}, \text{ measurable on } [0,\infty), \int_0^{\infty} |f(x)|\, dx < \infty \right\}$$

defined

$$K_n S_n^{**}(f;x) = n \sum_{k=0}^{\infty} s_{n,k}(x) \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(u)du$$

and so, using the Theorem 2.1 we obtain

$$|K_n S_n^{**}(f;x) - S_n^{**}(f;x)| \quad \leq \quad \frac{\|f''\|}{2} \cdot \frac{1}{12n^2},\ f \in \mathbf{C}_B^2[0,\infty).$$

C. Now, we consider the negative binomial probability or the Pascal probability

$$h_{n,k}(x) := \pi_{n,k}(x) = \frac{1}{n-1} b_{k+1,n-1}(x) = \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}}$$

$x \geq 0,\ k \geq 0,\ n > 1.$

C1. Let $\sigma_{n,k}(f) = (n-1) \int_0^{\infty} f(u)\pi_{n,k}(u)du = \int_0^{\infty} f(u)b_{k+1,n-1}(u)du$,be the linear functional which is defined whenever f is either a real-valued bounded measurable function on $[0,\infty)$ or a continuous function on $[0,\infty)$ such that $f(x) = O(x^r)$, $0 < r < n-1$. We obtain

$$x_{n,k} := \sigma_{n,k}(e_1) = \int_0^{\infty} u b_{k+1,n-1}(u)du = \frac{B(k+2,n-2)}{B(k+1,n-1)} = \frac{k+2}{n-2},\ n > 2.$$

We get with (1.1) the Baskakov-Durrmeyer operator, defined and investigated by Sahai A., Prasad G. [26]:

$$\begin{aligned} DM_n(f,x) &= (n-1) \sum_{k=0}^{\infty} \pi_{n,k}(x) \int_0^{\infty} f(u)\pi_{n,k}(u)du \\ &= \sum_{k=0}^{\infty} \pi_{n,k}(x) \int_0^{\infty} f(u)b_{k+1,n-1}(u)du,\ n > 1,\ x \geq 0 \end{aligned}$$

and with (1.4) a modification of Baskakov operator

$$M_n(f;x) = \sum_{k=0}^{\infty} \pi_{n,k}(x) f\left(\frac{k+1}{n-2}\right),\ n > 2.$$

So, for $n > 3$, $x \geq 0$, $f \in \mathbf{C}_B^2[0,\infty)$ with (2.1) we obtain the estimate

$$|DM_n(f;x) - M_n(f;x)| \leq \frac{\|f''\|}{2} \cdot \frac{n(n+1)x^2 + (n-2)(nx+1)}{(n-2)^2(n-3)}.$$

C2. If the functional (1.2) is

$$\nu_{n,k}(f) := (n+1)\int_0^\infty f(u)\pi_{n+2,k-1}(u)du = \int_0^\infty f(u)b_{k,n+1}(u)du,\ k \geq 1$$

then we get

$$x_{n,k} := \nu_{n,k}(e_1) = \begin{cases} \int_0^\infty u b_{k,n+1}(u)du = \dfrac{B(k+1,n)}{B(k,n+1)} = \dfrac{k}{n} & ,\ k > 0 \\ 0 & ,\ k = 0 \end{cases}$$

So, we have with (1.1) the genuine Baskakov-Durrmeyer operator

$$\begin{aligned} DM_n^*(f,x) &= f(0)\pi_{n,0}(x) + (n+1)\sum_{k=1}^\infty \pi_{n,k}(x)\int_0^\infty f(u)\pi_{n+2,k-1}(u)du \\ &= f(0)\pi_{n,0}(x) + (n+1)\sum_{k=1}^\infty \pi_{n,k}(x)\int_0^\infty f(u)b_{k,n+1}(u)du \end{aligned}$$

and with (1.4) the classical Baskakov operator

$$M_n^*(f;x) = \sum_{k=0}^\infty \pi_{n,k}(x) f\left(\frac{k}{n}\right).$$

Using the Inverse-Beta operator or the Stancu operator of the second kind [28]:

$$W_n(f;x) = \begin{cases} f(0) & ,\ x = 0 \\ \dfrac{1}{B(nx,n+1)} \int_0^\infty f(u)\dfrac{u^{nx-1}}{(1+u)^{nx+n+1}}du & ,\ x > 0, \end{cases}$$

for which

$$\begin{cases} W_n(e_0;x) = 1 \\ W_n(e_1;x) = x \\ W_n(e_2;x) = x^2 + \dfrac{x(x+1)}{n-1} & ,\ n > 1 \end{cases}$$

we have $DM_n^*(f) = M_n^*(W_n)(f;x)$. If $f \in \mathbf{C}_B^2[0,\infty)$, $n > 1$, $x \geq 0$, then we obtain the next estimation

$$\begin{aligned} &|DM_n^*(f;x) - M_n^*(f;x)| \\ &\leq \frac{\|f''\|}{2}\sum_{k=1}^\infty \pi_{n,k}(x)\left[W_n\left(e_2;\frac{k}{n}\right) - \left(W_n\left(e_1,\frac{k}{n}\right)\right)^2\right] \\ &\leq \frac{\|f''\|}{2} \cdot \frac{(n+1)x^2 + x}{n(n-1)} + \frac{x}{n-1}. \end{aligned}$$

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