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Stability in neutral nonlinear dynamic equations on time scale with unbounded delay

Abdelouaheb Ardjouni and Ahcene Djoudi

Abstract. Let $\mathbb{T}$ be a time scale which is unbounded above and below and such that $0 \in \mathbb{T}$. Let $id-r : \mathbb{T} \to \mathbb{T}$ be such that $(id-r)(\mathbb{T})$ is a time scale. We use the contraction mapping theorem to obtain stability results about the zero solution for the following neutral nonlinear dynamic equations with unbounded delay

$$x^{\triangle}(t) = -a(t)x^{\sigma}(t) + b(t)G\left(x^2(t), x^2(t-r(t))\right) + c(t)\left(x^2\right)^{\tilde{\triangle}}(t-r(t)),\ t \in \mathbb{T},$$

and

$$x^{\triangle}(t) = -a(t)x^{\sigma}(t) + b(t)G(x(t), x(t-r(t))) + c(t)x^{\tilde{\triangle}}(t-r(t)),\ t \in \mathbb{T},$$

where $f^{\triangle}$ is the $\triangle$-derivative on $\mathbb{T}$ and $f^{\tilde{\triangle}}$ is the $\triangle$-derivative on $(id-r)(\mathbb{T})$. We provide interesting examples to illustrate our claims.

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1. Introduction

The concept of time scales analysis is a fairly new idea. In 1988, it was introduced by the German mathematician Stefan Hilger in his Ph.D. thesis [11]. It combines the traditional areas of continuous and discrete analysis into one theory. After the publication of two textbooks in this area (by Bohner and Peterson, 2001, 2003, [5]-[6]), more and more researchers were getting involved in this fast-growing field of mathematics.

The study of dynamic equations brings together the traditional research areas of (ordinary and partial) differential and difference equations. It allows one to handle these two research areas at the same time, hence shedding light on the reasons for their seeming discrepancies. In fact, many new results for the continuous and discrete cases have been obtained by studying the more general time scales case (see [1]-[4], [8]-[13] and the references therein).

The reader can find more details on the materials and basic properties used in our section 2 in the first chapter of Bohner and Peterson book [5] pages 1-50 and can find good examples of dynamic equations in Chapter 2 [6] pages 17-46.

We have studied dynamic nonlinear equations with functional delay on a time scale and have obtained some interesting results concerning the existence of periodic solutions (see [1]-[3]) and this work is a continuation. Here, we focus on two neutral nonlinear dynamic equations which, for our delight, have not been yet studied by mean of fixed point technic on time scales.

There is no doubt that the Liapunov method have been used successfully to investigate stability properties of wide variety of ordinary, functional and partial equations. Nevertheless, the application of this method to problem of stability in differential equations with delay has encountered serious difficulties if the delay is unbounded or if the equation has unbounded term (see [7]-[10] and references therein). It has been noticed (see [8]-[10]) that some of theses difficulties vanish by using the fixed point technic. Other advantages of fixed point theory over Liapunov's method is that the conditions of the former are average while those of the latter are pointwise.

Below, we consider the following neutral nonlinear dynamic equations with unbounded delay given by

$$x^{\triangle}(t) = -a(t)\,x^{\sigma}(t) + b(t)\,G\left(x^{2}(t), x^{2}(t-r(t))\right) + c(t)\left(x^{2}\right)^{\widetilde{\triangle}}(t-r(t)),\ t \in \mathbb{T}, \tag{1.1}$$

and

$$x^{\triangle}(t) = -a(t)\,x^{\sigma}(t) + b(t)\,G(x(t), x(t-r(t))) + c(t)\,x^{\widetilde{\triangle}}(t-r(t)),\ t \in \mathbb{T}, \tag{1.2}$$

where $\mathbb{T}$ is an unbounded above and below time scale. Throughout this paper we assume that $0 \in \mathbb{T}$ for convenience. We also assume that $a, b : \mathbb{T} \to \mathbb{R}$ are continuous and that $c : \mathbb{T} \to \mathbb{R}$ is continuously delta-differentiable. In order for the function $x(t-r(t))$ to be well-defined and differentiable over $\mathbb{T}$, we assume that $r : \mathbb{T} \to \mathbb{R}$ is positive and twice continuously delta-differentiable, and that $id - r : \mathbb{T} \to \mathbb{T}$ is an increasing mapping such that $(id - r)(\mathbb{T})$ is closed where id is the identity function on the time scale $\mathbb{T}$. Throughout this paper, intervals subscripted with a $\mathbb{T}$ represent real intervals intersected with $\mathbb{T}$. For example, for $a, b \in \mathbb{T}$, $[a,b]_{\mathbb{T}} = [a,b] \cap \mathbb{T} = \{t \in \mathbb{T} : a \leq b\}$.

In recent years, when $\mathbb{T} = \mathbb{R}$, a number of investigators had studied stability of differential equations by mean of various fixed point techniques (see [7]-[10] and papers therein and we refer to [14] for fixed point theorems). In this work we use the fixed point technique based on the contraction mapping theorem to prove that the zero solution solution of equation 1.1 (respectively 1.2) is stable and illustrate our theory by giving examples.

In Section 2, we present some preliminary material that we will need through the remainder of the paper. We present our main results on stability by using the contraction mapping principle in Section 3 and we provide two examples to illustrate our work.

2. Preliminaries

In this section, we consider some advanced topics in the theory of dynamic equations on a time scales. Again, we remind that for a review of this topic we direct the reader to the monographs of Bohner and Peterson [5], Chapter 1 and Chapter 2, pages 1-78 and [6] pages 1-16.

A time scale $\mathbb{T}$ is a closed nonempty subset of $\mathbb{R}$. For $t \in \mathbb{T}$ the forward jump operator σ, and the backward jump operator ρ, respectively, are defined as $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$ and $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$. These operators allow elements in the time scale to be classified as follows. We say t is right scattered if $\sigma(t) > t$ and right dense if $\sigma(t) = t$. We say t is left scattered if $\rho(t) < t$ and left dense if $\rho(t) < t$. The graininess function $\mu : \mathbb{T} \to [0, \infty)$, is defined by $\mu(t) = \sigma(t) - t$ and gives the distance between an element and its successor. We set $\inf \emptyset = \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$. If $\mathbb{T}$ has a left scattered maximum M, we define $\mathbb{T}^k = \mathbb{T} \setminus \{M\}$. Otherwise, we define $\mathbb{T}^k = \mathbb{T}$. If $\mathbb{T}$ has a right scattered minimum m, we define $\mathbb{T}_k = \mathbb{T} \setminus \{m\}$. Otherwise, we define $\mathbb{T}_k = \mathbb{T}$.

Let $t \in \mathbb{T}^k$ and let $f : \mathbb{T} \to \mathbb{R}$. The delta derivative of $f(t)$, denoted $f^{\triangle}(t)$, is defined to be the number (when it exists), with the property that, for each $\epsilon > 0$, there is a neighborhood U of t such that

$$\left| f(\sigma(t)) - f(s) - f^{\triangle}(t)[\sigma(t) - s] \right| \leq \epsilon |\sigma(t) - s|,$$

for all $s \in U$. If $\mathbb{T} = \mathbb{R}$ then $f^{\triangle}(t) = f'(t)$ is the usual derivative. If $\mathbb{T} = \mathbb{Z}$ then $f^{\triangle}(t) = \triangle f(t) = f(t+1) - f(t)$ is the forward difference of f at t.

A function f is right dense continuous (rd-continuous), $f \in C_{rd} = C_{rd}(\mathbb{T}, \mathbb{R})$, if it is continuous at every right dense point $t \in \mathbb{T}$ and its left-hand limits exist at each left dense point $t \in \mathbb{T}$. The function $f : \mathbb{T} \to \mathbb{R}$ is differentiable on $\mathbb{T}^k$ provided $f^{\triangle}(t)$ exists for all $t \in \mathbb{T}^k$.

We are now ready to state some properties of the delta-derivative of f. Note $f^{\sigma}(t) = f(\sigma(t))$.

Theorem 2.1. [5, Theorem 1.20] *Assume $f, g : \mathbb{T} \to \mathbb{R}$ are differentiable at $t \in \mathbb{T}^k$ and let α be a scalar.*

(i) $(f+g)^{\triangle}(t) = f^{\triangle}(t) + g^{\triangle}(t)$.

(ii) $(\alpha f)^{\triangle}(t) = \alpha f^{\triangle}(t)$.

(ii) The product rules

$$\begin{aligned} (fg)^{\triangle}(t) &= f^{\triangle}(t)\, g(t) + f^{\sigma}(t)\, g^{\triangle}(t), \\ (fg)^{\triangle}(t) &= f(t)\, g^{\triangle}(t) + f^{\triangle}(t)\, g^{\sigma}(t). \end{aligned}$$

(iv) If $g(t)\, g^{\sigma}(t) \neq 0$ then

$$\left(\frac{f}{g}\right)^{\triangle}(t) = \frac{f^{\triangle}(t)\, g(t) - f(t)\, g^{\triangle}(t)}{g(t)\, g^{\sigma}(t)}.$$

The next theorem is the chain rule on time scales.

Theorem 2.2 (Chain Rule). [5, Theorem 1.93] *Assume* $\nu : \mathbb{T} \to \mathbb{R}$ *is strictly increasing and* $\widetilde{\mathbb{T}} := \nu(\mathbb{T})$ *is a time scale. Let* $\omega : \widetilde{\mathbb{T}} \to \mathbb{R}$. *If* $\nu^{\triangle}(t)$ *and* $\omega^{\widetilde{\triangle}}(\nu(t))$ *exist for* $t \in \mathbb{T}^k$, *then* $(\omega \circ \nu)^{\triangle} = \left(\omega^{\widetilde{\triangle}} \circ \nu\right) \nu^{\triangle}$.

In the sequel we will need to differentiate and integrate functions of the form $f(t - r(t)) = f(\nu(t))$ where, $\nu(t) := t - r(t)$. Our next theorem is the substitution rule.

Theorem 2.3 (Substitution). [5, Theorem 1.98]*Assume* $\nu : \mathbb{T} \to \mathbb{R}$ *is strictly increasing and* $\widetilde{\mathbb{T}} := \nu(\mathbb{T})$ *is a time scale. If* $f : \mathbb{T} \to \mathbb{R}$ *is rd-continuous function and* ν *is differentiable with rd-continuous derivative, then for* $a, b \in \mathbb{T}$,

$$\int_a^b f(t)\, \nu^{\triangle}(t)\, \triangle t = \int_{\nu(a)}^{\nu(b)} \left(f \circ \nu^{-1}\right)(s)\, \widetilde{\triangle} s.$$

A function $p : \mathbb{T} \to \mathbb{R}$ is said to be regressive provided $1 + \mu(t) p(t) \neq 0$ for all $t \in \mathbb{T}^k$. The set of all regressive rd-continuous function $f : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathcal{R}$. The set of all positively regressive functions $\mathcal{R}^+$, is given by $\mathcal{R}^+ = \{f \in \mathcal{R} : 1 + \mu(t) f(t) > 0 \text{ for all } t \in \mathbb{T}\}$.

Let $p \in \mathcal{R}$ and $\mu(t) \neq 0$ for all $t \in \mathbb{T}$. The exponential function on $\mathbb{T}$ is defined by

$$e_p(t, s) = \exp\left(\int_s^t \frac{1}{\mu(z)} Log\left(1 + \mu(z) p(z)\right) \triangle z\right). \tag{2.1}$$

It is well known that if $p \in \mathcal{R}^+$, then $e_p(t, s) > 0$ for all $t \in \mathbb{T}$. Also, the exponential function $y(t) = e_p(t, s)$ is the solution to the initial value problem $y^{\triangle} = p(t) y$, $y(s) = 1$. Other properties of the exponential function are given by the following lemma.

Lemma 2.4. [5, Theorem 2.36]*Let* $p, q \in \mathcal{R}$. *Then*

(*i*) $e_0(t, s) = 1$ *and* $e_p(t, t) = 1$;

(*ii*) $e_p(\sigma(t), s) = (1 + \mu(t) p(t))\, e_p(t, s)$;

(*iii*) $\dfrac{1}{e_p(t, s)} = e_{\ominus p}(t, s)$, *where* $\ominus p(t) = -\dfrac{p(t)}{1 + \mu(t) p(t)}$;

(*iv*) $e_p(t, s) = \dfrac{1}{e_p(s, t)} = e_{\ominus p}(s, t)$;

(*v*) $e_p(t, s)\, e_p(s, r) = e_p(t, r)$;

(*vi*) $e_p^{\triangle}(., s) = p e_p(., s)$ *and* $\left(\dfrac{1}{e_p(., s)}\right)^{\triangle} = -\dfrac{p(t)}{e_p^{\sigma}(., s)}$.

3. Stability by fixed point theory

We begin our work by considering the neutral nonlinear dynamic equation with an unbounded delay

$$x^{\triangle}(t) = -a(t) x^{\sigma}(t) + b(t) G(x(t), x(t - r(t))) + c(t) x^{\widetilde{\triangle}}(t - r(t)), \ t \in \mathbb{T}, \tag{3.1}$$

where a, b, c and r are defined as before. Here, we assume $G(x, y)$ is locally Lipschitz continuous in x and y. That is, there is a $L > 0$ so that if $|x|, |y|, |z|$ and $|w| \le L$, then

$$|G(x, y) - G(z, w)| \le k_1 |x - z| + k_2 |y - w|, \tag{3.2}$$

for some positive constants $k_1$and k_2.
Also, we assume

$$G(0, 0) = 0. \tag{3.3}$$

In addition to the conditions on r mentioned in Section 1, we need that

$$r^{\triangle}(t) \neq 1, \ \forall t \in \mathbb{T}. \tag{3.4}$$

Furthermore, the exponential function $e_{\ominus a}(t, 0)$ must satisfy

$$e_{\ominus a}(t, 0) \to 0 \text{ as } t \to \infty, \tag{3.5}$$

as well as the initial value problem $y^{\triangle}(t) = -a(t) y^{\sigma}(t)$, $y(0) = 1$. As such, we require that $a(t) \ge 0$ for all $t \in \mathbb{T}$. Since $a(t) \ge 0$ for all $t \in \mathbb{T}$, then $1 + \mu(t) a(t) \ge 1 > 0$ for all t and so $a \in \mathcal{R}^{+}$.

We begin by inverting equation (3.1) to obtain an equivalent equation . To do this, we use the variation of parameter formula to rewrite the equation as an integral mapping equation suitable for the contraction mapping theorem. So, in this step we need only to know what does a solution of (3.1) looks like. From now on, $\psi(t)$ denotes a real valued function with domain $(-\infty, 0]_{\mathbb{T}}$.

Lemma 3.1. *Suppose (3.4) holds. If $x(t)$ is a solution of equation (3.1) on an interval $[0, T)_{\mathbb{T}}$, $(T > 0)$ satisfying the initial condition $x(t) = \psi(t)$ for $t \in (-\infty, 0]_{\mathbb{T}}$, then $x(t)$ is a a solution of the integral equation*

$$\begin{aligned} x(t) = {} & \left(\psi(0) - \frac{c(0)}{1 - r^{\triangle}(0)} x(-r(0))\right) e_{\ominus a}(t, 0) + \frac{c(t)}{1 - r^{\triangle}(t)} x(t - r(t)) \\ & - \int_0^t \left[h(s) x^{\sigma}(s - r(s)) - b(s) G(x(s), x(s - r(s)))\right] e_{\ominus a}(t, s) \triangle s, \end{aligned} \tag{3.6}$$

where

$$h(s) = \frac{\left(c^{\triangle}(s) + c^{\sigma}(s) a(s)\right)\left(1 - r^{\triangle}(s)\right) + r^{\triangle\triangle}(s) c(s)}{\left(1 - r^{\triangle}(s)\right)\left(1 - r^{\triangle}(\sigma(s))\right)}. \tag{3.7}$$

Conversely, if a rd-continuous function $x(t)$ satisfies $x(t) = \psi(t)$ for $t \in (-\infty, 0]_{\mathbb{T}}$ and is a solution of (3.6) on some interval $[0, T)_{\mathbb{T}}$, $(T > 0)$, then $x(t)$ is a solution of equation (3.1) on $[0, T)_{\mathbb{T}}$.

Proof. We begin by rewriting (3.1) as

$$x^{\triangle}(t) + a(t) x^{\sigma}(t) = b(t) G(x(t), x(t - r(t))) + c(t) x^{\tilde{\triangle}}(t - r(t)).$$

Multiply both sides of the above equation by $e_a(t,0)$ and then we integrate from 0 to t to obtain

$$\begin{aligned}&\int_0^t \left(e_a(s,0)\,x(s)\right)^{\triangle} \triangle s\\&= \int_0^t \left[b(s)\,G\left(x(s),x(s-r(s))\right) + c(s)\,x^{\widetilde{\triangle}}(s-r(s))\right] e_a(s,0)\triangle s.\end{aligned}$$

As a consequence, we arrive at

$$\begin{aligned}&e_a(t,0)\,x(t) - x(0)\\&= \int_0^t \left[b(s)\,G\left(x(s),x(s-r(s))\right) + c(s)\,x^{\widetilde{\triangle}}(s-r(s))\right] e_a(s,0)\triangle s.\end{aligned}$$

Add $x(0)$ to both sides and multiply them by $e_{\ominus a}(t,0)$ to obtain

$$\begin{aligned}x(t) &= x(0)\,e_{\ominus a}(t,0)\\&+ \int_0^t \left[b(s)\,G\left(x(s),x(s-r(s))\right) + c(s)\,x^{\widetilde{\triangle}}(s-r(s))\right] e_{\ominus a}(t,s)\triangle s. \quad (3.8)\end{aligned}$$

Here we have used Lemma 2.4 to simplify the exponential. We want to pull the factor $x^{\widetilde{\triangle}}(s-r(s))$ from under the integral in (3.8). Clearly

$$\begin{aligned}&\int_0^t c(s)\,x^{\widetilde{\triangle}}(s-r(s))\,e_{\ominus a}(t,s)\triangle s\\&= \int_0^t x^{\widetilde{\triangle}}(s-r(s))\left(1-r^{\triangle}(s)\right)\frac{c(s)}{\left(1-r^{\triangle}(s)\right)}e_{\ominus a}(t,s)\triangle s.\end{aligned}$$

Using the integration by parts formula we get

$$\int_0^t f^{\triangle}(s)\,g(s)\triangle s = (fg)(t) - (fg)(0) - \int_0^t f^{\sigma}(s)\,g^{\triangle}(s)\triangle s,$$

and Theorems 2.2 and 2.3 implie

$$\begin{aligned}&\int_0^t c(s)\,x^{\widetilde{\triangle}}(s-r(s))\,e_{\ominus a}(t,s)\triangle s\\&= \frac{c(t)}{1-r^{\triangle}(t)}x(t-r(t)) - \frac{c(0)}{1-r^{\triangle}(0)}x(-r(0))\,e_{\ominus a}(t,0)\\&- \int_0^t h(s)\,x^{\sigma}(s-r(s))\,e_{\ominus a}(t,s)\triangle s, \quad (3.9)\end{aligned}$$

where h is given by (3.7). Finally, by substituting the right hand side of (3.9) into (3.8) we obtain (3.6). Conversely, suppose that a rd-continuous function $x(t)$ satisfying $x(t) = \psi(t)$ for $t \in (-\infty,0]_{\mathbb{T}}$ and is a solution of (3.6) on an interval $[0,T)_{\mathbb{T}}$. Then it is $\triangle$-differentiable on $[0,T)_{\mathbb{T}}$. By $\triangle$-differentiating (3.6) we obtain (3.1). □

Now, let $\psi : (-\infty,0]_{\mathbb{T}} \to \mathbb{R}$ be a given bounded $\triangle$-differentiable initial function. We say that $x := x(.,0,\psi)$ is a solution of (3.1) if $x(t) = \psi(t)$ for $t \le 0$ and satisfies (3.1) for $t \ge 0$.

We say the zero solution of (3.1) is stable at t_0 if for each $\epsilon > 0$, there is a $\delta = \delta(\epsilon) > 0$ such that [$\psi : (-\infty, 0]_{\mathbb{T}} \to \mathbb{R}$ with $\|\psi\| < \delta$] implies $|x(t, t_0, \psi)| < \epsilon$.

Let $C_{rd} = C_{rd}(\mathbb{T}, \mathbb{R})$ be the space of all rd-continuous functions from $\mathbb{T} \to \mathbb{R}$ and define the set S_ψ by

$$S_\psi = \{\varphi \in C_{rd} : \|\varphi\| \leq L,\ \varphi(t) = \psi(t) \text{ if } t \leq 0 \text{ and } \varphi(t) \to 0 \text{ as } t \to \infty\}.$$

Then $(S_\psi, \|.\|)$ is a Banach space where $\|.\|$ is the supremum norm (we refer to [7, Example 1.2.2, page 18] for the proof that S_ψ is a Banach space).

For the next theorem we assume there is an $\alpha > 0$ such that

$$\left|\frac{c(t)}{1 - r^{\triangle}(t)}\right| + \int_0^t (|h(s)| + (k_1 + k_2)|b(s)|)\, e_{\ominus a}(t, s) \triangle s \leq \alpha < 1,\ t \geq 0, \tag{3.10}$$

and

$$t - r(t) \to \infty \text{ as } t \to \infty. \tag{3.11}$$

Theorem 3.2. *If (3.2)-(3.5), (3.10) and (3.11) hold, then every solution $x(., 0, \psi)$ in C_{rd} of (3.1) with a small continuous initial function ψ, is bounded and tends to zero as $t \to \infty$. Moreover, the zero solution is stable at $t_0 = 0$.*

Proof. For α and L, find an appropriate $\delta > 0$ such that

$$\left|1 - \frac{c(0)}{1 - r^{\triangle}(0)}\right| \delta + \alpha L \leq L.$$

Let $\psi : (-\infty, 0]_{\mathbb{T}} \to \mathbb{R}$ be a given small bounded initial function with $\|\psi\| < \delta$. Define the mapping $P : S_\psi \to S_\psi$ by

$$(P\varphi)(t) = \psi(t),\ \text{if } t \leq 0,$$

and

$$(P\varphi)(t) = \left(\varphi(0) - \frac{c(0)}{1 - r^{\triangle}(0)}\varphi(-r(0))\right) e_{\ominus a}(t, 0) + \frac{c(t)}{1 - r^{\triangle}(t)}\varphi(t - r(t))$$
$$- \int_0^t [h(s)\varphi^{\sigma}(s - r(s)) - b(s) G(\varphi(s), \varphi(s - r(s)))]\, e_{\ominus a}(t, s) \triangle s,\ t \geq 0.$$

Clearly, $P\varphi$ is continuous when φ is such. Let $\varphi \in S_\psi$, then using (3.10) in the definition of $P\varphi$ and applying (3.2) and (3.3), we obtain

$$|(P\varphi)(t)| \leq \left|1 - \frac{c(0)}{1 - r^{\triangle}(0)}\right| \delta + \left|\frac{c(t)}{1 - r^{\triangle}(t)}\right| L$$
$$+ \int_0^t [|h(s)|\, |\varphi^{\sigma}(s - r(s))| + |b(s)|\, |G(\varphi(s), \varphi(s - r(s)))|]\, e_{\ominus a}(t, s) \triangle s$$
$$\leq \left|1 - \frac{c(0)}{1 - r^{\triangle}(0)}\right| \delta + L\left\{\left|\frac{c(t)}{1 - r^{\triangle}(t)}\right| + \int_0^t (|h(s)| + (k_1 + k_2)|b(s)|)\, e_{\ominus a}(t, s) \triangle s\right\}$$
$$\leq \left|1 - \frac{c(0)}{1 - r^{\triangle}(0)}\right| \delta + L\alpha,$$

which implies that $|(P\varphi)(t)| \leq L$ for the chosen δ. Thus we have $\|P\varphi\| \leq L$.

Next we show that $(P\varphi)(t) \to 0$ as $t \to \infty$. By (3.5) and (3.11), the first term in the definition of $(P\varphi)(t)$ tends to zero. Also, the second term on the right-hand side tends to zero because of (3.11) and the fact that $\varphi \in S_\psi$. It remains to show that the integral term tends to zero as $t \to \infty$.

Let $\epsilon > 0$ be arbitrary and $\varphi \in S_\psi$. Then $\|\varphi\| \le L$ and there exists $t_1 > 0$ such that $|\varphi(t)|$, $|\varphi(t - r(t))|$ and $|\varphi^\sigma(t - r(t))| < \epsilon$ for $t \ge t_1$. By condition (3.5), there exists $t_2 > t_1$ such that for $t > t_2$

$$e_{\ominus a}(t, t_1) < \frac{\epsilon}{\alpha L}.$$

For $t > t_2$, we have

$$\begin{aligned}
&\left| \int_0^t [h(s)\varphi^\sigma(s - r(s)) - b(s) G(\varphi(s), \varphi(s - r(s)))] e_{\ominus a}(t, s) \triangle s \right| \\
&\le \int_0^t [|h(s)| |\varphi^\sigma(s - r(s))| + |b(s)| |G(\varphi(s), \varphi(s - r(s)))|] e_{\ominus a}(t, s) \triangle s \\
&\le L \int_0^{t_1} (|h(s)| + (k_1 + k_2)|b(s)|) e_{\ominus a}(t, s) \triangle s \\
&+ \epsilon \int_{t_1}^{t_2} (|h(s)| + (k_1 + k_2)|b(s)|) e_{\ominus a}(t, s) \triangle s \\
&\le L e_{\ominus a}(t, t_1) \int_0^{t_1} (|h(s)| + (k_1 + k_2)|b(s)|) e_{\ominus a}(t_1, s) \triangle s + \alpha\epsilon \\
&\le \alpha L e_{\ominus a}(t, t_1) + \alpha\epsilon \le \epsilon + \alpha\epsilon.
\end{aligned}$$

Hence $(P\varphi)(t) \to 0$ as $t \to \infty$.

It remains to show that P is a contraction under the supremum norm. For this, let $\varphi, \phi \in S_\psi$ then

$$\begin{aligned}
|(P\varphi)(t) - (P\phi)(t)| &\le \left| \frac{c(t)}{1 - r^\triangle(t)} \right| \|\varphi - \phi\| \\
&+ \int_0^t |h(s)(\varphi^\sigma(s - r(s)) - \phi^\sigma(s - r(s)))| e_{\ominus a}(t, s) \triangle s \\
&+ \int_0^t |b(s)(G(\varphi(s), \varphi(s - r(s))) - G(\phi(s), \phi(s - r(s))))| e_{\ominus a}(t, s) \triangle s
\end{aligned}$$

$$\le \left\{ \left| \frac{c(t)}{1 - r^\triangle(t)} \right| + \int_0^t (|h(s)| + (k_1 + k_2)|b(s)|) e_{\ominus a}(t, s) \triangle s \right\} \|\varphi - \phi\| \le \alpha \|\varphi - \phi\|.$$

Thus, by the contraction mapping principle, P has a unique fixed point in S_ψ which solves (3.1), bounded and tends to zero as $t \to \infty$. The stability of the zero solution at $t_0 = 0$ follows from the above work by simply replacing L by ϵ. $\square$

Some stability result obtained on $\mathbb{R}$ for similar linear equations with delay via fixed point technic can be found in [8] (see also [7]). The authors in [13] have obtained results of stability for a nonlinear dynamic delay equation but with no neutral term.

Example 3.3. Let

$$\begin{aligned}\mathbb{T} &= (-\infty,-1] \cup \left\{(1/2)^{\mathbb{Z}} - 1\right\} \\ &= (-\infty,-1] \cup \{..., (1-2^n)/2^n, ..., -3/4, -1/2, 0, 1, 3, ..., 2^n - 1, ...\}.\end{aligned}$$

Then for any small continuous initial function $\psi : (-\infty, 0]_{\mathbb{T}} \to \mathbb{R}$, every solution $x(., 0, \psi)$ of the nonlinear neutral dynamic equation

$$\begin{aligned}x^{\triangle}(t) &= -3x^{\sigma}(t) + (3/2)c_0(\sin(x(t)) + \sin(x(t/2 - 1/2))) \\ &\quad + c_0 x^{\widetilde{\triangle}}(t/2 - 1/2), \end{aligned} \tag{3.12}$$

where c_0 is a positive constant, is bounded and goes to 0 as $t \to \infty$.

Indeed, in (3.12) we have $r(t) = t/2 + 1/2$. Let $t \in (1/2)^{\mathbb{Z}} - 1$. Then there exists an $n \in \mathbb{Z}$ such that $t = (1/2)^n - 1$. Hence

$$\begin{aligned}t - r(t) &= \frac{1}{2}\left(\left(\frac{1}{2}\right)^n - 1\right) - \frac{1}{2} \\ &= \left(\frac{1}{2}\right)^{n+1} - 1 \in \mathbb{T}.\end{aligned}$$

So, $id - r : \mathbb{T} \to \mathbb{T}$. Furthermore $(id - r)(\mathbb{T})$ is a time scale. Also, $t - r(t) = t/2 - 1/2 \to \infty$ as $t \to \infty$ and $(t - r(t))^{\triangle} = (t/2 - 1/2)^{\triangle} = 1/2$. Consequently, conditions (3.4) and (3.11) are satisfied. Since $1 + 3\mu(t) > 0$ for all $t \in \mathbb{T}$, then $3 \in \mathcal{R}^+$ and condition (3.5) is satisfied as well.
Also, in (3.12), we have

$$G(x(t), x(t/2 - 1/2)) = \sin(x(t)) + \sin(x(t/2 - 1/2)).$$

Clearly $G(0,0) = 0$ and $G(x,y)$ is locally Lipschitz continuous in x and y. That is, there is a $L > 0$ so that if $|x|, |y|, |z|$ and $|w| \le L$, then

$$\begin{aligned}|G(x,y) - G(z,w)| &= |\sin(x) + \sin(y) - (\sin(z) + \sin(w))| \\ &\le |\sin(x) - \sin(z)| + |\sin(y) - \sin(w)| \\ &\le |x - z| + |y - w|.\end{aligned}$$

One may easily check that $h(s) = 6c_0$. Also

$$\begin{aligned}
&\left|\frac{c(t)}{1-r^{\triangle}(t)}\right| + \int_0^t (|h(s)| + (k_1+k_2)|b(s)|) e_{\ominus a}(t,s) \triangle s \\
&= 2c_0 + 9c_0 \int_0^t e_{\ominus 3}(t,s) \triangle s \\
&= 2c_0 + 3c_0 - 3c_0 e_{\ominus 3}(t,0) \\
&\leq 5c_0.
\end{aligned}$$

Hence, (3.10) is satisfied for $c_0 \leq \frac{\alpha}{5}$, $\alpha \in (0,1)$. Let ψ be a given initial function which is continuous with $|\psi(t)| \leq \delta$ for all $t \in \mathbb{T}$ and define

$$S_\psi = \{\varphi \in C_{rd} : \|\varphi\| \leq L,\ \varphi(t) = \psi(t) \text{ if } t \leq 0 \text{ and } \varphi(t) \to 0 \text{ as } t \to \infty\}.$$

Define

$$(P\varphi)(t) = \psi(t) \text{ if } t \leq 0,$$

and

$$\begin{aligned}
(P\varphi)(t) &= (\psi(0) - 2c_0\psi(-1/2)) e_{\ominus 3}(t,0) + 2c_0\varphi(t/2-1/2) \\
&- \int_0^t [6c_0\varphi^\sigma(s/2-1/2) - (3/2)c_0(\sin(\varphi(s)) + \sin(\varphi(s/2-1/2)))] \\
&\times e_{\ominus 3}(t,s) \triangle s,\ t \geq 0.
\end{aligned}$$

Then, for $\varphi \in S_\psi$ with $\|\varphi\| \leq L$, we have

$$\|P\varphi\| \leq (1-2c_0)\delta + 5c_0 L \leq (1-2c_0)\delta + \alpha L.$$

This implies that $\|P\varphi\| \leq L$, for $L \geq \frac{(1-2c_0)\delta}{1-\alpha}$. To see that P defines a contraction mapping, we let $\varphi, \phi \in S_\psi$. Then

$$\begin{aligned}
|(P\varphi)(t) - (P\phi)(t)| &\leq 2c_0\|\varphi - \phi\| + 3c_0\|\varphi - \phi\| \\
&\leq \alpha\|\varphi - \phi\|.
\end{aligned}$$

Hence, by Theorem 3.2, every solution $x(.,0,\psi)$ of (3.12) with small continuous initial function $\psi : (-\infty, 0]_{\mathbb{T}} \to \mathbb{R}$, is in S_ψ, bounded and goes to zero as $t \to \infty$.

Next we turn our attention to the nonlinear neutral dynamic equation with unbounded delay

$$\begin{aligned}
x^{\triangle}(t) &= -a(t)x^\sigma(t) + b(t)G\left(x^2(t), x^2(t-r(t))\right) \\
&+ c(t)\left(x^2\right)^{\widetilde{\triangle}}(t-r(t)),\ t \in \mathbb{T}, \qquad (3.13)
\end{aligned}$$

where a, b, c, r and G are defined as before.

We use the variation of parameter to get the solution

$$x(t) = \left(x(0) - \frac{c(0)}{1 - r^{\triangle}(0)} x^2(-r(0)) \right) e_{\ominus a}(t, 0) + \frac{c(t)}{1 - r^{\triangle}(t)} x^2(t - r(t)) - \int_0^t \left[h(s) \left(x^2\right)^{\sigma}(s - r(s)) - b(s) G\left(x^2(s), x^2(s - r(s))\right) \right] e_{\ominus a}(t, s) \triangle s,$$

where h is given by (3.7).

Let

$$S_{\psi} = \{\varphi \in C_{rd} : \|\varphi\| \leq L,\ \varphi(t) = \psi(t) \text{ if } t \leq 0 \text{ and } \varphi(t) \to 0 \text{ as } t \to \infty\}.$$

For the next theorem we assume there is an $\alpha > 0$ such that

$$L \left\{ \left| \frac{c(t)}{1 - r^{\triangle}(t)} \right| + \int_0^t \left(|h(s)| + (k_1 + k_2) |b(s)| \right) e_{\ominus a}(t, s) \triangle s \right\} \leq \alpha < 1/2,\ t \geq 0. \tag{3.14}$$

Theorem 3.4. *If (3.2)-(3.5), (3.11) and (3.14) hold, then every solution $x(., 0, \psi)$ of (3.13) with a small continuous initial function ψ, is bounded and tends to zero as $t \to \infty$. Moreover, the zero solution is stable at $t_0 = 0$.*

Proof. For α and L, find an appropriate $\delta > 0$ such that

$$\delta + \left| \frac{c(0)}{1 - r^{\triangle}(0)} \right| \delta^2 + \alpha L \leq L.$$

Let $\psi : (-\infty, 0]_{\mathbb{T}} \to \mathbb{R}$ be a given small bounded initial function with $\|\psi\| < \delta$. Define the mapping $P : S_{\psi} \to S_{\psi}$ by

$$(P\varphi)(t) = \psi(t),\ \text{if } t \leq 0,$$

and

$$(P\varphi)(t) = \left(\varphi(0) - \frac{c(0)}{1 - r^{\triangle}(0)} \varphi^2(-r(0)) \right) e_{\ominus a}(t, 0) + \frac{c(t)}{1 - r^{\triangle}(t)} \varphi^2(t - r(t)) - \int_0^t \left[h(s) \left(\varphi^2\right)^{\sigma}(s - r(s)) - b(s) G\left(\varphi^2(s), \varphi^2(s - r(s))\right) \right] e_{\ominus a}(t, s) \triangle s,\ t \geq 0.$$

Clearly, $P\varphi$ is continuous when φ is such. Let $\varphi \in S_\psi$, then using (3.14) in the definition of $P\varphi$ and applying (3.2) and (3.3), we have

$$\begin{aligned}
&|(P\varphi)(t)| \\
&\leq \delta + \left|\frac{c(0)}{1-r^{\triangle}(0)}\right|\delta^2 + \left|\frac{c(t)}{1-r^{\triangle}(t)}\right| L \\
&+ \int_0^t \left[|h(s)|\left|\left(\varphi^2\right)^\sigma (s-r(s))\right| + |b(s)|\left|G\left(\varphi^2(s), \varphi^2(s-r(s))\right)\right|\right] e_{\ominus a}(t,s)\triangle s \\
&\leq \delta + \left|\frac{c(0)}{1-r^{\triangle}(0)}\right|\delta^2 \\
&+ L^2\left\{\left|\frac{c(t)}{1-r^{\triangle}(t)}\right| + \int_0^t \left(|h(s)| + (k_1+k_2)|b(s)|\right) e_{\ominus a}(t,s)\triangle s\right\} \\
&\leq \delta + \left|\frac{c(0)}{1-r^{\triangle}(0)}\right|\delta^2 + L\alpha,
\end{aligned}$$

which implies that $|(P\varphi)(t)| \leq L$ for the chosen δ. Thus we have $\|P\varphi\| \leq L$.

Next we show that $(P\varphi)(t) \to 0$ as $t \to \infty$. By (3.5) and (3.11), the first term in the definition of $(P\varphi)(t)$ tends to zero. Also, the second term on the right-hand side tends to zero because of (3.11) and the fact that $\varphi \in S_\psi$. It remains to show that the integral term tends to zero as $t \to \infty$.

Let $\epsilon > 0$ be arbitrary and $\varphi \in S_\psi$. Then $\|\varphi\| \leq L$ and there exists $t_1 > 0$ such that $|\varphi(t)|$, $|\varphi(t-r(t))|$ and $|\varphi^\sigma(t-r(t))| < \epsilon$ for $t \geq t_1$. By condition (3.5), there exists $t_2 > t_1$ such that for $t > t_2$

$$e_{\ominus a}(t,t_1) < \frac{\epsilon}{\alpha L^2}.$$

For $t > t_2$, we have

$$\begin{aligned}
&\left|\int_0^t \left[h(s)\left(\varphi^2\right)^\sigma (s-r(s)) - b(s)G\left(\varphi^2(s), \varphi^2(s-r(s))\right)\right] e_{\ominus a}(t,s)\triangle s\right| \\
&\leq \int_0^t [\,|h(s)|\left|\left(\varphi^2\right)^\sigma (s-r(s))\right| \\
&+ |b(s)|\left|G\left(\varphi^2(s), \varphi^2(s-r(s))\right)\right|\,] e_{\ominus a}(t,s)\triangle s \\
&\leq L^2 \int_0^{t_1} \left(|h(s)| + (k_1+k_2)|b(s)|\right) e_{\ominus a}(t,s)\triangle s \\
&+ \epsilon^2 \int_{t_1}^{t_2} \left(|h(s)| + (k_1+k_2)|b(s)|\right) e_{\ominus a}(t,s)\triangle s \\
&\leq L^2 e_{\ominus a}(t,t_1) \int_0^{t_1} \left(|h(s)| + (k_1+k_2)|b(s)|\right) e_{\ominus a}(t_1,s)\triangle s + \alpha\epsilon^2 \\
&\leq \alpha L^2 e_{\ominus a}(t,t_1) + \alpha\epsilon^2 \leq \epsilon + \alpha\epsilon^2.
\end{aligned}$$

Hence $(P\varphi)(t) \to 0$ as $t \to \infty$.

It remains to show that P is a contraction under the supremum norm. For this, let $\varphi, \phi \in S_\psi$ then

$$\begin{aligned}
&\left|(P\varphi)(t) - (P\phi)(t)\right| \\
&\leq \left\{\left|\frac{c(t)}{1-r^{\triangle}(t)}\right| + \int_0^t \left(|h(s)| + (k_1+k_2)|b(s)|\right) e_{\ominus a}(t,s) \triangle s\right\} \left\|\varphi^2 - \phi^2\right\| \\
&\leq (2L)\,\{ \left|\frac{c(t)}{1-r^{\triangle}(t)}\right| \\
&+ \int_0^t \left(|h(s)| + (k_1+k_2)|b(s)|\right) e_{\ominus a}(t,s) \triangle s \;\} \left\|\varphi - \phi\right\| \\
&\leq (2\alpha)\left\|\varphi - \phi\right\|.
\end{aligned}$$

Thus, by the contraction mapping principle, P has a unique fixed point in S_ψ which solves (3.13), is bounded and tends to zero as $t \to \infty$. The stability of the zero solution at $t_0 = 0$ follows from the above work by simply replacing L by ϵ. □

Example 3.5. Let

$$\begin{aligned}
\mathbb{T} &= (-\infty,-1] \cup \left\{(1/3)^{\mathbb{Z}} - 1\right\} \\
&= (-\infty,-1] \cup \left\{..., (1-3^n)/3^n, ..., -8/9, -2/3, 0, 2, 8, ..., 3^n - 1, ...\right\}.
\end{aligned}$$

Then for any small continuous initial function $\psi : (-\infty, 0]_{\mathbb{T}} \to \mathbb{R}$, every solution $x(.,0,\psi)$ of the nonlinear neutral dynamic equation

$$\begin{aligned}
x^{\triangle}(t) &= -2x^{\sigma}(t) + c_0\left(\sin\left(x^2(t)\right) + \cos\left(x^2(t/3-2/3)\right) - 1\right) \\
&\quad +2c_0\left(x^2\right)^{\widetilde{\triangle}}(t/3-2/3),
\end{aligned} \tag{3.15}$$

where c_0 is a positive constant, bounded and goes to 0 as $t \to \infty$.

Indeed, in (3.15) we have $r(t) = 2t/3 + 2/3$. Let $t \in (1/3)^{\mathbb{Z}} - 1$. Then there exists an $n \in \mathbb{Z}$ such that $t = (1/3)^n - 1$. Hence

$$\begin{aligned}
t - r(t) &= \frac{1}{3}\left(\left(\frac{1}{3}\right)^n - 1\right) - \frac{2}{3} \\
&= \left(\frac{1}{3}\right)^{n+1} - 1 \in \mathbb{T}.
\end{aligned}$$

So, $id - r : \mathbb{T} \to \mathbb{T}$. Furthermore $(id - r)(\mathbb{T})$ is a time scale. Also, $t - r(t) = t/3 - 2/3 \to \infty$ as $t \to \infty$ and $(t - r(t))^{\triangle} = (t/3 - 2/3)^{\triangle} = 1/3$. Consequently, conditions (3.4) and (3.11) are satisfied. Since $1 + 2\mu(t) > 0$ for all $t \in \mathbb{T}$, then $2 \in \mathcal{R}^+$ and condition (3.5) is satisfied as well.

Also, in (3.15), we have

$$G\left(x^2(t), x^2(t/3-2/3)\right) = \sin\left(x^2(t)\right) + \cos\left(x^2(t/3-2/3)\right) - 1.$$

Clearly $G(0,0) = 0$ and $G(x,y)$ is locally Lipschitz continuous in x and y. That is, there is a $L > 0$ so that if $|x|$, $|y|$, $|z|$ and $|w| \le L$, then

$$\begin{aligned} |G(x,y) - G(z,w)| &= |\sin(x) + \cos(y) - (\sin(z) + \cos(w))| \\ &\le |\sin(x) - \sin(z)| + |\cos(y) - \cos(w)| \\ &\le |x - z| + |y - w| . \end{aligned}$$

One may easily arrive at $h(s) = 6c_0$. Also

$$\begin{aligned} & L\left\{\left|\frac{c(t)}{1 - r^{\triangle}(t)}\right| + \int_0^t (|h(s)| + (k_1 + k_2)|b(s)|)\, e_{\ominus a}(t,s)\,\triangle s\right\} \\ & = L\left(3c_0 + 8c_0 \int_0^t e_{\ominus 2}(t,s)\,\triangle s\right) \\ & = L\{3c_0 + 4c_0 - 4c_0 e_{\ominus 2}(t,0)\} \\ & \le 7Lc_0. \end{aligned}$$

Hence, (3.14) is satisfied for $c_0 \le \frac{\alpha}{7L}$, $\alpha \in (0, 1/2)$. Let ψ be a given initial function that is continuous with $|\psi(t)| \le \delta$ for all $t \in \mathbb{T}$ and define

$$S_\psi = \{\varphi \in C_{rd} : \|\varphi\| \le L,\ \varphi(t) = \psi(t) \text{ if } t \le 0 \text{ and } \varphi(t) \to 0 \text{ as } t \to \infty\}.$$

Define

$$(P\varphi)(t) = \psi(t) \text{ if } t \le 0,$$

and

$$\begin{aligned} (P\varphi)(t) &= \left(\psi(0) - 3c_0\psi^2(-2/3)\right) e_{\ominus 2}(t,0) + 3c_0\varphi^2(t/3 - 2/3) \\ &\quad - \int_0^t \left[6c_0\left(\varphi^2\right)^\sigma (s/3 - 2/3) - c_0\left(\sin\left(\varphi^2(s)\right) + \cos\left(\varphi^2(s/3 - 2/3)\right) - 1\right)\right] \\ &\qquad \times e_{\ominus 2}(t,s)\,\triangle s,\ t \ge 0. \end{aligned}$$

Then, for $\varphi \in S_\psi$ with $\|\varphi\| \le L$, we have

$$\|P\varphi\| \le \delta + 3c_0\delta^2 + 7c_0L^2 \le \delta + 3c_0\delta^2 + \alpha L.$$

This implies that $\|P\varphi\| \le L$, for $L \ge \frac{\delta + 3c_0\delta^2}{1 - \alpha}$. To see that P defines a contraction mapping, we let $\varphi, \phi \in S_\psi$. Then

$$\begin{aligned} |(P\varphi)(t) - (P\phi)(t)| &\le 3c_0\left\|\varphi^2 - \phi^2\right\| + 4c_0\left\|\varphi^2 - \phi^2\right\| \\ &\le 14c_0L\|\varphi - \phi\| \\ &\le 2\alpha\|\varphi - \phi\|. \end{aligned}$$

Hence, by Theorem 3.4, every solution $x(.,0,\psi)$ of (3.15) with small continuous initial function $\psi : (-\infty, 0]_{\mathbb{T}} \to \mathbb{R}$, is in S_ψ, bounded and goes to zero as $t \to \infty$.

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Abdelouaheb Ardjouni
Laboratory of Applied Mathematics (LMA)
University of Annaba, Department of Mathematics
P.O.Box 12, 23000 Annaba, Algeria
e-mail: abd_ardjouni@yahoo.fr

Ahcene Djoudi
Laboratory of Applied Mathematics (LMA)
University of Annaba, Department of Mathematics
P.O.Box 12, 23000 Annaba, Algeria
e-mail: adjoudi@yahoo.com