Stud. Univ. Babeş-Bolyai Math. 57(2012), No. 3, 453–456

Book reviews

Joram Lindenstrauss, David Preiss and Jaroslav Tišer, *Fréchet differentiability of Lipschitz functions and porous sets in Banach spaces,* Annals of Mathematics Studies, vol. 179, Princeton University Press, Princeton, NJ, 2012, ix+425 pp. ISBN: 978-0-691-15355-1 (hardcover) – ISBN: 978-0-691-15356-8 (pbk.).

A classical result proved by Henri Lebesgue in his thesis (around 1900) asserts that every monotone function $f : \mathbb{R} \to \mathbb{R}$ is almost everywhere (a.e.) differentiable. As consequence, the functions with bounded variation and, in particular, the Lipschitz functions have this property. The result was extended in 1919 to Lipschitz functions $f : \mathbb{R}^n \to \mathbb{R}$ by Hans Rademacher. The case of the function $t \mapsto \chi_{[0,t]}$ from the interval $[0, 1]$ to $L^1[0, 1]$, which is nowhere differentiable, shows that some restrictions have to be imposed to the range space Y in order to extend this result to vector functions $f : U \to Y$, $U \subset \mathbb{R}^n$ an open subset and Y a Banach space: if the Banach space Y has the Radon-Nikodým Property (RNP), then the differentiability result is true. In fact, this property is equivalent to the RNP: if every Lipschitz function $f : \mathbb{R} \to Y$ is a.e. differentiable, then the Banach space Y has the RNP. The next step is to extend further the result to Lipschitz functions $f : U \to Y$, where U is an open subset of a Banach space X. It is clear that one needs first an appropriate notion of "almost everywhere" in an infinite dimensional Banach space. Since it is impossible to define a measure μ on an infinite dimensional Banach space X such that the class of sets of μ-measure 0 be a useful class of negligible sets, these must be defined by other means. There are several nonequivalent ways to define negligible sets in infinite dimension, leading to different classes – Haar null sets (J. P. R. Christensen, 1972), cube null sets (P. Mankiewicz, 1973) Gauss null sets (R. R. Phelps, 1978), Aronszajn null sets (N. Aronszajn, 1976), a.o., each of them forming a proper σ-ideal of Borel subsets of the Banach space X. Later, M. Csörnyei (1999) has shown that the classes of Gauss null, cube null and Aronszajn null sets agree and are properly contained in the class of Haar null sets. A good presentation of these result as well as of the a.e. Gâteaux differentiability of Lipschitz functions is given in the sixth chapter of the book by Y. Benyamini and J. Lindenstrauss, *Geometric Nonlinear Analysis*, Colloq. Publ. vol 48, AMS 2000. A brief presentation of these results is contained also in the second chapter of the present book. Supposing the space X separable and Y an RNP space, then every Lipschitz function $f : U \to Y$, $U \subset X$ open, is Gâteaux differentiable on U, excepting an Aronszajn null set.

The question of a.e. Fréchet differentiability is more delicate and the results are far from being complete. The norm $\|\cdot\|$ of ℓ^1 is nowhere Fréchet differentiable and a similar situation appears when X is separable with nonseparable conjugate, so a natural requirement on the domain space X is to be with separable conjugate (such a space is called an Asplund space). As the authors show in Section 3.6, some results can be extended to the nonseparable case by using separable reduction techniques. The first result, with an involved proof, in this direction was obtained by D. Preiss (1990) – any real-valued Lipschitz function on an Asplund space has a dense set of points of Fréchet differentiability. Another proof, simpler but still involved, was given by Lindenatrauss and Preiss in 2000. The authors consider a general conjecture, namely that for every Asplund space X there exists a nontrivial notion of negligible sets such that every locally Lipschitz function f defined on an open subset G of X and with values in an RNP space Y satisfies the conditions:

(C1) f is a.e. Gâteaux differentiable on G;

(C2) if $S \subset G$ has null complement and f is Gâteaux differentiable on S, then $\mathrm{Lip}(f) = \sup_S \|f'(x)\|$;

(C3) if for some $E \subset G$ the set $\{f'(x) : x \in E\}$ is separable (in $L(X, Y)$), then f is a.e. differentiable on E.

As the authors do mention, a challenging problem in this area is whether a countable family of Lipschitz functions defined on an Asplund space has a common point of Fréchet differentiability. In this book one shows that any Lipschitz mapping from a Hilbert space to $Y = \mathbb{R}^2$ is a.e. differentiable and satisfies a multi-dimensional mean value estimate (which is stronger that (C2)), the first known result of this kind. The result is obtained as a consequence of some more general results, but in the last chapter of the book, Chapter 16, a direct proof, essentially self-contained, based on specific properties of Hilbert spaces, is given. The differentiability result does not hold for $Y = \mathbb{R}^3$, or, more generally, for maps on ℓ^p to $\mathbb{R}^n$ with $n > p$.

Since the distance function $d(\cdot, E)$ to a subset E of a normed space X is nowhere F-differentiable iff the set E is porous, a natural requirement would be the containment of porous sets in these σ-ideals of negligible sets. Lindenstrauss and Preiss (2003) considered Γ-null sets, whose definition involves both Baire category and Lebesgue measure, and proved that each Lipschitz function from a separable Banach space X to an RNP space Y is G-differentiable excepting a Γ-null set, and that such a function is regularly G-differentiable (a notion stronger than Gâteaux differentiability) excepting a σ-porous set. Since a function is a.e. Fréchet differentiable on the set on which it is regularly Gâteaux differentiable, it follows that (C1)–(C3) hold for every Banach space X for which σ-porous sets are Γ-null. Example of such spaces are $C(K)$ with K countable and compact, subspaces of c_0, the Tsirelson space, but not Hilbert spaces.

In the present book, one considers also a more refined concept, namely Γ_n-null sets, which can be viewed as finite dimensional versions of Γ-null sets, one studies the relations between these classes and one presents new classes of Banach spaces for which strong Fréchet differentiability results hold. Of course, some geometric properties of the space X are involved, among which the notions of asymptotic uniform smoothness, asymptotic uniform convexity and their moduli, as well as the modulus of asymptotic smoothness associated to a function. The relevance of these geometric properties for the existence of bump functions with controlled moduli of asymptotic smoothness is discussed in detail in Chapter 8.

Most of the results in this book (Chapters 7–14) are new, leading to a better understanding of this difficult problem - the a.e. Fréchet differentiability of Lipschitz functions. The book is very well organized – each chapter starts with a brief information about its content, followed by an introductory section containing the most important results of the chapter and the proofs of some relevant corollaries. Also the authors explain the intuitive ideas behind the proofs (most of them long and difficult) and to the abstract notions considered in the book.

In conclusion, this is a remarkable book, opening new ways for further investigation of the Fréchet differentiability of Lipschitz functions. It is of great interest for researchers in functional analysis, mainly those interested in the applications of Banach space geometry.

S. Cobzaş

William J. Cook, *In Pursuit of the Traveling Salesman Problem: Mathematics at the Limits of Computation,* Princeton University Press, 2012, xiii + 228 pp., ISBN13: 978-0-691-15270-7

The Traveling Salesman Problem (TSP for short) is simply to formulate but very hard to solve. We are given a collection of cities and the distance to travel between each pair of them and one ask to find the shortest route to visit each city and to return to the starting point. The problem has many practical applications, presented in Chapter 3, *The salesman in action* - first of all routes for traveling salesmen (the GPS system often includes a TSP solver for small instances having a dozen of cities, which usually suffices for daily trips), the routing of buses and vans to pick up or deliver people and packages, to genome study, X-ray crystallography, computer chips, tests for microprocessors, organizing data, and even to music (organizing vast collections of computer-encoded music), speeding up video games, etc.

The difficulty of the problem arises from the big amount of possibilities to be examined in order to find the optimal one. For instance, the 33 cities problem, for which Procter & Gamble offered in 1962 a $10 000 prize for its solution, the number of possibilities are of the order of 10^{35}, for 50 cities the order is 10^{62}. A breakthrough in the field was made in 1954 by an ingenious application by George Dantzig, Ray Fulkerson and Selmer Johnson from RAND Corporation of linear programming to calculate (in a few weeks "by hand") the shortest route for 49 cities. This record lasted until 1975 when Panagiotis Miliotis solved the problem for 80 cities, followed in 1977 by Grötschel with 120 and Padberg and Crowder with 318 cities. In 1987 Grötschel and Padberg raised this number to 2392 cities. The author and Vašek Chvátal, assisted by the computational mathematicians David Applegate and Robert Bixby, started to work on the problem in 1988 and obtained astonishing results, culminating in 2006 with the solution for 85 900 cities, representing locations of the connections that must be cut by a laser to create a customized computer chip. The computer code used to solve the problem, called *Concorde*, is available on the internet. Some of these results are presented in the book by David L. Applegate, Robert E. Bixby, Vasek

Chvátal and William J. Cook, *The Traveling Salesman Problem: A Computational Study,* Princeton University Press, 2007.

It is unknown whether the the complexity of the TSP is polynomial (i.e. if it belongs to the class $\mathcal{P}$), the best estimation being $n^2 \cdot 2^n$. It is known that it belongs to the class $\mathcal{NP}$ (non-deterministic polynomial-time algorithms), and the TSP is $\mathcal{NP}$-complete, meaning that finding a good algorithm for TSP will prove the equality $\mathcal{P} = \mathcal{NP}$, one of the \$1 000 000 worth problems from Clay Mathematics Institute. An amusing discussion on the catastrophical consequences of this equality for the mankind, with quotations from the Charles Stross's story "Antibodies" and from Lance Fortnow (Comm. ACM, 2009), can be founded on page 10.

The author presents in a live and entertaining style the historical evolution of the problem and its interaction with other mathematical problems – linear programming (in Ch. 5), the cutting planes method (Ch. 6) and the branch-and-bound method (in Ch. 7) for integer programming – and computational ones – big computing and TPS on large scale (in Ch. 8), and the complexity problem in Chapter 9.

There are a lot of good quotations spread through the book, nice pictures, personal reminiscences and anecdotes. By its non-formal and amazing style, the book addresses a large audience interested to know something about the long and hard chest of generations of mathematicians and computer scientists to solve hard problems and how their solutions will influence our lives.

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