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More on pairwise extremally disconnected spaces

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Abstract. In [1] the authors, introduced the notion of pairwise extremally disconnected spaces and investigated its fundamental properties. In this paper, we investigate some more properties of pairwise extremally disconnected spaces.

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1. Introduction

The concept of bitopological spaces was first introduced by Kelly [4]. After the introduction of the definition of a bitopological space by Kelly, a large number of topologists have turned their attention to the generalization of different concepts of a single topological space in this space. In [1] the authors, introduced the notion of pairwise extremally disconnected spaces and investigated its fundamental properties. In this paper, we investigate some more properties of pairwise extremally disconnected spaces. Throughout this paper, the triple (X, τ_1, τ_2) where X is a set and τ_1 and τ_2 are topologies on X, will always denote a bitopological space. The τ_i-closure (resp. τ_i-interior) of a subset A of a bitopological space (X, τ_1, τ_2) is denoted by $\tau_i\text{-Cl}(A)$ (resp. $\tau_i\text{-Int}(A)$).

2. Preliminaries

Definition 2.1. *Let A be a subset of a bitopological space (X, τ_1, τ_2). Then A is called*

1. (τ_i, τ_j)*-regular open* [7] *if* $A = \tau_i\text{-Int}(\tau_j\text{-Cl}(A))$,
2. (τ_i, τ_j)*-semiopen* [2] *if* $A \subset \tau_j\text{-Cl}(\tau_i\text{-Int}(A))$,
3. (τ_i, τ_j)*-preopen* [5] *if* $A \subset \tau_i\text{-Int}(\tau_j\text{-Cl}(A))$,
4. (τ_i, τ_j)*-semipreopen* [5] *if* $A \subset \tau_j\text{-Cl}(\tau_i\text{-Int}(\tau_j\text{-Cl}(A)))$,

On each definition above, $i, j = 1, 2$ and $i \neq j$.

The complement of an (i, j)-regular open (resp. (τ_i, τ_j)-semiopen, (τ_i, τ_j)-preopen, (τ_i, τ_j)-semipreopen) set is called an (i, j)-regular closed (resp. (τ_i, τ_j)-semiclosed, (τ_i, τ_j)-preclosed, (τ_i, τ_j)-semipreclosed) set.

Definition 2.2. [2] *Let A be a subset of a bitopological space (X, τ_1, τ_2). Then*

1. *The intersection of all (i, j)-semiclosed sets of X containing A is called the (i, j)-semiclosure of A and is denoted by (i, j)-$s\,\mathrm{Cl}(A)$.*
2. *The union of all (i, j)-semiopen sets of X contained in A is called the (i, j)-semiinterior of A and is denoted by (i, j)-$s\,\mathrm{Int}(A)$.*

Theorem 2.3. *For a subset A of a bitopological space (X, τ_1, τ_2), the following are equivalent:*

1. *A is (τ_i, τ_j)-semiopen,*
2. *$A \subset \tau_j$-$\mathrm{Cl}(\tau_i$-$\mathrm{Int}(A))$,*
3. *τ_j-$\mathrm{Cl}(A) = \tau_j$-$\mathrm{Cl}(\tau_i$-$\mathrm{Int}(A))$.*

Theorem 2.4. [2] *For a set A of a bitopological space (X, τ_1, τ_2), the following are equivalent:*

1. *A is (τ_i, τ_j)-semiclosed,*
2. *τ_j-$\mathrm{Int}(\tau_i$-$\mathrm{Cl}(A)) \subset A$,*
3. *τ_j-$\mathrm{Int}(A) = \tau_j$-$\mathrm{Int}(\tau_i$-$\mathrm{Cl}(A))$.*

Theorem 2.5. [2] *For a subset A of a bitopological space (X, τ_1, τ_2),*

1. *a point $x \in (i, j)$-$s\,\mathrm{Cl}(A)$ if and only if $U \cap A \neq \emptyset$ for every $U \in (i, j)$-$SO(X, x)$.*
2. *(τ_i, τ_j)-$s\,\mathrm{Int}(A) = X \backslash (\tau_i, \tau_j)$-$s\,\mathrm{Cl}(X \backslash A)$,*
3. *(τ_i, τ_j)-$s\,\mathrm{Cl}(A) = X \backslash (\tau_i, \tau_j)$-$s\,\mathrm{Int}(X \backslash A)$.*

Definition 2.6. *A bitopological space (X, τ_1, τ_2) is said to be*

1. *(τ_i, τ_j)-extremally disconnected* [1] *if τ_j-closue of every τ_i-open set is τ_i-open in X,*
2. *pairwise extremally disconnetcted if (X, τ_1, τ_2) is (τ_1, τ_2)-extremally disconnected and (τ_2, τ_1)-extremally deisconnected.*

Theorem 2.7. [1] *A bitopological space (X, τ_1, τ_2) is pairwise extremally disconnected if and only if for each τ_i-open set A and each τ_j-open set B such that $A \cap B = \emptyset$, τ_j-$\mathrm{Cl}(A) \cap \tau_i$-$\mathrm{Cl}(B) = \emptyset$.*

3. Extremally disconnected bitopological spaces

Theorem 3.1. *The following are equivalent for a bitopological space (X, τ_1, τ_2):*

1. *(X, τ_1, τ_2) is pairwise extremally disconnected.*
2. *For each (τ_j, τ_i)-semiopen set A in X, τ_j-$\mathrm{Cl}(A)$ is τ_i-open set.*
3. *For each (τ_i, τ_j)-semiopen set A in X, (τ_j, τ_i)-$s\,\mathrm{Cl}(A)$ is τ_i-open set.*
4. *For each (τ_i, τ_j)-semiopen set A and each (τ_j, τ_i)-semiopen set B with $A \cap B = \emptyset$, τ_j-$\mathrm{Cl}(A) \cap \tau_i$-$\mathrm{Cl}(B) = \emptyset$.*
5. *For each (τ_j, τ_i)-semiopen set A in X, τ_j-$\mathrm{Cl}(A) = (\tau_j, \tau_i)$-$s\,\mathrm{Cl}(A)$.*
6. *For each (τ_i, τ_j)-semiopen set A in X, (τ_j, τ_i)-$s\,\mathrm{Cl}(A)$ is τ_j-closed set.*
7. *For each (τ_i, τ_j)-semiclosed set A in X, τ_j-$\mathrm{Int}(A) = (\tau_j, \tau_i)$-$s\,\mathrm{Int}(A)$.*
8. *For each (τ_i, τ_j)-semiclosed set A in X, (τ_j, τ_i)-$s\,\mathrm{Int}(A)$ is τ_j-open set.*

Proof. (1) $\Rightarrow$ (2): Follows from Theorem 2.3. (1) $\Rightarrow$ (5): Since (τ_j, τ_i)-$s\,\mathrm{Cl}(A) \subset \tau_j$-$\mathrm{Cl}(A)$ for any set A of X, it is sufficient to show that (τ_j, τ_i)-$s\,\mathrm{Cl}(A) \supset \tau_j$-$\mathrm{Cl}(A)$ for any (τ_i, τ_j)-semiopen set A of X. Let $x \notin (\tau_j, \tau_i)$-$s\,\mathrm{Cl}(A)$. Then there exists a (τ_j, τ_i)-semiopen set W with $x \in W$ such that $W \cap A = \emptyset$. Thus τ_j-$\mathrm{Int}(W)$ and τ_i-$\mathrm{Int}(A)$ are, respectively, τ_j-open and τ_i-open such that τ_j-$\mathrm{Int}(X) \cap \tau_i$-$\mathrm{Int}(A) = \emptyset$. By Theorem 2.7, τ_i-$\mathrm{Cl}(\tau_j$-$\mathrm{Int}(W)) \cap \tau_j$-$\mathrm{Cl}(\tau_i$-$\mathrm{Int}(A)) = \emptyset$ and then by Theorem 2.4, $x \notin \tau_j$-$\mathrm{Cl}(\tau_i$-$\mathrm{Int}(A)) = \tau_j$-$\mathrm{Cl}(A)$. Hence τ_j-$\mathrm{Cl}(A) \subset (\tau_j, \tau_i)$-$s\,\mathrm{Cl}(A)$. (5) $\Rightarrow$ (6): Obvious. (6) $\Rightarrow$ (5): For any set A in X, $A \subset (\tau_j, \tau_i)$-$s\,\mathrm{Cl}(A) \subset \tau_j$-$\mathrm{Cl}(A)$. Then τ_j-$\mathrm{Cl}(A) = \tau_j$-$\mathrm{Cl}((\tau_j, \tau_i)$-$s\,\mathrm{Cl}(A))$. Since A is (τ_i, τ_j)-semiopen, by (6), (τ_j, τ_i)-$s\,\mathrm{Cl}(A)$ is τ_j-closed. Hence, τ_j-$\mathrm{Cl}(A) = (\tau_j, \tau_i)$-$s\,\mathrm{Cl}(A)$. (6) $\Leftrightarrow$ (8): Follows from Theorem 2.5. (7) $\Rightarrow$ (8): Obvious. (8) $\Rightarrow$ (7): For any subset A of X, τ_j-$\mathrm{Int}(A) \subset (\tau_j, \tau_i)$-$s\,\mathrm{Int}(A) \subset A$ and hence τ_j-$\mathrm{Int}(A) = \tau_j$-$\mathrm{Int}((\tau_j, \tau_i)$-$s\,\mathrm{Int}(A))$. Since A is (τ_i, τ_j)-semiclosed, by (8), (τ_j, τ_i)-$s\,\mathrm{Int}(A)$ is τ_j-open. Hence τ_j-$\mathrm{Int}(A) = (\tau_j, \tau_i)$-$s\,\mathrm{Int}(A)$. (1) $\Rightarrow$ (4): Let A be a (τ_i, τ_j)-open set and B a (τ_j, τ_i)-semiopen set such that $A \cap B = \emptyset$. Then τ_i-$\mathrm{Int}(A) \cap \tau_j$-$\mathrm{Int}(B) = \emptyset$ and thus by Theorem 2.7, τ_j-$\mathrm{Cl}(\tau_j$-$\mathrm{Int}(A)) \cap \tau_i$-$\mathrm{Cl}(\tau_j$-$\mathrm{Int}(B)) = \emptyset$. Hence, by Theorem 2.3, τ_j-$\mathrm{Cl}(A) \cap \tau_i$-$\mathrm{Cl}(B) = \emptyset$. (4) $\Rightarrow$ (2): Let A be a (τ_i, τ_j)-semiopen subset of X. Then $X \backslash \tau_j$-$\mathrm{Cl}(A)$ is (τ_j, τ_i)-semiopen and $A \cap (X \backslash \tau_j$-$\mathrm{Cl}(A))$. Thus, by (4), τ_j-$\mathrm{Cl}(A) \cap \tau_i$-$\mathrm{Cl}(X \backslash \tau_j$-$\mathrm{Cl}(A)) = \emptyset$ which implies τ_j-$\mathrm{Cl}(A) \subset \tau_i$-$\mathrm{Int}(\tau_j$-$\mathrm{Cl}(A))$. Hence, τ_j-$\mathrm{Cl}(A) = \tau_i$-$\mathrm{Int}(\tau_j$-$\mathrm{Cl}(A))$ and consequently τ_j-$\mathrm{Cl}(A)$ is τ_i-open in X. (5) $\Rightarrow$ (4): Let A be a (τ_i, τ_j)-semiopen set and B be a (τ_j, τ_i)-semiopen set such that $A \cap B = \emptyset$. Then (τ_j, τ_i)-$s\,\mathrm{Cl}(A)$ is (τ_i, τ_j)-semiopen and (τ_i, τ_j)-$s\,\mathrm{Cl}(B)$ is (τ_j, τ_i)-semiopen in X and hence (τ_j, τ_i)-$s\,\mathrm{Cl}(A) \cap (\tau_j, \tau_i)$-$s\,\mathrm{Cl}(B) = \emptyset$. By (5), τ_j-$\mathrm{Cl}(A) \cap \tau_i$-$\mathrm{Cl}(B) = \emptyset$. (1) $\Rightarrow$ (3): Follows from Theorem 2.3 using the same method as (1) $\Rightarrow$ (5). (3) $\Rightarrow$ (1): Let A be a τ_i-open set in (X, τ_1, τ_2). It is sufficient to prove that τ_j-$\mathrm{Cl}(A) = (\tau_j, \tau_i)$-$s\,\mathrm{Cl}(A)$. Obviously, (τ_j, τ_i)-$s\,\mathrm{Cl}(A) \subset \tau_j$-$\mathrm{Cl}(A)$. Let $x \notin (\tau_j, \tau_i)$-$s\,\mathrm{Cl}(A)$. Then there exists a (τ_j, τ_i)-semiopen set U with $x \in U$ such that $A \cap U = \emptyset$. Hence (τ_i, τ_j)-$s\,\mathrm{Cl}(U) \subset (\tau_i, \tau_j)$-$s\,\mathrm{Cl}(X \backslash A) = X \backslash A$ and thus (τ_i, τ_j)-$s\,\mathrm{Cl}(U) \cap A = \emptyset$. Since (τ_i, τ_j)-$s\,\mathrm{Cl}(U)$ is a τ_j-open set with $x \in (\tau_i, \tau_j)$-$s\,\mathrm{Cl}(U)$, $x \notin \tau_j$-$\mathrm{Cl}(A)$. Hence τ_j-$\mathrm{Cl}(A) \subset (\tau_j, \tau_i)$-$\mathrm{Cl}(A)$. □

Definition 3.2. [3] *A point x in a bitoplogical space (X, τ_1, τ_2) is said to be (τ_i, τ_j)-θ-cluster point of a set A if for every τ_i-open, say, U containing x, τ_j-$\mathrm{Cl}(U) \cap A \neq \emptyset$. The set of all (τ_i, τ_j)-θ-closure of A and will be denoted by (τ_i, τ_j)-$\mathrm{Cl}_\theta(A)$. A set A is called (τ_i, τ_j)-θ-closed if $A = (\tau_i, \tau_j)$-$\mathrm{Cl}_\theta(A)$.*

Lemma 3.3. *For any (τ_j, τ_i)-preopen set A in a bitopological space (X, τ_1, τ_2), τ_i-$\mathrm{Cl}(A) = (\tau_i, \tau_j)$-$\mathrm{Cl}_\theta(A)$.*

Proof. It is obvious that τ_i-$\mathrm{Cl}(A) \subset (\tau_i, \tau_j)$-$\mathrm{Cl}_\theta(A)$, for any subset A of (X, τ_1, τ_2). Thus, it remains to be shown that (τ_i, τ_j)-$\mathrm{Cl}_\theta(A) \subset \tau_i$-$\mathrm{Cl}(A)$. If $x \notin \tau_i$-$\mathrm{Cl}(A)$, then there exists a τ_i-open set U containing x such that $U \cap A = \emptyset$ and thus $U \cap \tau_i$-$\mathrm{Cl}(A) = \emptyset$. But $U \cap \tau_j$-$\mathrm{Int}(\tau_i$-$\mathrm{Cl}(A)) = \emptyset$ which implies τ_j-$\mathrm{Cl}(U) \cap \tau_j$-$\mathrm{Int}(\tau_i$-$\mathrm{Cl}(A)) = \emptyset$ and so τ_j-$\mathrm{Cl}(U) \cap A = \emptyset$ since A is (τ_j, τ_i)-preopen. Hence $x \notin (\tau_j, \tau_i)$-$\mathrm{Cl}_\theta(A)$ and consequently (τ_j, τ_i)-$\mathrm{Cl}_\theta(A) \subset \tau_i$-$\mathrm{Cl}(A)$. □

Theorem 3.4. *The following are equivalent for a bitopological space (X, τ_1, τ_2):*

1. *(X, τ_1, τ_2) is pairwise extremally disconnected.*

2. *The* τ_j*-closure of every* (τ_i, τ_j)*-semipreopen set of* X *is* τ_i*-open set.*
3. *The* (τ_j, τ_i)*-*θ*-closure of every* (τ_i, τ_j)*-preopen set of* X *is* τ_i*-open set.*
4. *The* τ_j*-closure of every* (τ_i, τ_j)*-preopen set of* X *is* τ_i*-open set.*

Proof. (1) $\Rightarrow$ (2): Let A be a (τ_i, τ_j)-semipreopen set. Then τ_j-Cl$(A) = \tau_j$-Cl$(\tau_i$-Int$(\tau_j$-Cl$(A)))$. Since (X, τ_1, τ_2) is pairwise extremally disconnected. τ_j-Cl(A) is a τ_i-open set. (2) $\Rightarrow$ (4): Follows from the fact that every (τ_i, τ_j)-preopen set is (τ_i, τ_j)-semipreopen. (4) $\Rightarrow$ (1): Clear. (3) $\Leftrightarrow$ (4): Follows from Lemma 3.3. □

Theorem 3.5. *A bitopological space* (X, τ_1, τ_2) *is pairwise extremally disconnected if and only if every* (τ_i, τ_j)*-semiopen set is a* (τ_i, τ_j)*-preopen set.*

Proof. Let A be a (τ_i, τ_j)-semiopen set. Then $A \subset \tau_j$-Cl$(\tau_i$-Int$(A))$. Since X is pairwise extremally disconnected, τ_j-Cl$(\tau_i$-Int$(A))$ is a τ_i-open set and then $A \subset \tau_j$-Cl$(\tau_i$-Int$(A)) = \tau_i$-Int$(\tau_j$-Cl$(\tau_i$-Int$(A))) \subset \tau_i$-Int$(\tau_j$-Cl$(A))$. Hence A is a (τ_i, τ_j)-preopen set. Conversely, let A be a τ_i-open set. Since τ_j-Cl$(A) = \tau_j$-Cl$(\tau_i$-Int$(A))$, we have τ_j-Cl$(A) = \tau_j$-Cl$(\tau_i$-Int$(\tau_j$-Cl$(A)))$. Then τ_j-Cl(A) is (τ_j, τ_i)-regular closed and hence A is (τ_i, τ_j)-semiopen. By hypothesis, A is (τ_i, τ_j)-propen so that τ_j-Cl$(A) = \tau_i$-Int$(\tau_j$-Cl$(A))$. Then τ_j-Cl(A) is τ_i-open in X and hence X is pairwise extremally disconnected. □

Lemma 3.6. *For a subset* A *of a bitopological space* (X, τ_1, τ_2)*,*

1. τ_j-Int$(\tau_i$-Cl$(A)) \subset (\tau_i, \tau_j)$-$s\,Cl(A)$,[6]
2. τ_j-Int$((\tau_i, \tau_j)$-$s\,$Cl$(A)) = \tau_j$-Int$(\tau_i$-Cl$(A))$.

Proof. (2) Follows easily from (1). □

Theorem 3.7. *Let* A *be a subset of a bitopological space* (X, τ_1, τ_2)*. Then* A *is* (τ_i, τ_j)*-regular open if and only if* A *is* τ_i*-open and* τ_j*-closed.*

Proof. Let A be a (τ_i, τ_j)-regular open set of a bitoplogical space (X, τ_1, τ_2). Then τ_i-Int$(\tau_j$-Cl$(A)) = A$. Now, $X \backslash \tau_j$-Cl(A) and A are, respectively, τ_j-open and τ_i-open such that $(X \backslash \tau_j$-Cl$(A)) \cap A = \emptyset$. Since (X, τ_1, τ_2) is pairwise extremally disconnected, by Theorem 2.7, τ_i-Cl$(X \backslash \tau_j$-Cl$(A)) \cap \tau_j$-Cl$(A) = \emptyset$. Then τ_i-Cl$(X \backslash \tau_j$-Cl$(A)) = X \backslash \tau_j$-Cl$(A)$ and $X \backslash \tau_j$-Cl(A) is τ_i-closed. Hence, τ_j-Cl(A) is τ_i-open, so that τ_j-Cl$(A) = \tau_i$-Int$(\tau_j$-Cl$(A)) = A$ is τ_i-open and τ_j-closed. The converse is clear. □

Lemma 3.8. *Let* A *be a subset of a bitopological space* (X, τ_1, τ_2)*. Then we have*

1. A *is* (τ_i, τ_j)*-preopen if and only if* (τ_j, τ_i)*-*$s\,$Cl$(A) = \tau_i$-Int$(\tau_j$-Cl$(A))$.
2. A *is* (τ_i, τ_j)*-preopen if and only if* (τ_j, τ_i)*-*$s\,$Cl(A) *is* (τ_i, τ_j)*-regular open.*
3. A *is* (τ_i, τ_j)*-regular open if and only if* A *is* (τ_i, τ_j)*-preopen and* (τ_j, τ_i)*-semiclosed.*

Proof. (1) Let A b e a (τ_i, τ_j)-preopen set. Then (τ_j, τ_i)-$s\,$Cl$(A) \subset (\tau_j, \tau_i)$-$s\,Cl(\tau_i$-Int$(\tau_j$-Cl$(A)))$. Since τ_i-Int$(\tau_j$-Cl$(A))$ is (τ_j, τ_i)-semiclosed, (τ_j, τ_i)-$s\,$Cl$(A) \subset \tau_i$-Int$(\tau_j$-Cl$(A))$. Hence, by Lemma 3.6 (1), (τ_j, τ_i)-$s\,$Cl$(A) = \tau_i$-Int$(\tau_j$-Cl$(A))$. The converse is obvious. (2) Let (τ_j, τ_i)-$s\,$Cl(A) be a (τ_i, τ_j)-regular open set. Then we have

(τ_j,τ_i)-$s\,\mathrm{Cl}(A) = \tau_i$-Int($\tau_j$-Cl$((\tau_j,\tau_i)$-$s\,\mathrm{Cl}(A))$ and hence (τ_j,τ_i)-$s\,\mathrm{Cl}(A) \subset \tau_i$-Int($\tau_j$-Cl($\tau_j$-Cl($A$))) $= \tau_i$- Int(τ_j-Cl(A)). By Lemma 3.6 (1), we have (τ_j,τ_i)-$s\,\mathrm{Cl}(A) = \tau_i$-Int($\tau_j$-Cl($A$)). Hence, A is a (τ_i,τ_j)-preopen set from (1). The converse follows from (1). (3) Let A be a (τ_i,τ_j)-preopen and a (τ_j,τ_i)-semiclosed set. Then by (2), A is (τ_i,τ_j)-regular open in X. Conversely, let A be a (τ_i,τ_j)-regular open set. Then $A = \tau_i$-Int(τ_j-Cl(A)) and thus τ_i-Int(τ_j-Cl(A)) $= (\tau_j,\tau_i)$-$s\,\mathrm{Cl}(A) = A$. Hence A is (τ_i,τ_j)-preopen and (τ_j,τ_i)-semiclosed. □

Theorem 3.9. *In a bitopological space (X,τ_1,τ_2), the following are equivalent:*

1. *X is pairwise extremally disconnected.*
2. *(τ_j,τ_i)-$s\,\mathrm{Cl}(A) = (\tau_j,\tau_i)$-$\mathrm{Cl}_\theta(A)$ for every (τ_i,τ_j)-preopen (or (τ_i,τ_j)-semiopen) set A in X.*
3. *(τ_j,τ_i)-$s\,\mathrm{Cl}(A) = \tau_j$-Cl($A$) for every (τ_i,τ_j)-semipreopen set A in X.*

Proof. (1) ⇒ (2): Since (τ_j,τ_i)-$s\,\mathrm{Cl}(A) \subset (\tau_j,\tau_i)$-$\mathrm{Cl}_\theta(A)$ for any subset A of X, it is sufficient to show that (τ_j,τ_i)-$\mathrm{Cl}_\theta(A) \subset (\tau_j,\tau_i)$-$s\,\mathrm{Cl}(A)$ for any (τ_i,τ_j)-preopen or (τ_i,τ_j)-semiopen set A of X. Let $x \notin (\tau_j,\tau_i)$-$s\,\mathrm{Cl}(A)$. Then there exists a (τ_j,τ_i)-semiopen set U with $x \in U$ such that $U \cap A = \emptyset$ and thus there exists a τ_j-open set V such that $V \subset U \subset \tau_j$-Cl($V$) with $V \cap A = \emptyset$ which implies $V \cap \tau_j$-Cl(A) $= \emptyset$. This means $V \cap \tau_i$-Int(τ_j-Cl(A)) $= \emptyset$ and hence τ_i-Cl(V) $\cap\, \tau_i$-Int(τ_j-Cl(A)) $= \emptyset$. Now, if A is (τ_i,τ_j)-preopen, then $A \subset \tau_i$-Int(τ_j-Cl(A)) and hence τ_i-Cl(V) $\cap A = \emptyset$. If A is (τ_i,τ_j)-semiopen, since X is pairwise extremally disconnected, τ_i-Cl(V) is τ_j-open and thus τ_i-Cl(V) $\cap\, \tau_j$-Cl(τ_i-Int(τ_j-Cl(A))) $= \emptyset$ which implies τ_i-Cl(V) $\cap A = \emptyset$. Hence, in any case, $x \notin (\tau_j,\tau_i)$-$\mathrm{Cl}_\theta(A)$. (2) ⇒ (1): First let A be a (τ_i,τ_j)-preopen set in X. By Lemmas 3.8 and 3.3, we have τ_i-Int(τ_j-Cl(A)) $= (\tau_j,\tau_i)$-$s\,\mathrm{Cl}(A) = (\tau_j,\tau_i)$-$\mathrm{Cl}_\theta(A) = \tau_j$-Cl($A$). Then τ_j-Cl(A) is τ_i-open and hence by Theorem 3.4, X is pairwise extremally disconnected. Next, let A be a (τ_i,τ_j)-semiopen set in X. Then (τ_j,τ_i)-Cl(A) $\subset \tau_j$-Cl(A) $\subset (\tau_j,\tau_i)$-$\mathrm{Cl}_\theta(A) = (\tau_j,\tau_i)$-$s\,\mathrm{Cl}(A)$ and thus (τ_j,τ_i)-$s\,\mathrm{Cl}(A) = \tau_j$-Cl($A$). Hence, X is pairwise extremally disconnected from Theorem 3.4. (1) ⇒ (3): Let A be a (τ_i,τ_j)-semipreopen set in X. Since X is pairwise extremally disconnected, by Theorem 3.4, τ_j-Cl(A) is τ_i-open in X. Hence, by Lemma 3.8, (τ_j,τ_i)-$s\,\mathrm{Cl}(A) = \tau_j$-Cl($A$). (3) ⇒(1): Let U and V, respectively, be τ_i-open and τ_j-open sets such that $U \cap V = \emptyset$. Then $U \subset X\backslash V$ which implies (τ_j,τ_i)-$s\,\mathrm{Cl}(U) \subset (\tau_j,\tau_i)$-$s\,\mathrm{Cl}(X\backslash V) = X\backslash V$ and hence (τ_j,τ_i)-$s\,\mathrm{Cl}(U) \cap V = \emptyset$. Since (τ_j,τ_i)-$s\,\mathrm{Cl}(U)$ is (τ_i,τ_j)-semiopen in X, (τ_j,τ_i)-$s\,\mathrm{Cl}(U) \cap (\tau_i,\tau_j)$-$s\,\mathrm{Cl}(V) = \emptyset$. Then by (3) τ_j-Cl(U) $\cap\, \tau_i$-Cl(V) $= \emptyset$ and hence by Theorem 2.7, X is pairwise extremally disconnected. □

Theorem 3.10. *In a bitopological space (X,τ_1,τ_2), the following are equivalent:*

1. *X is pairwise extremally disconnected.*
2. *For each (τ_i,τ_j)-semipreopen set A in X and each (τ_j,τ_i)-semiopen set B in X such that $A \cap B = \emptyset$, τ_i-Cl(A) $\cap\, \tau_j$-Cl(B) $= \emptyset$*
3. *For each (τ_i,τ_j)-preopen set A in X and each (τ_j,τ_i)-semiopen set B in X such that $A \cap B = \emptyset$, τ_i-Cl(A) $\cap\, \tau - j$-Cl(B) $= \emptyset$.*

Proof. (1)⇒ (2): Let A be a (τ_i,τ_j)-semipreopen set and B a (τ_j,τ_i)-semiopen set such that $A \cap B = \emptyset$. Then $A \cap \tau_j$-Int(B) $= \emptyset$ and hence τ_j-Cl(A) $\cap\, \tau_j$-Int(B) $= \emptyset$. By Theorem 3.4, τ_j-Cl(A) is a τ_i-open set in X and hence τ_j-Cl(A)$\cap\tau_i$-Cl(τ_j-Int(B)) $= \emptyset$.

Since B is (τ_j,τ_i)-semiopen in X, by Theorem 2.3, τ_i-Cl$(B) = \tau_i$-Cl$(\tau_j$-Int$(B))$. Thus τ_j-Cl$(A) \cap \tau_i$-Cl$(B) = \emptyset$. (2) $\Rightarrow$ (3): Straightforward. (3) $\Rightarrow$ (1): Let A be a τ_i-open set and B a τ_j-open set such that $A \cap B = \emptyset$. Since every τ_i-open set is a (τ_i,τ_j)-semiopen set and every τ_j-open set is a (τ_i,τ_j)-semiopen set and every τ_j-open set is a (τ_j,τ_i)-preopen set, τ_j-Cl$(A) \cap \tau_i$-Cl$(B) = \emptyset$. Hence by Theorem 2.7, X is pairwise extremally disconnected. □

Definition 3.11. *A function $f : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ is said to be*

1. *pairwise semicontinuous* [2] *if $f^{-1}(V)$ is a (τ_i,τ_j)-semiopen set in X for each σ_i-open set V in Y.*
2. *pairwise almost open if $f(U)$ is a σ_i-open set in Y for each (τ_i,τ_j)-regular open set U in X.*

Lemma 3.12. *A function $f : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ is pairwise almost open if and only if for each (τ_j,τ_i)-semiclosed set A in X, $f(\tau_i\text{-Int}(A)) \subset \sigma_i\text{-Int}(f(A))$.*

Proof. Let A be a (τ_j,τ_i)-semiclosed set in X. Then τ_i-Int$(\tau_j$-Cl$(A))$ is (τ_i,τ_j)-regular open and hence $f(\tau - i\text{-Int}(\tau_j\text{-Cl}(A)))$ is σ_i-open in Y. Now by Theorem 2.4, τ_i-Int$(A) = \tau_i$-Int$(\tau_j$-Cl$(A)) \subset A$ which implies that $f(\tau_i\text{-Int}(A)) = f(\tau_i\text{-Int}(\tau_j\text{-Cl}(A))) = \sigma_i\text{-Int}(f(\tau_i\text{-Int}(\tau_j\text{-Cl}(A)))) \subset \sigma_i\text{-Int}(f(A))$. Hence $f(\tau_i\text{-Int}(A)) \subset \sigma_i\text{-Int}(f(A))$. Conversely, let A be a (τ_i,τ_j)-regular open set in X. Then A is (τ_j,τ_i)-semiclosed and hence $f(\tau_i\text{-Int}(A)) \subset \sigma_i\text{-Int}(f(A))$. Now, $A = \tau_i$-Int(A) and thus $f(A) = f(\tau_i\text{-Int}(A)) \subset \sigma_i\text{-Int}(f(A))$, so that $f(A)$ is σ_i-open in Y. Hence f is pairwise almost open. □

Lemma 3.13. *If $f : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ is a pairwise semicontinuous and a pairwise almost open mapping, then $f(A)$ is a (σ_i,σ_j)-preopen set in Y for each (τ_i,τ_j)-preopen set A in X.*

Proof. Let A be a (τ_i,τ_j)-preopen set in X. Since f is pairwise semicontinuous, $f(A) \subset f((\tau_j,\tau_i)\text{-}s\,\text{Cl}(A)) \subset \sigma_j\text{-Cl}(f(A))$. By Lemma 3.8 (2), (τ_j,τ_i)-$s\,$Cl(A) is (τ_j,τ_i)-regular open set in X and thus $f((\tau_j,\tau_i)\text{-}s\,\text{Cl}(A))$ is a (σ_i,σ_j)-preopen set in Y because f is pairwise almost open. By Lemma 3.8 (1), $(\sigma_j,\sigma_i)\text{-}s\,\text{Cl}(f((\tau_j,\tau_i)\text{-}s\,\text{Cl}(A))) = \sigma_i\text{-Int}(\sigma_j\text{-Cl}(f((\tau_j,\tau_i)\text{-}s\,\text{Cl}(A))))$. Hence, $(\sigma_j,\sigma_i)\text{-}s\,\text{Cl}(f(A)) \subset (\sigma_j,\sigma_i)\text{-}s\,\text{Cl}(f((\tau_j,\tau_i)\text{-}s\,\text{Cl}(A))) = \sigma_i\text{-Int}(\sigma_j\text{-Cl}(f((\tau_j,\tau_i)\text{-}s\,\text{Cl}(A)))) \subset \sigma_j\text{-Cl}(f(A))$. Since $\sigma_i\text{-Int}(\sigma_j\text{-Cl}(f(A))) = \sigma_i\text{-Int}(\sigma_j\text{-Cl}(f((\tau_j,\tau_i)\text{-}s\,\text{Cl}(A))))$, we have $f(A) \subset (\sigma_j,\sigma_i)\text{-}s\,\text{Cl}(f(A)) \subset \sigma_i\text{-Int}(\sigma_j\text{-Cl}(f(A)))$. Hence $f(A)$ is (σ_i,σ_j)-preopen in Y. □

Lemma 3.14. *If $f : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ is a pairwise semicontinuous and a pairwise almost open mapping we have*

1. *$f^{-1}(B)$ is a (τ_i,τ_j)-semiclosed set in X for each (σ_i,σ_j)-semiclosed set in B in Y.*
2. *$f^{-1}(B)$ is a (τ_i,τ_j)-semiopen set in X for each (σ_i,σ_j)-semiopen set in B in Y.*

Proof. (1) Let B be a (σ_i,σ_j)-semiclosed set in Y. Since f is pairwise semicontinuous and σ_i-Cl(B) is a σ_i-closed set, $f^{-1}(\sigma_i\text{-Cl}(B))$ is (τ_i,τ_j)-semiclosed in X. Hence, $\tau_i\text{-Int}(\tau_j\text{-Cl}(f^{-1}(\sigma_i\text{-Cl}(B)))) \subset \tau_j\text{- Int}(f^{-1}(\sigma_i\text{-Cl}(B)))$. Since f is pairwise almost open by Lemma 3.12 $f(\tau_j\text{-Int}(f^{-1}(\sigma_i\text{-Cl}(B)))) \subset \tau_j\text{-Int}(f(f^{-1}(\sigma_i\text{-Cl}(B)))) \subset \sigma_j\text{-Int}(\sigma_i\text{-Cl}(B)) \subset B$. Which implies that $\tau_j\text{-Int}(f^{-1}(\sigma_i\text{-Cl}(B))) \subset f^{-1}(B)$. Now, $\tau_j\text{-Int}(\tau_i\text{-Cl}(f^{-1}(B))) \subset \tau_j\text{-Int}(\tau_i\text{- Cl}(f^{-1}(\sigma_i\text{-Cl}(B)))) \subset \tau_j\text{-Int}(f^{-1}(\sigma_i\text{- Cl}(B))) \subset f^{-1}(B)$. Hence $f^{-1}(B)$ is a (τ_i, τ_j)-semiclosed set in X. (2) Follows easily from (1) by taking the complement. □

Theorem 3.15. *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a pairwise semicontinuous and a pairwise almost open surjection. If (X, τ_1, τ_2) is pairwise extremally disconnected, then (Y, σ_1, σ_2) is also pairwise extremally disconnected.*

Proof. Let B be a (σ_i, σ_j)-semiopen set in Y. By Lemma 3.14, $f^{-1}(B)$ is (τ_i, τ_j)-semiopen in X. Since X is pairwise extremally disconnected, by Theorem 3.5, $f^{-1}(B)$ is (τ_i, τ_j)-preopen in X. By Lemma 3.13, B is (σ_i, σ_j)-preopen in Y and hence by Theorem 3.5, Y is pairwise extremally disconnected. □

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