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# Variational analysis of a contact problem with friction between two deformable bodies

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**Abstract.** This paper deals with the study of a nonlinear problem of friction contact between two deformable bodies. The elastic constitutive law is a assumed to be nonlinear and the contact is modeled with Signorini's conditions and version of Coulomb's law of dry friction. We present two variational formulations, noted $\mathbb{P}_1$, $\mathbb{P}_2$, of the considered problem, where $\mathbb{P}_1$ depends on the displacement field and $\mathbb{P}_2$ depends on the stress field. We establish existence and uniqueness results, using arguments of elliptic variational inequalities and a fixed point property and *Lions, Stampachia* theorem.

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## 1. Introduction

Frictional contact between deformable bodies can be frequently found in industry and everyday life such as train wheels with the rails, a shoe with the floor, tectonic plates, the car's braking system, etc. Considerable progress has been made with the modeling and analysis of static contact problems. The mathematical, mechanical and numerical state of the art can be found in the recent proceedings Raous [21]. Only recently, however, have the quasistatic and dynamic problems been considered. The reason lies in the considerable difficulties that the process of frictional contact presents in the modeling and analysis because of the complicated surface phenomena involved. General models for thermoelastic frictional contact, derived from thermodynamical principles, have been obtained in [25]. Quasistatic contact problems with normal compliance and friction have been considered in [3], where the existence of weak solutions has been proven. The existence of a weak solution to the, technically very complicated, problem with Signorini's contact condition has been established recently in [10]. The quasistatic frictional contact problem for viscoelastic materials can be found in [23] and the one for elastoviscoplastic materials in [22]. Dynamic problems with normal compliance were first considered in [19]. The existence of weak solutions to dynamic thermoelastic contact problems with frictional heat generation have been proven in [1] and when wear is taken into account in [2].

In this work we consider the process of frictional contact which is acted upon by volume forces and surface tractions, between two elastics bodies. The material's constitutive law is assumed to be nonlinear elastic. The contact is modeled with a normal compliance and the friction with the associated Coulomb's law of dry friction. The normal compliance contact condition was proposed in [19] and used in [1] and [15]. This condition allows the interpenetration of the body's surface into the foundation. In [19] normal compliance was justified by considering the interpenetration and deformation of surface asperities. It was assumed to have the form of a power law. We refer to [18] for the existence of static problems with Signorini's and Coulomb's conditions. We use a general expression for the normal compliance, similarly to the one in [2]. In part, the introduction of the normal compliance contact condition, in evolution problems, is motivated by the observation that Signorini's condition, while elegant and easy to explain, leads to discontinuous surface velocities which are associated with infinite tractions on the contact surface. This clearly is physically unrealistic; it leads to severe mathematical and numerical difficulties which do not necessarily represent the physical process. The normal compliance condition predicts large, but finite, contact forces. At any rate, we do not have a completely satisfactory contact condition yet, and maybe it is unrealistic to expect one single condition to model the wide variety of phenomena encountered in frictional contact.

The paper is organized as follows. Section 2 contains the notations and some preliminary material. In Section 3 we describe the model for the process, set it in a variational form, list the assumptions on the problem data and state our main results. In Section 4, basing on the theory of elliptic variational inequalities and application of fixed point theorems, we show the existence and uniqueness of a solution.

## 2. Notations and preliminaries

In this short section we present the notations and some preliminary material. For further details we refer the reader to [11] or [15]. We denote by $\mathbb{S}_N$ the space of second order symmetric tensors on $\mathbb{R}^N$, or equivalently, the space of the symmetric matrices of order $N$. The inner products and the corresponding norms on $\mathbb{R}^N$ and $\mathbb{S}_N$ are

$$u^\ell.v^\ell = u_i^\ell.v_i^\ell, \quad \|v^\ell\| = (v^\ell.v^\ell)^{\frac{1}{2}} \quad \forall u^\ell, v^\ell \in \mathbb{R}^N,$$
$$\sigma^\ell.\tau^\ell = \sigma_{ij}^\ell.\tau_{ij}^\ell, \quad \|\tau^\ell\| = (\tau^\ell.\tau^\ell)^{\frac{1}{2}} \quad \forall \sigma^\ell, \tau^\ell \in \mathbb{S}_N.$$

Here and below, $i, j = 1, 2, ..., N$, and the summation convention over repeated indices is adopted. Let two bounded domains $\Omega^\ell$, $\ell = 1, 2$ of the space $\mathbb{R}^N (N = 2, 3)$ be a bounded domain with a Lipschitz boundary $\Gamma^\ell$ and let $\eta^\ell = (\eta_i^\ell)$ denote the normal unit outward vector on $\Gamma^\ell$. We shall use the notations

$$H^\ell = \{u^\ell = (u_i^\ell)/\ u_i^\ell \in \mathbb{L}^2(\Omega^\ell)\}\ ,\quad \mathcal{H}^\ell = \{\sigma^\ell = (\sigma_{ij}^\ell)/\sigma_{ij}^\ell = \sigma_{ji}^\ell \in \mathbb{L}^2(\Omega^\ell)\},$$
$$H_1^\ell = \{u^\ell = (u_i^\ell)/\ u_i^\ell \in H^1(\Omega^\ell)\}\ ,\quad \mathcal{H}_1^\ell = \{\sigma^\ell \in \mathcal{H}^\ell/\sigma_{ij,j}^\ell \in H^\ell\},$$
$$H = H^1 \times H^2 \quad ,\quad H_1 = H_1^1 \times H_1^2 \quad ,\quad \mathcal{H} = \mathcal{H}^1 \times \mathcal{H}^2 \quad ,\quad \mathcal{H}_1 = \mathcal{H}_1^1 \times \mathcal{H}_1^2.$$

The spaces $H^\ell, H_1^\ell, \mathcal{H}^\ell$ and $\mathcal{H}_1^\ell$ are real Hilbert spaces endowed with the inner products given by

$$\langle u^\ell, v^\ell\rangle_{H^\ell} = \int_{\Omega^\ell} u_i^\ell v_i^\ell dx\ ,\quad \langle u^\ell, v^\ell\rangle_{H_1^\ell} = \langle u^\ell, v^\ell\rangle_{H^\ell} + \langle \epsilon(u^\ell), \epsilon(v^\ell)\rangle_{\mathcal{H}^\ell},$$
$$\langle \sigma^\ell, \tau^\ell\rangle_{\mathcal{H}^\ell} = \int_{\Omega^\ell} \sigma_{ij}^\ell \tau_{ij}^\ell dx\ ,\quad \langle \sigma^\ell, \tau^\ell\rangle_{\mathcal{H}_1^\ell} = \langle \sigma^\ell, \tau^\ell\rangle_{\mathcal{H}^\ell} + \langle div\sigma^\ell, div\tau^\ell\rangle_{H^\ell},$$

respectively. Here $\epsilon : H_1^\ell \to \mathcal{H}^\ell$ and $div : \mathcal{H}_1^\ell \to H^\ell$ are the *deformation* and *divergence* operators, defined by

$$\epsilon(u^\ell) = \frac{1}{2}\big(\nabla u^\ell + (\nabla u^\ell)^T\big), \qquad div\sigma^\ell = (\sigma_{ij,j}^\ell).$$

The associated norms on the spaces $H^\ell$, $H_1^\ell$, $\mathcal{H}^\ell$ and $\mathcal{H}_1^\ell$ are denoted by $\|.\|_{H^\ell}, \|.\|_{H_1^\ell}, \|.\|_{\mathcal{H}^\ell}$ and $\|.\|_{\mathcal{H}_1^\ell}$, respectively.

Let $H_{\Gamma^\ell} = H^{\frac{1}{2}}(\Gamma^\ell)^N$ and let $\gamma^\ell : H_1^\ell \to H_{\Gamma^\ell}$ be the trace map. For every element $v^\ell \in H_1^\ell$, we also use the notation $v^\ell$ for the trace $\gamma^\ell v^\ell$ of $v^\ell$ on $\Gamma^\ell$ and we denote by $v_\eta^\ell$ and $v_\tau^\ell$ the *normal* and *tangential* components of $v^\ell$ on $\Gamma^\ell$ given by

$$v_\eta^\ell = v^\ell.\eta^\ell, \quad v_\tau^\ell = v^\ell - v_\eta^\ell \eta^\ell. \tag{2.1}$$

Let $H'_{\Gamma^\ell}$ be the dual of $H_{\Gamma^\ell}$ and let $\langle .,.\rangle$ denote the duality pairing between $H'_{\Gamma^\ell}$ and $H_{\Gamma^\ell}$. For every element $\sigma^\ell \in \mathcal{H}_1^\ell$ let $\sigma^\ell\eta^\ell$ be the element of $H'_{\Gamma^\ell}$ given by

$$\langle \sigma^\ell\eta^\ell, \gamma^\ell v^\ell\rangle = \langle \sigma^\ell, \epsilon(v^\ell)\rangle_{\mathcal{H}^\ell} + \langle div\sigma^\ell, v^\ell\rangle_{H^\ell} \quad \forall v^\ell \in H_1^\ell. \tag{2.2}$$

We also denote by $\sigma_\eta^\ell$ and $\sigma_\tau^\ell$ the *normal* and *tangential* traces of $\sigma^\ell$, respectively. If $\sigma^\ell$ is continuously differentiable on $\overline{\Omega}^\ell$, then

$$\sigma_\eta^\ell = (\sigma^\ell\eta^\ell).\eta^\ell, \qquad \sigma_\tau^\ell = \sigma^\ell\eta^\ell - \sigma_\eta^\ell\eta^\ell, \tag{2.3}$$
$$\langle \sigma^\ell\eta^\ell, \gamma^\ell v^\ell\rangle = \int_{\Gamma^\ell} \sigma^\ell\eta^\ell.\gamma^\ell v^\ell da \tag{2.4}$$

for all $v^\ell \in H_1^\ell$, where $da$ is the surface measure element.

## 3. The model and statement of results

In this section we describe a model for the process, present its variational formulation, list the assumptions on the problem data and state our main results.

Let us consider two elastic bodies, occupying two bounded domains $\Omega^1$, $\Omega^2$ of the space $\mathbb{R}^N (N = 2, 3)$. The boundary $\Gamma^\ell = \partial\Omega^\ell$ is assumed piecewise continuous, and composed of three complementary parts $\Gamma_1^\ell, \Gamma_2^\ell$ and $\Gamma_3^\ell$. The body $\overline{\Omega}^\ell$ is fixed on the set $\Gamma_1^\ell$ of positive measure. The $\Gamma_2^\ell$ boundary is submitted to a density of forces noted $g^\ell$.

In the initial configuration, both bodies have a common contact portion $\Gamma_3^1 = \Gamma_3^2 = \Gamma_3$. The $\Omega^\ell$ body is submitted to $f^\ell$ forces. The normal unit outward vector on $\Gamma^\ell$ is denoted $\eta^\ell = (\eta_i^\ell)$. On the contact zone the normal vector $\eta = \eta^1 = -\eta^2$ is assumed to be constant.

We denot by $u^\ell = (u_i^\ell)_{1\leq i\leq N}$ the displacement fields of the body $\Omega^\ell$, $\sigma^\ell = (\sigma_{ij}^\ell)_{1\leq i,j\leq N}$ the stress field of the body $\Omega^\ell$ and $\epsilon^\ell = \epsilon(u^\ell)$ the linearized strain tensor. The elastic constitutive law of the material is assumed to be

$$\sigma^\ell = F^\ell\left(\epsilon(u^\ell)\right) \quad \text{in} \quad \Omega^\ell \tag{3.1}$$

in which $F^\ell$ is a given nonlinear function. The elastic equilibrium condition can be written as

$$\begin{cases} div\sigma^\ell + f^\ell = 0 & \text{in} \quad \Omega^\ell, \\ u^\ell = 0 & \text{on} \quad \Gamma_1^\ell, \qquad \ell = 1,2 \\ \sigma^\ell\eta^\ell = g^\ell & \text{on} \quad \Gamma_2^\ell, \end{cases} \tag{3.2}$$

where $u = (u^1, u^2)$. In addition to (3.2) and $\sigma^1\eta^1 = \sigma^2\eta^2$ on $\Gamma_3$, we have to satisfy the linearized non-penetration condition. The conditions on the boundary part $\Gamma_3$ constrained by *Coulomb* friction unilateral contact conditions incorporate the *Signorini conditions* :

$$[u_\eta] \leq 0, \quad \sigma_\eta \leq 0, \quad \sigma_\eta[u_\eta] = 0, \tag{3.3}$$

$$\begin{cases} |\sigma_\tau| \leq -\mu\sigma_\eta & \text{if} \quad [u_\tau] = 0, \\ \sigma_\tau = \mu\sigma_\eta \frac{[u_\tau]}{\left|[u_\tau]\right|} & \text{if} \quad [u_\tau] \neq 0 \end{cases} \tag{3.4}$$

where $\sigma_\eta$ and $\sigma_\tau$ is the normal and tangential component, respectively, of the boundary stress, and $[u_\eta] = u_\eta^1 + u_\eta^2$ stands for the jump of the displacements in normal direction: either contact (i.e. $[u_\eta] = 0$ )or separation (i.e. $[u_\eta] < 0$ )are allowed.in other word ( $[u_\eta] \leq 0$ ) is the nonpenetration condition, $[u_\tau] = u_\tau^1 + u_\tau^2$ stands for the jump of the displacements in tangential direction and $\mu \geq 0$ is the friction coefficient. This is a static version of *Coulomb's* law of dry friction and should be seen either as a mechanical model suitable for the proportional loadings case or as a first approximation of a more realistic model, based on a friction law involving the time derivative of $u^1$, $u^2$ (see for instance *Shillor* and *Sofonea*(1997), *Rochdi*(1998) ). The friction law (3.4) states that the tangential shear cannot exceed the maximum frictional resistance $-\mu\sigma_\eta$. Then, if the inequality holds, the surfaces adheres and is so-called *stick* state, and the equality holds there is relative sliding, the so-called *slip* state. Therefore, the contact surface $\Gamma_3$ is divided into three zones: the stick zone, the slip zone and the zone of separation in which $[u_\eta] < 0$, i.e, there is no contact. The boundaries of these zones are *free boundaries* since they are unknown a priori, and are part of the problem. There is virtually no literature dealing with these free boundaries.

It is possible to express equivalently the contact and friction conditions considering the two following multivalued functions:

$$J_\eta(\xi) = \begin{cases} \{0\} & \text{if } \xi < 0, \\ [0, +\infty[ & \text{if } \xi = 0, \\ \emptyset & \text{if } \xi > 0, \end{cases}$$

$$Dir_\tau(v) = \begin{cases} \{\frac{v}{|v|}\} & \text{if } v \in \mathbb{R}^{N-1} \text{ and } v \neq 0, \\ \{\omega \in \mathbb{R}^{N-1} / |\omega| \ \leq 1\,;\, \omega_\eta = 0\} & \text{if } v = 0. \end{cases}$$

$J_\eta$ and $Dir_\tau$ are maximal monotone maps representing sub-gradients of the indicator function of interval $]-\infty, 0]$ and the function $v \mapsto |v_T|$ respectively. With these maps, unilateral contact and *Coulomb* friction conditions can be rewritten as:

$$\begin{cases} -\sigma_\eta \in J_\eta([u_\eta]), \\ -\sigma_\tau \in \mu\sigma_\eta Dir_\tau([u_\tau]). \end{cases}$$

Using (3.1)-(3.4), the mechanical problem non linear of the unilateral contact with *Coulomb* friction between two deformable bodies may be formulated as classically as follows:

**Problem$\mathbb{P}$:** For $\ell = 1, 2$, find the displacement field $u^\ell : \Omega^\ell \longrightarrow \mathbb{R}^N$ and the stress field $\sigma^\ell : \Omega^\ell \longrightarrow \mathbb{S}_N$ such that

$$\sigma^\ell = F^\ell(\epsilon(u^\ell)) \quad \text{in} \quad \Omega^\ell, \tag{3.5}$$

$$div\sigma^\ell + f^\ell = 0 \quad \text{in} \quad \Omega^\ell, \tag{3.6}$$

$$u^\ell = 0 \quad \text{on} \quad \Gamma_1^\ell, \tag{3.7}$$

$$\sigma^\ell\eta^\ell = g^\ell \quad \text{on} \quad \Gamma_2^\ell, \tag{3.8}$$

$$\begin{cases} (a) & \sigma^1\eta^1 = \sigma^2\eta^2, \\ (b) & [u_\eta] \leq 0,\ \sigma_\eta \leq 0,\ \sigma_\eta[u_\eta] = 0, \\ (c) & |\sigma_\tau| \leq -\mu\sigma_\eta, \\ (d) & |\sigma_\tau| < -\mu\sigma_\eta \Rightarrow [u_\tau] = 0, \\ (e) & |\sigma_\tau| = -\mu\sigma_\eta \Rightarrow \exists\dot{\lambda} \geq 0; \quad \sigma_\tau = -\dot{\lambda}[u_\tau], \end{cases} \quad \text{on} \quad \Gamma_3. \tag{3.9}$$

To obtain a variational formulation for problem (3.5)-(3.9) we need the following additional notations. Let $V$ denote the closed subspace of $H_1$ given by

$$V = V(\Omega^1) \times V(\Omega^2) \tag{3.10}$$

where

$$V(\Omega^\ell) = \left\{v^\ell \in H_1^\ell \;\middle|\; v^\ell = 0 \text{ on } \Gamma_1^\ell\right\} \tag{3.11}$$

and let denote the closed subspace of $\mathcal{H}_1$ given by

$$\widehat{\mathcal{H}}_1 = \Big\{\sigma = (\sigma^1, \sigma^2) \in \mathcal{H}_1 \;\Big|\; \sigma^1\eta^1 = \sigma^2\eta^2 \text{ on } \Gamma_3\Big\}. \tag{3.12}$$

Since $meas\Gamma_1^\ell > 0$, the following *Korn's* inequality holds:

$$\|\epsilon(v^\ell)\|_{\mathcal{H}^\ell} \geq c\|v^\ell\|_{H_1^\ell}, \quad \forall v^\ell \in V(\Omega^\ell) \quad \ell = 1.2. \tag{3.13}$$

Here $c$ denotes a positive constant which may depends only on $\Omega^\ell, \Gamma_1^\ell, \ell = 1, 2$. We equip $V$ with the scalar product

$$\langle\nu, \omega\rangle_V = \langle\epsilon(\nu^1), \epsilon(\omega^1)\rangle_{\mathcal{H}^1} + \langle\epsilon(\nu^2), \epsilon(\omega^2)\rangle_{\mathcal{H}^2} \tag{3.14}$$

and $\|.\|_V$ is the associated norm. It follows from *Korn's* inequality (3.13) that the norms $\|.\|_{H_1}$ and $\|.\|_V$ are equivalent on $V$. Then $(V, \|.\|_V)$ is a real Hilbert space.

Moreover, by the *Sobolev's* trace theorem and (3.13)we have a positive constant $c_0$ depending only on the domain $\Omega^\ell, \Gamma_1^\ell, \ell = 1, 2$ and $\Gamma_3$ such that

$$\|v^\ell\|_{\mathbb{L}^2(\Gamma_3)^N} \leq c_0 \|v^\ell\|_V \qquad \forall v \in V. \tag{3.15}$$

In the study of the mechanical problem (3.5)-(3.9) we assume that operators $F^\ell$ : $\Omega^\ell \times \mathbb{S}_N \to \mathbb{S}_N$ satisfy

$$\begin{cases} (a) & \text{There exists } m > 0 \text{ such that} \\ & (F^\ell(x,\varepsilon_1) - F^\ell(x,\varepsilon_2)).(\varepsilon_1 - \varepsilon_2) \geq m|\varepsilon_1 - \varepsilon_2|^2 \\ & \quad \forall \varepsilon_1, \varepsilon_2 \in \mathbb{S}_N, \text{ a.e. } x \in \Omega^\ell. \\ (b) & \text{There exists } L > 0 \text{ such that} \\ & |F^\ell(x,\varepsilon_1) - F^\ell(x,\varepsilon_2)| \leq L|\varepsilon_1 - \varepsilon_2| \\ & \quad \forall \varepsilon_1, \varepsilon_2 \in \mathbb{S}_N, \text{ a.e. } x \in \Omega^\ell. \\ (c) & \text{For any } \varepsilon \in \mathbb{S}_N, \ x \to F^\ell(x,\varepsilon) \text{ is Lebesgue measurable on } \Omega^\ell. \\ (d) & \text{The mapping } x \mapsto F^\ell(x,0) \in \mathcal{H}^\ell. \end{cases} \tag{3.16}$$

**Remark 3.1.** Using (3.16) we obtain that for all $\varepsilon^\ell \in \mathcal{H}^\ell$ the function $x \mapsto F^\ell(x, \varepsilon^\ell(x))$ belongs to $\mathcal{H}^\ell$ and hence we may consider $F^\ell$ as an operator defined on $\mathcal{H}^\ell$ with the range on $\mathcal{H}^\ell$. Moreover, $F^\ell : \mathcal{H}^\ell \to \mathcal{H}^\ell$ is a strongly monotone Lipschitz continuous operator and therefore $F^\ell$ is invertible and its inverse $(F^\ell)^{-1} : \mathcal{H}^\ell \to \mathcal{H}^\ell$ is also a strongly monotone Lipschitz continuous operator.

We also suppose that the forces and the tractions have the regularity

$$f^\ell \in H^\ell, \qquad g^\ell \in \mathbb{L}^2(\Gamma_2^\ell)^N \tag{3.17}$$

while the coefficient of friction $\mu$ is such that

$$\mu \in L^\infty(\Gamma_3), \quad \mu \geq 0 \text{ on } \Gamma_3. \tag{3.18}$$

For $(u, v) \in V$, we define the bilinear form of virtual works produced by the displacement $u$ by

$$a(u,v) = \sum_{\ell=1}^{2} \int_{\Omega^\ell} F^\ell \epsilon(u^\ell).\epsilon(v^\ell) d\Omega^\ell \tag{3.19}$$

and the linear form of virtual works due to volume forces and surface traction by

$$\langle \varphi^\ell, v^\ell \rangle_{V(\overline{\Omega^\ell})} = \int_{\Omega^\ell} f^\ell.v^\ell d\Omega^\ell + \int_{\Gamma_2^\ell} g^\ell.v^\ell \eta^\ell d\Gamma_2^\ell \ , \ \forall v^\ell \in V(\Omega^\ell) \tag{3.20}$$

where $\varphi = (\varphi^1, \varphi^2) \in V$.
and let $j : \mathcal{H}_1^\ell \times V \longrightarrow \mathbb{R}$ be the functional

$$j(\sigma, v) = -\int_{\Gamma_3} \mu \sigma_\eta |[v_\tau]| d\Gamma_3 \tag{3.21}$$

where $|.|$ denotes the Euclidean norm. Let $\sigma \in \mathcal{H}_1^\ell$, the functional $j(\sigma, .)$ is continuous, convex and non-differentiable. Thus, $j(\sigma, .)$ is convex and lower semi-continuous on $V$. Finally, we denote in the sequel by $U_{ad}$ the set of *geometrically admissible displacement fields* defined by

$$U_{ad} = \left\{ v = (v^1, v^2) \in V \ \mid \ [v_\eta] \leq 0 \text{ on } \Gamma_3 \ \right\} \tag{3.22}$$

The set $U_{ad}$ is nonempty ($0 \in U_{ad}$), closed and convex.
For all $g \in \widehat{\mathcal{H}}_1$, let $\Sigma_{ad}(g)$ denote the set of *statically admissible stress fields* given by:

$$\Sigma_{ad}(g) = \Big\{ \tau \in \widehat{\mathcal{H}}_1 \mid \sum_{\ell=1}^{2} \langle \tau^\ell, \epsilon(v^\ell) \rangle_{\mathcal{H}^\ell} + j(g,v) \geq \langle g, v \rangle, \ \forall v \in U_{ad} \Big\} \tag{3.23}$$

also, for all $g \in \widehat{\mathcal{H}}_1$ with $g^1_\eta\big|_{\Gamma_3} \leq 0$, the set $\Sigma_{ad}(g)$ is nonempty $\big(g \in \Sigma_{ad}(g)\big)$, closed and convex.
Using (2.1)-(2.4) we have the following result.

**Lemma 3.2.** *If* $(u, \sigma)$ *are sufficiently regular functions satisfying* (3.5)*-*(3.9)*, then:*

$$u \in U_{ad}, \qquad \sigma \in \Sigma_{ad}(\sigma), \tag{3.24}$$

$$\sum_{\ell=1}^{2} \langle \sigma^\ell, \epsilon(v^\ell) - \epsilon(u^\ell) \rangle_{\mathcal{H}^\ell} + j(\sigma, v) - j(\sigma, u) \geq \langle \varphi, v - u \rangle_V \quad \forall v \in U_{ad}, \tag{3.25}$$

$$\sum_{\ell=1}^{2} \langle \tau^\ell - \sigma^\ell, \epsilon(u^\ell) \rangle_{\mathcal{H}^\ell} \geq 0 \quad \forall \tau \in \Sigma_{ad}(\sigma). \tag{3.26}$$

*Proof.* The regularity $u \in U_{ad}$ follows from (3.7) and (3.9). By applying Green formula in (3.6) and from (3.7),(3.8),(3.20), (3.21) we have (3.25). Choosing now $v = 2u \in U_{ad}$ and $v = 0 \in U_{ad}$ in (3.25), we find

$$\sum_{\ell=1}^{2} \langle \sigma^\ell, \epsilon(u^\ell) \rangle_{\mathcal{H}^\ell} + j(\sigma, u) = \langle \varphi, u \rangle_V . \tag{3.27}$$

Using (3.25) we deduce

$$\sum_{\ell=1}^{2} \langle \sigma^\ell, \epsilon(v^\ell) \rangle_{\mathcal{H}^\ell} + j(\sigma, v) \geq \langle \varphi, v \rangle_V \qquad \forall v \in U_{ad}. \tag{3.28}$$

The regularity $\sigma \in \Sigma_{ad}(\sigma)$ is now a consequence of (3.23) and (3.28). Moreover, from (3.23) and (3.27) we obtain

$$\sum_{\ell=1}^{2} \langle \tau^\ell - \sigma^\ell, \epsilon(u^\ell) \rangle_{\mathcal{H}^\ell} \geq \langle \varphi, u \rangle_V - \langle \varphi, u \rangle_V = 0 \qquad \forall \tau \in \Sigma_{ad}.$$

Therefore (3.26). □

Lemma 3.2 and (3.5) lead us to consider the following two variational problems.
**Problem** $\mathbb{P}_1$: For $\ell = 1, 2$, find the displacement fields $u^\ell : \Omega^\ell \longrightarrow \mathbb{R}^N$, such that

$$\begin{cases} u \in U_{ad}, \qquad F^1(\epsilon(u^1)).\eta^1 = F^2(\epsilon(u^2)).\eta^2 \text{ on } \Gamma_3, \\ a(u, v-u) + j(F(\epsilon(u)), v) - j(F(\epsilon(u)), u) \geq \langle \varphi, v - u \rangle_V , \quad \forall v \in U_{ad} \end{cases} \tag{3.29}$$

where

$$F(\epsilon(u)) = F^1(\epsilon(u^1)) \text{ or } F(\epsilon(u)) = F^2(\epsilon(u^2)).$$

**Problem** $\mathbb{P}_2$: For $\ell = 1, 2$, find the stress fields $\sigma^\ell : \Omega^\ell \longrightarrow \mathbb{S}_N$, such that

$$\sigma \in \Sigma_{ad}(\sigma), \qquad \sum_{\ell=1}^{2} \langle \tau^\ell - \sigma^\ell, \left(F^\ell\right)^{-1}(\sigma^\ell)\rangle_{\mathcal{H}^\ell} \geq 0\,, \quad \forall \tau \in \Sigma_{ad}(\sigma). \tag{3.30}$$

Details of such correspondences can be found in [13]. So, problem (3.29) can be rewritten as the following direct hybrid formulation:

**Problem** $\overline{\mathbb{P}}_1$: For $\ell = 1, 2$, find the displacement fields $u^\ell : \Omega^\ell \longrightarrow \mathbb{R}^N$, such that

$$u \in U_{ad}, \quad F^1(\epsilon(u^1))\eta^1 = F^2(\epsilon(u^2))\eta^2 \equiv \sigma\eta \quad \text{on } \Gamma_3, \tag{3.31}$$

$$a(u, v) = \langle \varphi, v\rangle + \langle \sigma_\eta, [v_\eta]\rangle_{-\frac{1}{2},\frac{1}{2},\Gamma_3} + \langle \sigma_\tau, [v_\tau]\rangle_{-\frac{1}{2},\frac{1}{2},\Gamma_3} \quad \forall v \in V, \tag{3.32}$$

$$\langle \sigma_\eta, [v_\eta] - [u_\eta]\rangle_{-\frac{1}{2},\frac{1}{2},\Gamma_3} \geq 0 \qquad \forall v \in U_{ad}, \tag{3.33}$$

$$\langle \sigma_\tau, [v_\tau] - [u_\tau]\rangle_{-\frac{1}{2},\frac{1}{2},\Gamma_3} - \langle \mu\sigma_\eta, |[v_\tau]| - |[u_\tau]|\rangle_{-\frac{1}{2},\frac{1}{2},\Gamma_3} \geq 0 \quad \forall v \in V. \tag{3.34}$$

For each body $\Omega^\ell$, we define the total potential energy functional $J^\ell$ by

$$J^\ell(v^\ell) = \frac{1}{2}a(v^\ell, v^\ell) - \langle \varphi^\ell, v^\ell\rangle_{V^\ell}\,, \quad \forall v^\ell \in V^\ell$$

and we set

$$J(v) = J^1(v^1) + J^2(v^2)\,, \quad \forall v \in V \tag{3.35}$$

the total potential energy of the two-body system. With the assumption $mes(\Gamma_1^\ell) > 0$, the functional $J$ is convex, G-differentiable and coercive on $V$. The following theorem (see e.g. [15], Theorem 3.8) allows us to replace the variational inequality (3.29) by a minimization problem.

**Theorem 3.3.** *Let $\theta \in \widehat{\mathcal{H}}_1$ and suppose $G : U_{ad} \to \mathbb{R}$ is of the form $G(v) = J(v) + j(\theta, v)$, where $J(.)$ and $j(\theta, .)$ are convex and lower semi-continuous and $J(.)$ is G-differentiable on $U_{ad}$. Then, if $u_\theta$ is a minimizer of $G$ on $U_{ad}$,*

$$\langle DJ(u_\theta), v - u_\theta\rangle + j(\theta, v) - j(\theta, u_\theta) \geq 0, \quad \forall v \in U_{ad}. \tag{3.36}$$

*Conversely, if (*3.36 *) holds for $u_\theta \in U_{ad}$, then $u_\theta$ is a minimizer of $G$.*

In (3.36 ), $DJ(u_\theta)$ is the gradient of $J$. Since $J$ is a quadratic functional, (3.36 ) is precisely

$$u_\theta \in U_{ad},\ a(u_\theta, v - u_\theta) + j(\theta, v) - j(\theta, u_\theta) \geq \langle \varphi, v - u_\theta\rangle_V\,, \quad \forall v \in U_{ad}. \tag{3.37}$$

With the assumption $mes(\Gamma_1^\ell) > 0$, the functional $J(.) + j(\theta, .)$ is strictly convex and coercive, then there exists a unique solution to (3.36).

With the above preparations, the unilateral contact problem with Coulomb friction can be formulated as the constrained minimization problem.

**Problem** $\widehat{\mathbb{P}}_1$: For $\ell = 1, 2$, find the displacement fields $u^\ell : \Omega^\ell \longrightarrow \mathbb{R}^N$, such that

$$\begin{cases} u \in U_{ad}, \quad F^1(\epsilon(u^1)).\eta^1 = F^2(\epsilon(u^2)).\eta^2 \text{ on } \Gamma_3, \\ J(u) + j(F(\epsilon(u)), u) \leq J(v) + j(F(\epsilon(u)), v) \qquad \forall v \in U_{ad}. \end{cases} \tag{3.38}$$

**Theorem 3.4.** *Assume the hypothesis* (3.16), (3.17). *Let* $u = (u^1, u^2) \in V$ *be a solution of the variational problem* $\mathbb{P}_1$ *and* $\sigma = (\sigma^1, \sigma^2)$ *is defined by* $\sigma^\ell = F^\ell\left(\epsilon(u^\ell)\right)$, $\ell = 1, 2$, *then* $(u, \sigma)$ *is a solution of the problem* $\mathbb{P}$.

*Proof.* For all $\Phi^\ell \in D^\ell \equiv (D(\Omega^\ell))^N$ be arbitrary, $v = u \pm \Phi \in U_{ad}$, where $\Phi = (\Phi^1, \Phi^2)$, and $\Phi^{3-\ell} = 0$, then using (3.29) and $\sigma^\ell = F^\ell\left(\epsilon(u^\ell)\right)$ we have:

$$\begin{aligned} 0 &\leq \int_{\Omega^\ell} \sigma^\ell.\epsilon(v^\ell - u^\ell)d\Omega^\ell - \int_{\Omega^\ell} f^\ell.(v^\ell - u^\ell)d\Omega^\ell - \int_{\Gamma_2^\ell} g^\ell(v^\ell - u^\ell)d\Gamma_2^\ell \\ &= \int_{\Gamma^\ell} \sigma^\ell.(v^\ell - u^\ell)d\Gamma^\ell - \int_{\Omega^\ell} (div\sigma^\ell + f^\ell).(v^\ell - u^\ell)d\Omega^\ell \\ &= \pm \int_{\Omega^\ell} (div\sigma^\ell + f^\ell).\Phi^\ell d\Omega^\ell \end{aligned}$$

which implies (3.6).
By applying *Green's* formula and using (3.29) (3.6), we have

$$\sum_{\ell=1}^{2} \langle \sigma^\ell \eta^\ell, (v^\ell - u^\ell)\eta^\ell \rangle_{H'_{\Gamma^\ell} \times H_{\Gamma^\ell}} + j(\sigma, v) - j(\sigma, u) \geq \sum_{\ell=1}^{2} \langle g^\ell, (v^\ell - u^\ell)\eta^\ell \rangle_{H'_{\Gamma_2^\ell} \times H_{\Gamma_2^\ell}}, \quad \forall v \in U_{ad}. \tag{3.39}$$

Taking $v = u \pm (\omega^1, \omega^2) \in U_{ad}$, with $\omega^\ell \in D(\Omega^\ell \cup \Gamma_2^\ell)^N$ and $\omega^{3-\ell} = 0$ in (3.39), it follows that

$$\langle \sigma^\ell \eta^\ell, \omega^\ell \eta^\ell \rangle_{H'_{\Gamma_2^\ell} \times H_{\Gamma_2^\ell}} = \langle g^\ell, \omega^\ell \eta^\ell \rangle_{H'_{\Gamma_2^\ell} \times H_{\Gamma_2^\ell}}$$

which implies (3.8).
Let $(\omega^1, \omega^2) \in H_1$ with $\omega_\eta^\ell = 0$, $\omega^\ell\big|_{\Gamma_1^\ell \cup \Gamma_2^\ell} = 0$ and $\omega_\tau^1\big|_{\Gamma_3} = -\omega_\tau^2\big|_{\Gamma_3}$.
Then $v = u \pm (\omega^1, \omega^2) \in U_{ad}$ and (3.39) gives:

$$\sum_{\ell=1}^{2} \int_{\Gamma_3} \sigma_\tau^\ell.\omega_\tau^\ell d\Gamma_3 = 0.$$

From where, it follows

$$\int_{\Gamma_3} \sigma_\tau^1.\omega_\tau^1 d\Gamma_3 = \int_{\Gamma_3} \sigma_\tau^2.\omega_\tau^1 d\Gamma_3.$$

This implies $\sigma_\tau^1\big|_{\Gamma_3} = \sigma_\tau^2\big|_{\Gamma_3}$ and from (3.29), we have (3.9.a).
Taking $v = u \pm (\omega^1, \omega^2) \in U_{ad}$, with $\omega^\ell \in D(\Omega^\ell \cup \Gamma_3)^N$, $\omega_\tau^\ell = 0$ on $\Gamma_3$ and $\omega^{3-\ell} = 0$ in (3.39), it follows that

$$\langle \sigma_\eta^\ell, \omega_\eta^\ell \rangle_{H'_{\Gamma_3} \times H_{\Gamma_3}} \geq 0.$$

Furthermore $\sigma_\eta^\ell \leq 0$ on $\Gamma_3$.
Now, by $u \in U_{ad}$, we have $[u_\eta] \leq 0$ on $\Gamma_3$.
Taking now $v \in U_{ad}$ such that $v_\tau = u_\tau$ and $v_\eta = 0$ in (3.39), we obtain:

$$\int_{\Gamma_3} \sigma_\eta [u_\eta] d\Gamma_3 \leq 0$$

and from $\sigma_\eta \leq 0$, $[u_\eta] \leq 0$ on $\Gamma_3$, we deduce $\sigma_\eta[u_\eta] = 0$ on $\Gamma_3$. Therefore, (3.9.b) holds.
Suppose that $v \in U_{ad}$, with $v_\eta = u_\eta$ on $\Gamma_3$, and using (3.8),(3.9.a.b) in (3.39), we obtain:

$$\int_{\Gamma_3} (\sigma_\tau[v_\tau] - \mu\sigma_\eta|[v_\tau]|)d\Gamma_3 - \int_{\Gamma_3} (\sigma_\tau[u_\tau] - \mu\sigma_\eta|[u_\tau]|)d\Gamma_3 \geq 0 \tag{3.40}$$

and choosing $v_\tau = 2u_\tau$ $\big($resp. $v_\tau = 0$ $\big)$ in (3.40), we deduce

$$\int_{\Gamma_3} (\sigma_\tau[u_\tau] - \mu\sigma_\eta|[u_\tau]|)d\Gamma_3 = 0. \tag{3.41}$$

Compining (3.40) and (3.41), we have

$$\int_{\Gamma_3} (\sigma_\tau[v_\tau] - \mu\sigma_\eta|[v_\tau]|)d\Gamma_3 \geq 0 \quad \forall v \in U_{ad} \tag{3.42}$$

and let $N = \big\{ x \in \Gamma_3/|\sigma_\tau| > -\mu\sigma_\eta \big\}$. From $v \in U_{ad}$ with $[v_\tau]|_{\Gamma_3 - N} = 0$ and $[v_\tau]|_N = -\sigma_\tau$ in (3.42), we deduce:

$$\int_N (-|\sigma_\tau|^2 - \mu\sigma_\eta|\sigma_\tau|)d\Gamma_3 \geq 0. \tag{3.43}$$

Since $|\sigma_\tau| > -\mu\sigma_\eta$ and $\sigma_\eta \leq 0$ on $N$, then $-|\sigma_\tau| - \mu\sigma_\eta < 0$ and $|\sigma_\tau| \neq 0$ on $N$, which implies

$$-|\sigma_\tau|^2 - \mu\sigma_\eta|\sigma_\tau| > 0 \text{ on } N. \tag{3.44}$$

Using (3.43) and (3.44), we obtain $mes(N) = 0$, we deduce

$$|\sigma_\tau| \leq -\mu\sigma_\eta \quad p.p \text{ on } \Gamma_3$$

and hence (3.9.c) holds.
Using now (3.9.c) and (3.41) we deduce

$$\sigma_\tau[u_\tau] - \mu\sigma_\eta\big|[u_\tau]\big| = 0 \quad p.p \text{ on } \Gamma_3. \tag{3.45}$$

Moreover, from (3.9.c) we obtain

$$0 = \sigma_\tau.[u_\tau] - \mu|[u_\tau]|\sigma_\eta \geq -|\sigma_\tau|.|[u_\tau]| - \mu|[u_\tau]|\sigma_\eta \geq -|[u_\tau]|(|\sigma_\tau| + \mu\sigma_\eta) \geq 0.$$

Therefore,

$$-|[u_\tau]|(|\sigma_\tau| + \mu\sigma_\eta) = 0. \tag{3.46}$$

For $|\sigma_\tau| < -\mu\sigma_\eta$: from (3.46), we deduce $[u_\tau] = 0$, hence (3.9.d) holds.
For $|\sigma_\tau| = -\mu\sigma_\eta$: from (3.45),we deduce

$$\sigma_\tau.[u_\tau] = \mu|[u_\tau]|\sigma_\eta = -|\sigma_\tau|.|[u_\tau]|.$$

So we deduce that there exists a constant $\lambda \geq 0$ such that $[u_\tau] = -\lambda\sigma_\tau$, hence (3.9.e) holds. □

**Theorem 3.5.** *Assume the hypothesis* (3.16)*,*(3.17)*. Let* $\sigma = (\sigma^1, \sigma^2)$ *be a solution of the variational problem* $\mathbb{P}_2$*, and* $u = (u^1, u^2) \in V$ *is given by* $\sigma^\ell = F^\ell\big(\epsilon(u^\ell)\big)$*,* $\ell = 1, 2$*, then* $u$ *is a solution of the variational problem* $\mathbb{P}_1$*.*

*Proof.* Firstly we prove $u \in U_{ad}$. Supposing that $u \notin U_{ad}$, and let $u_*$ the projection of $u$ on $U_{ad}$, we have the existence of $\alpha \in \mathbb{R}$ such that

$$\langle u_* - u, v\rangle_V > \alpha > \langle u_* - u, u\rangle_V \qquad \forall v \in U_{ad}.$$

We introduce the functional $\tau_*$ defined by: $\tau_* = (\epsilon(u_*^1 - u^1), \epsilon(u_*^2 - u^2)) \in \mathcal{H}$, and we use inner products defined by (3.14), we deduce:

$$\sum_{\ell=1}^{2}\langle \tau_*^\ell, \epsilon(v^\ell)\rangle_{\mathcal{H}^\ell} > \alpha > \sum_{\ell=1}^{2}\langle \tau_*^\ell, \epsilon(u^\ell)\rangle_{\mathcal{H}^\ell} \qquad \forall v \in U_{ad}. \tag{3.47}$$

Taking $v = 0 \in U_{ad}$ in (3.47), we obtain $\alpha < 0$, it is easy to verify that

$$\langle \tau_*^1, \epsilon(v^1)\rangle_{\mathcal{H}^1} + \langle \tau_*^2, \epsilon(v^2)\rangle_{\mathcal{H}^2} \geq 0 \qquad \forall v \in U_{ad}. \tag{3.48}$$

Really, we suppose the existence of $v_* = (v_*^1, v_*^2) \in U_{ad}$ where

$$\langle \tau_*^1, \epsilon(v_*^1)\rangle_{\mathcal{H}^1} + \langle \tau_*^2, \epsilon(v_*^2)\rangle_{\mathcal{H}^2} < 0. \tag{3.49}$$

As $\beta v_* \in U_{ad}$, $\forall \beta > 0$, if we replace $v = \beta v_*$ in (3.47) we obtain

$$\beta\left(\langle \tau_*^1, \epsilon(v_*^1)\rangle_{\mathcal{H}^1} + \langle \tau_*^2, \epsilon(v_*^2)\rangle_{\mathcal{H}^2}\right) > \alpha, \quad \forall \beta > 0.$$

And making $\beta \to +\infty$ with (3.49), we have $\alpha \leq -\infty$, this constitutes a contradiction with the fact that $\alpha$ is real. So we deduce (3.48). Now, using (3.30), (3.23) we deduce

$$\sum_{\ell=1}^{2}\langle \sigma^\ell, \epsilon(v^\ell)\rangle_{\mathcal{H}^\ell} + j(\sigma, v) \geq \langle \varphi, v\rangle_V \qquad \forall v \in U_{ad} \tag{3.50}$$

and, using (3.48) we obtain

$$\sum_{\ell=1}^{2}\langle \tau_*^\ell + \sigma^\ell, \epsilon(v^\ell)\rangle_{\mathcal{H}^\ell} + j(\sigma, v) \geq \langle \varphi, v\rangle_V \qquad \forall v \in U_{ad}$$

saying

$$\tau_* + \sigma \in \Sigma_{ad}(\sigma). \tag{3.51}$$

Choosing $\tau = \tau_* + \sigma \in \Sigma_{ad}(\sigma)$ in (3.30), and $\sigma^\ell = F^\ell(\epsilon(u^\ell))$ we obtain

$$\sum_{\ell=1}^{2}\langle \tau_*^\ell, \epsilon(u^\ell)\rangle_{\mathcal{H}^\ell} \geq 0. \tag{3.52}$$

Using now (3.47) and $\alpha < 0$, we find

$$\sum_{\ell=1}^{2}\langle \tau_*^\ell, \epsilon(u^\ell)\rangle_{\mathcal{H}^\ell} < 0. \tag{3.53}$$

The relations (3.52) and (3.53) constitute a contradiction, so we deduce that $u \in U_{ad}$. It remains to prove the inequality given in (3.29).
Using Riesz's representation theorem we define the nonlinear operator $R : V \to V$ by

$$\langle Rv, w\rangle_V = \sum_{\ell=1}^{2}\langle F^\ell(\epsilon(v^\ell)), \epsilon(w^\ell)\rangle_{\mathcal{H}^\ell}.$$

Then hypotheses (3.16) on $F^\ell$ imply that $R$ is strictly monotone, coercive and lipschitzian operator, on the other hand the functional $j(\sigma,.)$ is proper, convex and lower continuous on V. Then results from the theory of elliptic variational inequalities [4] of the second kind, we have the existence of $bar\tau = (\bar{\tau}^1, \bar{\tau}^2) \in \mathcal{H}$ such that

$$\sum_{\ell=1}^{2} \langle \bar{\tau}^\ell, \epsilon(v^\ell) - \epsilon(u^\ell) \rangle_{\mathcal{H}^\ell} + j(\sigma, v) - j(\sigma, u) \geq \langle \varphi, v - u \rangle_V, \quad \forall v \in V. \tag{3.54}$$

Taking $v = 2u$ and $v = 0$ in (3.54), then

$$\sum_{\ell=1}^{2} \langle \bar{\tau}^\ell, \epsilon(u^\ell) \rangle_{\mathcal{H}^\ell} + j(\sigma, u) = \langle \varphi, u \rangle_V. \tag{3.55}$$

Subtracting (3.55) from (3.54), this means that $\bar{\tau} \in \Sigma_{ad}(\sigma)$.
Therefore, from (3.30), (3.55) and $\sigma^\ell = F^\ell(\epsilon(u^\ell))$, we derive

$$\langle \varphi, u \rangle_V \geq \sum_{\ell=1}^{2} \langle \sigma^\ell, \epsilon(u^\ell) \rangle_{\mathcal{H}^\ell} + j(\sigma, u).$$

The converse inequality follows from (3.23) since $\sigma \in \Sigma_{ad}(\sigma)$ and $u \in U_{ad}$.
Therefore, we conclude that

$$\langle \varphi, u \rangle_V = \sum_{\ell=1}^{2} \langle \sigma^\ell, \epsilon(u^\ell) \rangle_{\mathcal{H}^\ell} + j(\sigma, u). \tag{3.56}$$

Using again (3.23), we have

$$\sum_{\ell=1}^{2} \langle \sigma^\ell, \epsilon(v^\ell) - \epsilon(u^\ell) \rangle_{\mathcal{H}^\ell} + j(\sigma, v) - j(\sigma, u) \geq \langle \varphi, v - u \rangle_V, \ \forall v \in U_{ad} \tag{3.57}$$

and $\sigma^\ell = F^\ell(\epsilon(u^\ell))$, $\sigma = (\sigma^1, \sigma^2) \in \widehat{\mathcal{H}}_1$ it results that $u$ is a solution of the problem $\mathbb{P}_1$. □

Theorem 3.4 and Theorem 3.5, allow to deduce the following results

**Corollary 3.6.** *Assume the hypothesis (3.16),(3.17). Let $\sigma = (\sigma^1, \sigma^2)$ be a solution of the variational problem $\mathbb{P}_2$, and $u = (u^1, u^2) \in V$ is given by $\sigma^\ell = F^\ell\left(\epsilon(u^\ell)\right)$, $\ell = 1, 2$. then $(u, \sigma)$ is a solution of the problem $\mathbb{P}$.*

Also Theorem 3.4 and Lemma 3.2, allow to deduce the following results

**Corollary 3.7.** *Assume the hypothesis (3.16),(3.17). Let $u = (u^1, u^2) \in V$ is a solution of the problem $\mathbb{P}_1$, and setting $\sigma^\ell = F^\ell\left(\epsilon(u^\ell)\right)$, $\ell = 1, 2$ we have $\sigma = (\sigma^1, \sigma^2)$ a solution of the problem $\mathbb{P}_2$.*

**Theorem 3.8.** *Under the hypotheses (3.16)-(3.17). Then there exists $C_o > 0$ which depends only on $\Omega^\ell$, $\Gamma^\ell$ and $F^\ell$, $\ell = 1, 2$ such that if $\|\mu\|_{L^\infty(\Gamma_3)} \leq C_o$ then there exists a unique solution $(u, \sigma)$ of problem $\mathbb{P}$. Moreover, the solution satisfies*

$$u \in V, \quad \sigma \in \mathcal{H}_1.$$

**Proposition 3.9.** *Let $\theta \in \widehat{\mathcal{H}}_1$ and let $(u^1_\theta, u^2_\theta)$ be the solution of (3.37), then:*

$$\big(F^1(\epsilon(u^1_\theta)), F^2(\epsilon(u^2_\theta))\big) \in \widehat{\mathcal{H}}_1. \tag{3.58}$$

*Proof.* Let $\omega = (\omega_1, \omega_2)$ where $\omega^\ell \in D(\Omega^\ell \cup \Gamma_3)^N$ and $[\omega\eta] = 0$ on $\Gamma_3$. Then $v = u \pm \omega \in U_{ad}$ in (3.37) gives:

$$\sum_{\ell=1}^{2} \langle F^\ell(\epsilon(u^\ell_\theta)), \epsilon(\omega^\ell)\rangle_{\mathcal{H}^\ell} = \langle \varphi, \omega\rangle$$

and using (3.20), with $\omega^\ell \eta^\ell = -\omega^{3-\ell}\eta^{3-\ell}$ on $\Gamma_3$, we have

$$\int_{\Gamma_3} \{F^1(\epsilon(u^1_\theta))\eta^1 - F^2(\epsilon(u^2_\theta))\eta^2\}.\omega^1\eta^1 d\Gamma_3 = 0.$$

Therefore, we conclude that $F^1(\epsilon(u^1_\theta))\eta^1 = F^2(\epsilon(u^2_\theta))\eta^2$ on $\Gamma_3$
Then (3.58). □

Let us consider now the operator $A : \widehat{\mathcal{H}}_1 \longrightarrow \widehat{\mathcal{H}}_1$ defined by

$$A(\theta) = (F^1(\epsilon(u^1_\theta)), F^2(\epsilon(u^2_\theta)). \tag{3.59}$$

We have the following result.

**Proposition 3.10.** *There exists $C_0 > 0$, such that, $\|\mu\|_{L^\infty(\Gamma_3)} \leq C_0$, The operator $A$ has a unique fixed point $\theta^* \in \widehat{\mathcal{H}}_1$.*

*Proof.* Let $\theta_i \in \widehat{\mathcal{H}}_1$, for $i = 1, 2$, and let $u_i$ the solutions of (3.37), we have

$$\begin{cases} a(u_1, u_2 - u_1) + j(\theta_1, u_2) - j(\theta_1, u_1) \geq \langle \varphi, u_2 - u_1\rangle, \\ \\ a(u_2, u_1 - u_2) + j(\theta_2, u_1) - j(\theta_2, u_2) \geq \langle \varphi, u_1 - u_2\rangle. \end{cases}$$

Thus, using (3.19),we deduce that

$$\begin{cases} \sum_{\ell=1}^{2} \int_{\Omega^\ell} (F^\ell(\epsilon(u^\ell_1)) - F^\ell(\epsilon(u^\ell_2))(\epsilon(u^\ell_1) - \epsilon(u^\ell_2)) d\Omega^\ell \leq \\ \\ \qquad j(\theta_1, u_2) - j(\theta_1, u_1) + j(\theta_2, u_1) - j(\theta_2, u_2). \end{cases} \tag{3.60}$$

From the *Korn's* inequality and (3.16), yields

$$\sum_{\ell=1}^{2} \langle F^\ell(\epsilon(u^\ell_1) - F^\ell(\epsilon(u^\ell_2)) \,,\, \epsilon(u^\ell_1) - \epsilon(u^\ell_2)\rangle \;\geq C_1 \|u_1 - u_2\|^2_V. \tag{3.61}$$

Using (3.21), we obtain

$$j(\theta_1, u_2) - j(\theta_1, u_1) + j(\theta_2, u_1) - j(\theta_2, u_2) = -\int_{\Gamma_3} \mu(\theta_{1\eta} - \theta_{2\eta})(|[u_{1\tau}]| - |[u_{2\tau}]|) d\Gamma_3.$$

So that

$$j(\theta_1, u_2) - j(\theta_1, u_1) + j(\theta_2, u_1) - j(\theta_2, u_2) \leq C_2 \|\mu\|_{L^\infty(\Gamma_3)} \|\theta_1 - \theta_2\|_{\mathcal{H}_1} . \|u_1 - u_2\|_V$$

and using (3.60),(3.61) and using the trace theorem, we have

$$\|u_1 - u_2\|_V \leq C_3 \|\mu\|_{L^\infty(\Gamma_3)} \|\theta_1 - \theta_2\|_{\mathcal{H}_1}. \tag{3.62}$$

Putting (3.16) and (3.60), it yields:

$$\|A\theta_1 - A\theta_2\|^2_{\mathcal{H}_1} \le C_4 \sum_{\ell=1}^{2} \|\epsilon(u_1^\ell) - \epsilon(u_2^\ell)\|^2_{\mathcal{H}_1^\ell}. \tag{3.63}$$

Moreover, from (3.62) and (3.63), we obtain:

$$\|A\theta_1 - A\theta_2\|_{\mathcal{H}_1} \le C_5 \|\mu\|_{L(\Gamma_3)^\infty} \|\theta_1 - \theta_2\|_{\mathcal{H}_1}.$$

We conclude that the operator $A$ is a contradiction if $\|\mu\|_{L(\Gamma_3)^\infty} < \frac{1}{C_5}$. By the Banach fixed point theorem, we obtain that this operator has a unique fixed point $\theta^* \in \widehat{\mathcal{H}}_1$. □

**Proposition 3.11.** *For each $\theta \in \widehat{\mathcal{H}}_1$, there exists a unique $\sigma_\theta \in \widehat{\mathcal{H}}_1$, such that*

$$\sigma_\theta \in \Sigma_{ad}(\theta), \qquad \sum_{\ell=1}^{2} \langle (F^\ell)^{-1}(\sigma_\theta), \tau^\ell - \sigma_\theta^\ell \rangle_{\mathcal{H}^\ell} \ge 0 \quad \forall \tau \in \Sigma_{ad}(\theta). \tag{3.64}$$

*Proof.* Let $\sigma \in \widehat{\mathcal{H}}_1$, it is easy to check that the application

$$\tau \longmapsto \sum_{\ell=1}^{2} \langle (F^\ell)^{-1}(\sigma^\ell), \tau^\ell \rangle_{\mathcal{H}^\ell}$$

is a continuous linear form on $\widehat{\mathcal{H}}_1$ ( for $\sigma$ fixe ). Moreover, using *Riesz's* representation theorem we may define the operator $E : \widehat{\mathcal{H}}_1 \longrightarrow \widehat{\mathcal{H}}_1$ by the relation

$$\langle E\sigma, \tau \rangle_{\mathcal{H}_1} = \sum_{\ell=1}^{2} \langle (F^\ell)^{-1}(\sigma^\ell), \tau^\ell \rangle_{\mathcal{H}^\ell} \qquad \forall \sigma, \tau \in \widehat{\mathcal{H}}_1. \tag{3.65}$$

Keeping in mind (3.16) and *Korn's* inequality, we deduce that the operator $E$ is strongly monotone and *Lipschitz* contiunous on $E$. Also , $\Sigma_{ad}(\theta)$ is a closed, convex and nonempty subset of $\widehat{\mathcal{H}}_1$.
According to the *Lions, Stampacchia* theorem, we obtain the existence and uniqueness of the element $\sigma_\theta \in \widehat{\mathcal{H}}_1$ such that

$$\sigma_\theta \in \Sigma_{ad}(\theta), \qquad \langle E\sigma_\theta, \tau - \sigma_\theta \rangle_V \ge 0 \qquad \forall \tau \in \Sigma_{ad}.$$

Then

$$\sigma_\theta \in \Sigma_{ad}(\theta), \qquad \sum_{\ell=1}^{2} \langle (F^\ell)^{-1}(\sigma_\theta), \tau^\ell - \sigma_\theta^\ell \rangle_{\mathcal{H}^\ell} \ge 0 \quad \forall \tau \in \Sigma_{ad}(\theta).$$

□

Let us consider now the operator $B : \widehat{\mathcal{H}}_1 \longrightarrow \widehat{\mathcal{H}}_1$ defined by

$$B\theta = \sigma_\theta : \quad \forall \theta \in \widehat{\mathcal{H}}_1. \tag{3.66}$$

## 4. Proof of Theorem 3.8

*Proof. Existence.* Let $u^* = (u^{*1}, u^{*2}) \in V$ the solutions of (3.37) with $\theta = \theta^*$. Taking $v = 0 \in V$ and $v = 2u^* \in V$ in (3.37) we obtain

$$a(u^*, u^*) + j(\theta^*, u^*) = \langle \varphi, u^* \rangle \tag{4.1}$$

and from (3.37), (4.1), we have

$$a(u^*, v^*) + j(\theta^*, v^*) \geq \langle \varphi, v^* \rangle \quad \forall v \in U_{ad}. \tag{4.2}$$

From (3.23), (4.2) and $\theta^* = A(\theta^*)$ , it follows that

$$\theta^* \in \Sigma_{ad}(\theta^*). \tag{4.3}$$

Taking now $v = u \pm \phi \in V$ with $\phi = (\phi^1, \phi^2)$ and $\phi^\ell \in (D(\Omega^\ell))^N$, $\phi^{3-\ell} = 0$ in (3.37), it follows that

$$\langle F^\ell(\epsilon(u^\ell_\theta)), \epsilon(\phi^\ell) \rangle_{\mathcal{H}^\ell} = \langle \varphi^\ell, \phi^\ell \rangle. \tag{4.4}$$

Moreover, from (3.20), (4.4) and applying *Green's* formula, we have

$$-div(F^\ell(\epsilon(u^\ell_\theta))) = f^\ell \quad \text{ in } \Omega^\ell. \tag{4.5}$$

Using (4.1) and (3.20), we deduce that

$$\sum_{\ell=1}^{2} \int_{\Omega^\ell} div(F^\ell(\epsilon(u^{*\ell}))).u^{*\ell} d\Omega^\ell + \sum_{\ell=1}^{2} \int_{\Gamma^\ell} F^\ell(\epsilon(u^{*\ell}))\eta^\ell.u^{*\ell}\eta^\ell d\Gamma^\ell + j(\theta^*, u^*)$$

$$= \sum_{\ell=1}^{2} \int_{\Omega^\ell} f^\ell.u^{*\ell} d\Omega^\ell + \sum_{\ell=1}^{2} \int_{\Gamma_2^\ell} g^\ell.u^{*\ell}\eta^\ell d\Gamma_2^\ell.$$

Using now (4.5) and $u^*|_{\Gamma_1^\ell} \equiv 0$, we have

$$\int_{\Gamma_3} \theta^*\eta.[u^*\eta] d\Gamma_3 + j(\theta^*, u^*) = \sum_{\ell=1}^{2} \int_{\Gamma_2^\ell} g^\ell.u^{*\ell}\eta^\ell d\Gamma_2^\ell - \sum_{\ell=1}^{2} \int_{\Gamma_2^\ell} F^\ell(\epsilon(u^{*\ell}))\eta^\ell.u^{*\ell}\eta^\ell d\Gamma^\ell. \tag{4.6}$$

Taking $v = u \pm \phi \in V$ with $\phi = (\phi^1, \phi^2) \in V$, $\phi^{3-\ell} = 0$ and $\phi^\ell\eta^\ell = 0$ on $\Gamma_1^\ell \cup \Gamma_3$ in (3.37), it follows that

$$\int_{\Omega^\ell} F^\ell(\epsilon(u^{*\ell})).\epsilon(\phi^\ell) d\Omega^\ell = \int_{\Omega^\ell} f^\ell.\phi^\ell d\Omega^\ell + \int_{\Gamma_2^\ell} g^\ell.\phi^\ell\eta^\ell d\Gamma_2^\ell. \tag{4.7}$$

By applying *Green's* formula in (4.7) and using (4.5), we obtain

$$F^\ell(\epsilon(u^{*\ell}))\eta^\ell = g^\ell \ \texttt{ on } \ \Gamma_2^\ell. \tag{4.8}$$

Combining (4.6) and (4.8), it follows that

$$\int_{\Gamma_3} \theta^*\eta.[u^*\eta] d\Gamma_3 = -j(\theta^*, u^*) \tag{4.9}$$

and, for any $\tau \in \Sigma_{ad}(\theta^*)$

$$\sum_{\ell=1}^{2} \int_{\Omega^\ell} \tau^\ell.\epsilon(u^{*\ell}) d\Omega^\ell \geq \langle \varphi, u^* \rangle - j(\theta^*, u^*). \tag{4.10}$$

Using (4.5) and (4.8) with $F^1(\epsilon(u^{*1}))\eta^1 = F^2(\epsilon(u^{*2}))\eta^2$ on $\Gamma_3$ , we deduce that

$$\sum_{\ell=1}^{2} \int_{\Omega^\ell} F^\ell(\epsilon(u^{*\ell})).\epsilon(u^{*\ell}) d\Omega^\ell = \langle \varphi, u^* \rangle + \int_{\Gamma_3} \theta^* \eta.[u^* \eta] d\Gamma_3. \tag{4.11}$$

Moreover, from (4.10), (4.11) and (4.9), we deduce the inequality in (3.30) witch proves that $\theta^*$ is a solution of problem $\mathbb{P}_2$.
It follows from Corollary3.6 that $(u^*, \theta^*)$ is a solution to problem$\mathbb{P}$.
*Uniqueness.* To prove the uniqueness of the solution let $(u^*, \theta^*)$ be the solution of problem$\mathbb{P}$ obtained above and let $(u, \sigma)$ be another solution such that $u \in V$ and $\sigma \in \widehat{\mathcal{H}}_1$.
for all $\theta \in \widehat{\mathcal{H}}_1$. Therefore, choosing $\tilde{\sigma}_\theta = A\theta$ and using (3.37) and (3.59), we get

$$\sum_{\ell=1}^{2} \langle \tilde{\sigma}_\theta^\ell, \epsilon(v^\ell) - \epsilon(u_\theta^\ell) \rangle_{\mathcal{H}^\ell} + j(\theta, v) - j(\theta, u_\theta) \geq \langle \varphi, v - u_\theta \rangle_V, \ \forall v \in V. \tag{4.12}$$

Taking $v = 2u_\theta$ and $v = 0$ in (4.12), we obtain

$$\sum_{\ell=1}^{2} \langle \tilde{\sigma}_\theta^\ell, \epsilon(u_\theta^\ell) \rangle_{\mathcal{H}^\ell} + j(\theta, u_\theta) = \langle \varphi, u_\theta \rangle_V. \tag{4.13}$$

Using now (4.12) and (4.13), we have

$$\tilde{\sigma}_\theta \in \Sigma_{ad}(\theta) \tag{4.14}$$

and from (3.59), (4.13) and (3.23) it follows that

$$\sum_{\ell=1}^{2} \langle (F^\ell)^{-1}(\tilde{\sigma}_\theta^\ell), \tau^\ell - \tilde{\sigma}_\theta^\ell \rangle_{\mathcal{H}^\ell} \geq 0 \qquad \forall \tau \in \Sigma_{ad}(\theta). \tag{4.15}$$

Moreover, from (4.14) and (4.15), it results that $\tilde{\sigma}_\theta$ is a solution of problem (3.64). and by the uniqueness of the solution, we deduce $\tilde{\sigma}_\theta = \sigma_\theta$, then we have

$$A\theta = B\theta : \quad \forall \theta \in \widehat{\mathcal{H}}_1. \tag{4.16}$$

Using now Lemma 3.2, with

$$\theta^* = (F^1(\epsilon(u^{*^1})), F^2(\epsilon(u^{*^2})))$$

and

$$\sigma = (F^1(\epsilon(u^1)), F^2(\epsilon(u^2))),$$

such that

$$\begin{cases} \theta^* \in \Sigma_{ad}(\theta^*), \quad \sum_{\ell=1}^{2} \langle \tau^\ell - \theta^{*^\ell}, (F^\ell)^{-1}(\theta^{*^\ell}) \rangle_{\mathcal{H}^\ell} \geq 0, \ \forall \tau \in \Sigma_{ad}(\theta^*), \\ \sigma \in \Sigma_{ad}(\sigma), \quad \sum_{\ell=1}^{2} \langle \tau^\ell - \sigma^\ell, (F^\ell)^{-1}(\sigma^\ell) \rangle_{\mathcal{H}^\ell} \geq 0, \ \forall \tau \in \Sigma_{ad}(\sigma) \end{cases} \tag{4.17}$$

and from (3.64) and (3.66), we obtain

$$B\theta^* = \theta^*, \quad B\sigma = \sigma. \tag{4.18}$$

Moreover, from (4.18) and (4.16) and proposition3.10, it follows that

$$\theta^* = \sigma. \tag{4.19}$$

Hence

$$F^\ell(\epsilon(u^{*^\ell})) = F^\ell(\epsilon(u^\ell)) \quad \ell = 1, 2. \tag{4.20}$$

Therefore, by (3.16) and (4.20), we have

$$u^* = u.$$

The proof of Theorem 3.8 is complete. □

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