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On characterization of dual focal curves of spacelike biharmonic curves with timelike binormal in the dual Lorentzian $\mathbb{D}_1^3$

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Abstract. In this paper, we study dual focal curves of spacelike biharmonic curves with timelike binormal in the dual Lorentzian 3-space $\mathbb{D}_1^3$. We characterize dual focal curves in terms of their dual focal curvatures.

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1. Introduction

The application of dual numbers to the lines of the 3-space is carried out by the principle of transference which has been formulated by Study and Kotelnikov. It allows a complete generalization of the mathematical expression for the spherical point geometry to the spatial line geometry by means of dual-number extension, i.e. replacing all ordinary quantities by the corresponding dual-number quantities.

Harmonic maps $f : (M, g) \longrightarrow (N, h)$ between Riemannian manifolds are the critical points of the energy

$$E(f) = \frac{1}{2}\int_M |df|^2 v_g, \tag{1.1}$$

and they are therefore the solutions of the corresponding Euler–Lagrange equation. This equation is given by the vanishing of the tension field

$$\tau(f) = \mathrm{trace}\nabla df. \tag{1.2}$$

Bienergy of a map f by

$$E_2(f) = \frac{1}{2}\int_M |\tau(f)|^2 v_g, \tag{1.3}$$

and say that is biharmonic if it is a critical point of the bienergy.

Jiang derived the first and the second variation formula for the bienergy in [13], showing that the Euler–Lagrange equation associated to E_2 is

$$\begin{aligned}\tau_2(f) &= -\mathcal{J}^f(\tau(f)) = -\Delta\tau(f) - \operatorname{trace}R^N(df, \tau(f))\,df \\ &= 0,\end{aligned} \tag{1.4}$$

where $\mathcal{J}^f$ is the Jacobi operator of f. The equation $\tau_2(f) = 0$ is called the biharmonic equation. Since $\mathcal{J}^f$ is linear, any harmonic map is biharmonic. Therefore, we are interested in proper biharmonic maps, that is non-harmonic biharmonic maps.

In this paper, we study dual focal curves in the dual 3-space $\mathbb{D}^3$. We characterize dual focal curves in terms of their focal curvatures.

2. Preliminaries

If a and a^* are real numbers, the combination

$$A = a + \varepsilon a^* \tag{2.1}$$

is called a dual number. Here ε is the dual unit. Dual numbers are considered as polynomials in ε, subject to the rules $\varepsilon \neq 0$, $\varepsilon^2 = 0$, $\varepsilon.1 = 1.\varepsilon = \varepsilon$. W. K. Clifford defined the dual numbers, the set of dual numbers forms a commutative ring having the numbers $\varepsilon a^*(a^*$ real) as divisors of zero, not a field. No number εa^* has an inverse in the algebra. But the other laws of the algebra of dual numbers are the same as the laws of algebra of complex numbers. For example, two dual numbers A and $B = b+\varepsilon b^*$ are added componentwise.

$$A + B = (a + b) + \varepsilon(a^* + b^*), \tag{2.2}$$

and they are multiplied by

$$AB = ab + \varepsilon(a^*b + ab^*). \tag{2.3}$$

For the equality of A and B we have

$$A = B \Leftrightarrow a = b, \quad \text{and} \quad a^* = b^*. \tag{2.4}$$

An oriented line in $\mathbb{E}^3$ may be given by two points x and y on it. If μ is any non-zero constant [2], the parametric equation of the line is:

$$y = x + \mu a, \tag{2.5}$$

a is a unit vector along the line. The moment of a with respect to the origin is

$$a^* = x \times a = y \times a. \tag{2.6}$$

This means that a and a^* are not independent of the choice of the points on the line and these vectors are not independent of one another; satisfy the following equations:

$$< a, a >= 1, \quad < a, a^* >= 0. \tag{2.7}$$

The six components a_i, a_i^* $(i = 1, 2, 3)$ of the vectors a and a^* are known to be Plücker homogeneous line coordinates. These two vectors a and a^* determine an oriented line in $\mathbb{E}^3$. A point z lies on this line if and only if

$$z \times a = a^*. \tag{2.8}$$

The set of oriented lines in $\mathbb{E}^3$ is in one-to-one correspondence with pairs of vectors subject to the conditions (2.7), and so we may expect to represent it as a certain fourdimensional set in $\Re^6$ of sixtuples of real numbers; we may take the space $\mathbb{D}^3$ of triples of dual numbers with coordinates:

$$X_i = x_i + \varepsilon x_i^* \quad (i = 1, 2, 3). \tag{2.9}$$

Each line in $\mathbb{E}^3$ may be represented by a dual unit vector

$$A = a + \varepsilon a^*; \tag{2.10}$$

in $\mathbb{D}^3$. It is clear that this dual unit vector has the property

$$< A, A >=< a, a > +2\varepsilon < a, a^* >= 1. \tag{2.11}$$

The Lorentzian inner product of dual vectors $\hat{\varphi}$ and $\hat{\psi}$ in $\mathbb{D}^3$ is defined by

$$\left\langle \hat{\Omega}, \hat{\psi} \right\rangle = \langle \Omega, \psi \rangle + \varepsilon \left(\langle \Omega, \psi^* \rangle + \langle \Omega^*, \psi \rangle \right), \tag{2.12}$$

with the Lorentzian inner product φ and ψ

$$\langle \Omega, \psi \rangle = -\Omega_1 \psi_1 + \Omega_2 \psi_2 + \Omega_3 \psi_3, \tag{2.13}$$

where $\Omega = (\Omega_1, \Omega_2, \Omega_3)$ and $\psi = (\psi_1, \psi_2, \psi_3)$.

Theorem 2.1. (E. Study) The oriented lines in $\mathbb{E}^3$ are in one-to-one correspondence with points of the dual unit sphere $< X, X >= 1$ in $\mathbb{D}^3$.

3. Dual spacelike biharmonic curves with timelike binormal in the dual Lorentzian space $\mathbb{D}_1^3$

Let $\hat{\gamma} = \gamma + \varepsilon \gamma^* : I \subset R \to \mathbb{D}_1^3$ be a C^4 dual spacelike curve with arc length parameter s. Then the unit tangent vector $\hat{\gamma}' = \hat{\mathbf{T}}$ is defined, and the principal normal is $\hat{\mathbf{N}} = \frac{1}{\hat{\kappa}} \hat{\mathbf{T}}'$, where $\hat{\kappa}$ is never a pure-dual. The function $\hat{\kappa} = \left\| \hat{\mathbf{T}}' \right\| = \kappa + \varepsilon \kappa^*$ is called the dual curvature of the dual curve $\hat{\gamma}$. Then the binormal of $\hat{\gamma}$ is given by the dual vector $\hat{\mathbf{b}} = \hat{\mathbf{T}} \times \hat{\mathbf{N}}$. Hence, the triple $\left\{ \hat{\mathbf{T}}, \hat{\mathbf{N}}, \hat{\mathbf{B}} \right\}$ is called the Frenet frame fields and the Frenet formulas may be expressed

$$\begin{bmatrix} \hat{\mathbf{T}}' \\ \hat{\mathbf{N}}' \\ \hat{\mathbf{B}}' \end{bmatrix} = \begin{bmatrix} 0 & \hat{\kappa} & 0 \\ -\hat{\kappa} & 0 & \hat{\tau} \\ 0 & \hat{\tau} & 0 \end{bmatrix} \begin{bmatrix} \hat{\mathbf{T}} \\ \hat{\mathbf{N}} \\ \hat{\mathbf{B}} \end{bmatrix}, \tag{3.1}$$

where $\hat{\tau} = \tau + \varepsilon \tau^*$ is the dual torsion of the spacelike dual curve $\hat{\gamma}$. Here, we suppose that the dual torsion $\hat{\tau}$ is never pure-dual.

Theorem 3.1. *(see [5]) Let $\hat{\gamma} : I \longrightarrow \mathbb{D}_1^3$ be a dual spacelike biharmonic curves with timelike binormal parametrized by arc length. $\hat{\gamma}$ is a dual timelike biharmonic curve if and only if*

$$\begin{gathered} \kappa = \text{constant and } \kappa^* = \text{constant}, \\ \tau = \text{constant and } \tau^* = \text{constant}, \\ \kappa^2 - \tau^2 + \varepsilon \left(2\kappa\kappa^* - 2\tau\tau^* \right) = 0. \end{gathered} \tag{3.2}$$

Corollary 3.2. *(see [5]) Let $\hat{\gamma} : I \longrightarrow \mathbb{D}_1^3$ be a dual spacelike biharmonic curves with timelike binormal parametrized by arc length. $\hat{\gamma}$ is a dual spacelike elastic biharmonic curves with timelike binormal if and only if*

$$\kappa^2 = \tau^2, \tag{3.3}$$

$$\kappa\kappa^* = \tau\tau^*. \tag{3.4}$$

4. Dual focal curves of dual spacelike biharmonic curves with timelike binormal in the dual Lorentzian space $\mathbb{D}_1^3$

Denoting the dual focal curve by $\hat{\wp}$ we can write

$$\hat{\wp}(s) = (\hat{\gamma} + \hat{\mathfrak{m}}_1\hat{\mathbf{N}} + \hat{\mathfrak{m}}_2\hat{\mathbf{B}})(s), \tag{4.1}$$

where the coefficients $\hat{\mathfrak{m}}_1$, $\hat{\mathfrak{m}}_2$ are smooth functions of the parameter of the curve $\hat{\gamma}$, called the first and second dual focal curvatures of $\hat{\gamma}$, respectively.

The formula (4.1) is separated into the real and dual part, we have

$$\begin{aligned} \wp(s) = &\quad (\gamma + \mathfrak{m}_1\mathbf{N} + \mathfrak{m}_2\mathbf{B})(s), \\ \wp^*(s) = &\quad \left(\gamma^* + \mathfrak{m}_1\mathbf{N}^* + \mathfrak{m}_1^*\mathbf{N} + \mathfrak{m}_2\mathbf{B}^* + \mathfrak{m}_2^*\mathbf{B}\right)(s). \end{aligned} \tag{4.2}$$

Theorem 4.1. *Let $\hat{\gamma} : I \longrightarrow \mathbb{D}_1^3$ be a unit speed dual spacelike curve and $\hat{\wp}$ its dual focal curve on $\mathbb{D}_1^3$. Then,*

$$\wp = \gamma + \frac{1}{\kappa}\mathbf{N}, \tag{4.3}$$

$$\begin{aligned} \wp^* = \gamma^* + \frac{1}{\kappa}\mathbf{N}^* - \frac{\kappa^*}{\kappa^2}\mathbf{N} + \frac{\kappa'}{\kappa^2\tau}\mathbf{B}^* \\ + \left(\frac{(\kappa^*)'\kappa^2 - 2\kappa\kappa^*\kappa'}{\kappa^4\tau} - \frac{\tau^*\kappa'}{\kappa^2\tau^2}\right)\mathbf{B}. \end{aligned} \tag{4.4}$$

Proof. Assume that $\hat{\gamma}$ is a unit speed dual spacelike curve and $\hat{\wp}$ its dual focal curve on $\mathbb{D}_1^3$.

So, by differentiating of the formula (4.1), we get

$$\hat{\wp}(s)' = (1 - \hat{\kappa}\hat{\mathfrak{m}}_1)\hat{\mathbf{T}} + (\hat{\mathfrak{m}}_1' + \hat{\tau}\hat{\mathfrak{m}}_2)\hat{\mathbf{N}} + (\hat{\tau}\hat{\mathfrak{m}}_1 + \hat{\mathfrak{m}}_2')\hat{\mathbf{B}}. \tag{4.5}$$

Using above equation, the first 2 components vanish, we have using above equation,

$$\begin{aligned} \kappa\mathfrak{m}_1 &= 1, \\ \kappa\mathfrak{m}_1^* + \kappa_1^*\mathfrak{m} &= 0, \\ \mathfrak{m}_1' + \tau\mathfrak{m}_2 &= 0, \\ (\mathfrak{m}_1^*)' + \tau\mathfrak{m}_2^* + \tau^*\mathfrak{m}_2 &= 0. \end{aligned}$$

Considering equations above system, we have

$$\begin{aligned}\mathfrak{m}_1 &= \frac{1}{\kappa},\\ \mathfrak{m}_1^* &= -\frac{\kappa^*}{\kappa^2},\\ \mathfrak{m}_2 &= \frac{\kappa'}{\kappa^2\tau},\\ \mathfrak{m}_2^* &= \frac{(\kappa^*)'\kappa^2+2\kappa\kappa^*\kappa'}{\kappa^4\tau}-\frac{\tau^*\kappa'}{\kappa^2\tau^2}.\end{aligned}$$

By means of obtained equations, we express (4.3) and (4.4). This completes the proof.

Corollary 4.2. *Let $\hat{\gamma}: I \longrightarrow \mathbb{D}_1^3$ be a unit speed dual spacelike curve and $\hat{\wp}$ its dual focal curve on $\mathbb{D}_1^3$. Then, the dual focal curvatures of $\hat{\wp}$ are*

$$\begin{aligned}\mathfrak{m}_1 &= \frac{1}{\kappa},\\ \mathfrak{m}_1^* &= -\frac{\kappa^*}{\kappa^2},\\ \mathfrak{m}_2 &= \frac{\kappa'}{\kappa^2\tau},\\ \mathfrak{m}_2^* &= \frac{(\kappa^*)'\kappa^2+2\kappa\kappa^*\kappa'}{\kappa^4\tau}-\frac{\tau^*\kappa'}{\kappa^2\tau^2}.\end{aligned}$$

In the light of Theorem 4.1, we express the following corollary without proof:

Corollary 4.3. *Let $\hat{\gamma}: I \longrightarrow \mathbb{D}_1^3$ be a unit speed dual spacelike biharmonic curve and $\hat{\wp}$ its dual focal curve on $\mathbb{D}_1^3$. Then,*

$$\begin{aligned}\kappa &= \mp\frac{1}{\mathfrak{m}_1},\\ \tau &= \mp\frac{1}{\mathfrak{m}_1},\\ \kappa^* &= \mp\frac{\mathfrak{m}_1^*}{\mathfrak{m}_1^2},\\ \tau^* &= \mp\frac{\mathfrak{m}_1^*}{\mathfrak{m}_1^2}.\end{aligned}$$

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