Stud. Univ. Babeş-Bolyai Math. 57(2012), No. 3, 409–420

Approximate character amenability of Banach algebras

Ali Jabbari

Abstract. New notion of character amenability of Banach algebras is introduced. Let $\mathfrak{A}$ be a Banach algebra and $\varphi \in \Delta(\mathfrak{A})$. We say $\mathfrak{A}$ is approximately φ-amenable if there exist a bounded linear functional m on $\mathfrak{A}^*$ and a net (m_α) in $\mathfrak{A}^{**}$, such that $m(\varphi) = 1$ and

$$m(f.a) = \lim_\alpha m_\alpha(f.a) = \lim_\alpha \varphi(a) m_\alpha(f)$$

for all $a \in \mathfrak{A}$ and $f \in \mathfrak{A}^*$. The corresponding class of Banach algebras is larger than that for the classical character amenable (φ-amenable) algebras. General theory is developed for this notion, and we show that this notion is different from that character amenability and φ-amenability.

Mathematics Subject Classification (2010): 46H20, 43A20.

Keywords: Amenability, approximate amenability, approximate identity, Banach algebras, character amenability.

1. Introduction

The concept of amenable Banach algebra was introduced by Johnson in 1972 [16], and has proved to be of enormous importance in Banach algebra theory. Johnson showed that locally compact group G is amenable if and only if $L^1(G)$ is amenable as a Banach algebra.

Let $\mathfrak{A}$ be a Banach algebra, and let X be a Banach $\mathfrak{A}$-bimodule. A derivation is a linear map $D : \mathfrak{A} \longrightarrow X$ such that

$$D(ab) = a.D(b) + D(a).b \qquad (a, b \in \mathfrak{A}).$$

Throughout this paper, unless otherwise stated, by a derivation we means that a continuous derivation. For $x \in X$, set $ad_x : a \mapsto a.x - x.a$, $\mathfrak{A} \longrightarrow X$. Then ad_x is the inner derivation induced by x.

Denote the linear space of bounded derivations from $\mathfrak{A}$ into X by $Z^1(\mathfrak{A}, X)$ and the linear subspace of inner derivations by $N^1(\mathfrak{A}, X)$, we consider the quotient space $H^1(\mathfrak{A}, X) = Z^1(\mathfrak{A}, X)/N^1(\mathfrak{A}, X)$, called the first Hochschild cohomology group of $\mathfrak{A}$

with coefficients in X. The Banach algebra $\mathfrak{A}$ is said to be amenable if $H^1(\mathfrak{A}, X^*) = \{0\}$ for all Banach $\mathfrak{A}$-bimodules X.

The concept of approximate amenability of Banach algebras was introduced by F. Ghahramani and R. J. Loy in 2004 [11]. They showed that locally compact group G is amenable if and only if $L^1(G)$ is approximately amenable as a Banach algebra.

The derivation $D : \mathfrak{A} \longrightarrow X$ is approximately inner if there is a net (x_α) in X such that

$$D(a) = \lim_\alpha (a.x_\alpha - x_\alpha.a) \qquad (a \in \mathfrak{A}),$$

so that $D = \lim_\alpha ad_{x_\alpha}$ in the strong-operator topology of $B(\mathfrak{A}, X)$. The Banach algebra $\mathfrak{A}$ is approximately amenable if for any $\mathfrak{A}$-bimodule X, every derivation $D : \mathfrak{A} \longrightarrow X^*$ is approximately inner. There are many alternative formulations of the notion of amenability, of which we note the following, for further details see [6, 5, 9, 12, 13, 14]. Of course, amenable Banach algebra is approximately amenable. Some approximately amenable Banach algebras, which are not amenable constructed in [11]. Further examples have been shown by Ghahramani and Stokke in [13]: the Fourier algebra $A(G)$ is approximately amenable for each amenable, discrete group G, but it is known that these algebras are not always amenable.

The notion of character amenability of Banach algebras was defined by Sangani Monfared in [21]. This notion improved by Kaniuth, Lau, and Pym in [18, 19] and some new results were presented by Azimifard in [2]. Let $\mathfrak{A}$ be an arbitrary Banach algebra and φ be a homomorphism from $\mathfrak{A}$ onto $\mathbb{C}$. We denote that the space of all non-zero multiplicative linear functionals from $\mathfrak{A}$ onto $\mathbb{C}$ by $\Delta(A)$ ($\Delta(A)$ is the maximal ideal space of $\mathfrak{A}$). If $\varphi \in \Delta(\mathfrak{A}) \cup \{0\}$ and X is an arbitrary Banach space, then X can be viewed as Banach left or right $\mathfrak{A}$-module by the following actions

$$a.x = \varphi(a)x \qquad \textit{and} \qquad x.a = \varphi(a)x \qquad (a \in \mathfrak{A}, x \in X).$$

The Banach algebra $\mathfrak{A}$ is said to be left character amenable (LCA) if for all $\varphi \in \Delta(\mathfrak{A}) \cup \{0\}$ and all Banach $\mathfrak{A}$-bimodules X for which the right module action is given by $a.x = \varphi(a)x$ ($a \in \mathfrak{A}, x \in X$), every continuous derivation $D : \mathfrak{A} \longrightarrow X^*$ is inner. Right character amenability (RCA) is defined similarly by considering Banach $\mathfrak{A}$-bimodules X for which the left module action is given by $x.a = \varphi(a)x$, and $\mathfrak{A}$ is called character amenable (CA) if it is both left and right character amenable.

Also, according to [18], the Banach algebra $\mathfrak{A}$ is φ-amenable ($\varphi \in \Delta(\mathfrak{A})$) if there exists a bounded linear functional m on $\mathfrak{A}^*$ satisfying $m(\varphi) = 1$ and $m(f.a) = \varphi(a)m(f)$ for all $a \in \mathfrak{A}$ and $f \in \mathfrak{A}^*$. Therefore the Banach algebra $\mathfrak{A}$ is character amenable if and only if $\mathfrak{A}$ is φ-amenable, for every $\varphi \in \Delta(\mathfrak{A}) \cup \{0\}$.

It is clear that if $\mathfrak{A}$ is amenable then $\mathfrak{A}$ is φ-amenable for every $\varphi \in \Delta(\mathfrak{A})$, but converse is not true, because for example let G be a locally compact group and $A(G)$ is a Fourier algebra on G. Fourier algebra $A(G)$ is character amenable (Example 2.6, [18]), but it is not amenable even when G is compact (see [17]).

Recently in [25], authors defined and studied approximate character amenability of Banach algebras. The Banach algebra $\mathfrak{A}$ is said to be approximately left character amenable if for all $\varphi \in \Delta(\mathfrak{A}) \cup \{0\}$ and all Banach $\mathfrak{A}$-bimodules X for which the right module action is given by $a.x = \varphi(a)x$ ($a \in \mathfrak{A}, x \in X$), every continuous derivation $D : \mathfrak{A} \longrightarrow X^*$ is approximately inner. Approximately right character amenability is defined similarly by considering Banach $\mathfrak{A}$-bimodules X for which the left module action is given by $x.a = \varphi(a)x$, and $\mathfrak{A}$ is called approximately character amenable if it is both approximately left and right character amenable.

In this work, after establishing some background definition (our definition is deferent from [25]) and notation, we discuss some results for general Banach algebras. We generalize the character amenability (φ-amenability) of Banach algebras to approximate character amenability (approximate φ-amenability) of Banach algebras. By approximate character amenability, we provide some results, which they concluded by character amenability. A Banach algebra $\mathfrak{A}$ is φ-amenable, if and only if there exists a bounded net (u_α) in $\mathfrak{A}$ such that $\|a.u_\alpha - \varphi(a).u_\alpha\| \to 0$ for all $a \in \mathfrak{A}$ and $\varphi(u_\alpha) = 1$ for all α (Theorem 1.4 of [18]). By this notion, we construct such net and by this net, we give some results about existing of bounded approximate identity for Banach algebra $\mathfrak{A}$.

Also by giving an example, we show that there exists an approximately character amenable non-character amenable Banach algebra. Finally we consider some hereditary properties of approximate φ-amenability.

2. Main results

Definition 2.1. *Let $\mathfrak{A}$ be a Banach algebra and $\varphi \in \Delta(\mathfrak{A})$. We say $\mathfrak{A}$ is approximately φ-amenable if there exist a bounded linear functional m on $\mathfrak{A}^*$ and a net (m_α) in $\mathfrak{A}^{**}$, such that $m(\varphi) = 1$ and*

$$m(f.a) = \lim_\alpha m_\alpha(f.a) = \lim_\alpha \varphi(a) m_\alpha(f)$$

for all $a \in \mathfrak{A}$ and $f \in \mathfrak{A}^*$.

When $\mathfrak{A}$ is φ-amenable, then it is approximately $\varphi-$amenable. In [18, 19], authors showed that the relation between φ-amenability and amenability of Banach algebras in special case. Banach algebra $\mathfrak{A}$ is φ-amenable if and only if $H^1(\mathfrak{A}, X^*) = \{0\}$ for each Banach $\mathfrak{A}$-bimodule X such that $a.x = \varphi(a)x$, for all $x \in X$ and $a \in \mathfrak{A}$. In the following Theorem, we generalize the Theorem 1.1 of [18] as follows:

Theorem 2.2. *Let $\mathfrak{A}$ be a Banach algebra, and $\varphi \in \Delta(\mathfrak{A})$. Then the following statements are equivalent.*

(i) $\mathfrak{A}$ is approximately φ-amenable.

(ii) If X is a Banach $\mathfrak{A}$-bimodule such that $a.x = \varphi(a).x$ for all $x \in X$ and $a \in \mathfrak{A}$, then every derivation from $\mathfrak{A}$ into X^* is approximately inner.

Proof. $(ii) \to (i)$, let $\varphi \in \Delta(\mathfrak{A})$. It is clear that $\mathfrak{A}^*$ is a Banach $\mathfrak{A}$-bimodule by following action

$$a.f = \varphi(a).f,$$

for all $f \in \mathfrak{A}^*$ and $a \in \mathfrak{A}$. Also since $\varphi \in \mathfrak{A}^*$, so we have

$$a.\varphi = \varphi.a = \varphi(a)\varphi. \tag{2.1}$$

Therefore $\mathbb{C}\varphi$ is a closed $\mathfrak{A}$-submodule of $\mathfrak{A}^*$. Take $X = \mathfrak{A}^* \backslash \mathbb{C}\varphi$, and consider $i : \mathfrak{A}^* \longrightarrow X$ that is a canonical mapping. Let δ be a derivation from $\mathfrak{A}$ into $\mathfrak{A}^{**}$.

Since every derivation from $\mathfrak{A}$ into each Banach $\mathfrak{A}$-bimodule is approximately inner, then there exists a net $(v_\alpha)_\alpha$ in $\mathfrak{A}^{**}$, such that

$$\delta(a) = \lim_\alpha a.v_\alpha - v_\alpha.a.$$

According to (2.1) for given $a \in \mathfrak{A}$, we have

$$\delta(a)(\varphi) \;=\; \lim_\alpha (a.v_\alpha)\varphi - (v_\alpha.a)\varphi = \lim_\alpha v_\alpha(\varphi.a) - v_\alpha(\varphi.a) = 0.$$

Therefore $\delta(a) \in i^*X^*$, and since i^* is a monomorphism, thus there exists a unique element $D(a) \in X^*$, such that $i^*(D(a)) = \delta(a)$. This shows that D is a derivation from $\mathfrak{A}$ into X^*. Therefore there exists a net $(\xi_\beta)_\beta$ in X^* such that

$$D(a) = \lim_\beta a.\xi_\beta - \xi_\beta.a,$$

for all $a \in \mathfrak{A}$. Then we have

$$\begin{aligned}\lim_\beta a.(i^*\xi_\beta) - (i^*\xi_\beta).a \;&=\; \lim_\beta i^*(a.\xi_\beta - \xi_\beta.a)\\ &=\; i^*(D(a)) = \delta(a) = \lim_\alpha a.v_\alpha - v_\alpha.a.\end{aligned}$$

Let $m_{\alpha,\beta} = v_\alpha - i^*\xi_\beta$, then $m_{\alpha,\beta} \in \mathfrak{A}^{**}$, $m_{\alpha,\beta}(\varphi) = 1$ and $a.m_{\alpha,\beta} = m_{\alpha,\beta}.a$. Therefore we have

$$m_{\alpha,\beta}(f.a) = m_{\alpha,\beta}(a.f) = \varphi(a)m_{\alpha,\beta}(f) \qquad (a \in \mathfrak{A}, f \in \mathfrak{A}^*).$$

Let I and J be the index sets for nets (v_α) and (ξ_β), respectively. We construct the required net (m_k) using an iterated limit construction (see [20]). Our indexing directed set is defined to be $K = I \times \Pi_{\alpha\in I} J$, equipped with the product ordering, and for each $k = (\alpha, f) \in K$, we define $m_k = m_{\alpha,f(\alpha)}$. Now let $m = \lim_k m_k$ and this complete the proof.

For $(i) \to (ii)$, let (m_α) and m be as in Definition 2.1. Let $D : \mathfrak{A} \longrightarrow X^*$ be a derivation, and let $D' = D^*|_X : X \longrightarrow \mathfrak{A}^*$ and $g_\alpha = (D')^*(m_\alpha) \in X^*$, such that $\lim_\alpha g_\alpha = \lim_\alpha (D')^*(m_\alpha) = (D')^*(m)$. Then, for all $a, b \in \mathfrak{A}$ and $x \in X$,

$$\langle b, D'(a.x)\rangle \;=\; \langle a.x, D(b)\rangle = \varphi(a)\langle x, D(b)\rangle = \varphi(a)\langle b, D'(x)\rangle,$$

and, hence $D(a.x) = \varphi(a)D'(x)$. This implies that

$$\begin{aligned}\langle x, g_\alpha.a\rangle \;&=\; \langle a.x, g_\alpha\rangle = \langle D'(a.x), m_\alpha\rangle\\ &=\; \varphi(a)\langle D'(x), m_\alpha\rangle = \varphi(a)\langle x, g_\alpha\rangle.\end{aligned}$$

Since D is a derivation, so we have

$$\begin{aligned}\langle b, D'(x.a)\rangle \;&=\; \langle x.a, D(b)\rangle = \langle x, a.D(b)\rangle = \langle ab, D'(x)\rangle - \langle b.x, D(x)\rangle\\ &=\; \langle b, D'(x).a\rangle - \varphi(b)\langle x, D(a)\rangle\end{aligned}$$

for all $a, b \in \mathfrak{A}$ and $x \in X$. Thus

$$D'(x.a) = D'(x).a - \langle x, D(a)\rangle\varphi,$$

for all $a \in \mathfrak{A}$ and $x \in X$. Consequently

$$\begin{aligned}\langle D'(x.a), m\rangle &= \langle x.a, (D')^*(m)\rangle = \lim_\alpha \langle x, a.g_\alpha\rangle = \lim_\alpha \langle x.a, g_\alpha\rangle = \lim_\alpha \langle D'(x.a), m_\alpha\rangle \\ &= \lim_\alpha \langle D'(x).a, m_\alpha\rangle - \lim_\alpha \langle x, D(a)\rangle\langle \varphi, m_\alpha\rangle \\ &= \lim_\alpha \varphi(a)\langle x, g_\alpha\rangle - \langle x, D(a)\rangle,\end{aligned}$$

and hence $D(a) = \lim_\alpha \varphi(a)g_\alpha - a.g_\alpha$. Therefore we have

$$D(a) = \lim_\alpha a.(-g_\alpha) - (-g_\alpha).a,$$

for all $a \in \mathfrak{A}$. □

According to the Theorem 1.4 of [18], the Banach algebra $\mathfrak{A}$ is φ-amenable if and only if there exists a bounded net (u_α) in $\mathfrak{A}$ such that $\|a.u_\alpha - \varphi(a).u_\alpha\| \to 0$ for all $a \in \mathfrak{A}$ and $\varphi(u_\alpha) = 1$ for all α. Now, by using of technique of Theorem 1.4 of [18], we have the following Theorem.

Theorem 2.3. *Let $\mathfrak{A}$ be a Banach algebra and $\varphi \in \Delta(\mathfrak{A})$. If $\mathfrak{A}$ is approximately φ-amenable then there exists a bounded net (u_α) in $\mathfrak{A}$ such that $\|a.u_\alpha - \varphi(a).u_\alpha\| \to 0$ for all $a \in \mathfrak{A}$.*

Proof. Suppose that $\mathfrak{A}$ is approximately φ-amenable. Let $m \in \mathfrak{A}^{**}$ and net $(m_\alpha) \in \mathfrak{A}^{**}$ be as in Definition 2.1.

Fix α, then by the Goldstaine Theorem (Theorem A.3.29 of [7]) there exists a net $(\nu_{\alpha,\beta}) \subset \mathfrak{A}$, such that $\nu_{\alpha,\beta} \xrightarrow{w^*} m_\alpha$, and $\|\nu_{\alpha,\beta}\| \le \|m_\alpha\| + 1$ for all β. Consider the product space $\mathfrak{A}^{\mathfrak{A}}$ endowed with the product of the norm topologies. Then $\mathfrak{A}^{\mathfrak{A}}$ is a locally convex topological vector space. Define a linear map $T : \mathfrak{A} \longrightarrow \mathfrak{A}^{\mathfrak{A}}$ by

$$T(u) = (au - \varphi(a)u)_{a\in\mathfrak{A}} \qquad (u \in \mathfrak{A}),$$

and a subset C_α of $\mathfrak{A}$ by

$$C_\alpha = \{u \in \mathfrak{A} : \|u\| \le \|m_\alpha\| + 1\}.$$

Then C_α is convex and hence $T(C_\alpha)$ is a convex subset of $\mathfrak{A}^{\mathfrak{A}}$. The fact that $\langle a\nu_{\alpha,\beta} - \varphi(a)\nu_{\alpha,\beta}, f\rangle \longrightarrow 0$, for all $f \in \mathfrak{A}^*$ and $a \in \mathfrak{A}$. This shows that the zero element of $\mathfrak{A}^{\mathfrak{A}}$ is contained in the closure of $T(C_\alpha)$ with respect to the product of the weak topologies. Now this product of the weak topologies coincides with the weak topology on $\mathfrak{A}^{\mathfrak{A}}$ and since $\mathfrak{A}^{\mathfrak{A}}$ is a locally convex space and $T(C_\alpha)$ is convex, the weak closure of $T(C_\alpha)$ equals the closure of $T(C_\alpha)$ in the given topology on $\mathfrak{A}^{\mathfrak{A}}$, that is, the product of the norm topologies (see [24, 23]). It follows that there exists a bounded net (u_γ) in $\mathfrak{A}$ such that $\|a.u_\gamma - \varphi(a).u_\gamma\| \to 0$ for all $a \in \mathfrak{A}$. □

Note that according to the proof of the above Theorem, existence of such net is not unique. for given character amenable Banach algebra $\mathfrak{A}$, set the character amenability constant $C(\mathfrak{A})$ of $\mathfrak{A}$ to be infimum of the norms of nets which obtained by Theorem 1.4 of [18]. We will say $\mathfrak{A}$ is K- character amenable if its character amenability constant is at most K.

We should show that approximate character amenability of Banach algebras is different from character amenability of Banach algebras. To showing this difference, by using of technique of Example 6.1 of [11], we consider the following Example.

Example 2.4. Let $(\mathfrak{A}_n)$ be a sequence of character amenable Banach algebras such that $C(\mathfrak{A}_n) \to \infty$. Then $B = c_0(\mathfrak{A}_n^\sharp)$ is approximately character amenable non-character amenable Banach algebra.

Proof. Let B be character amenable, and let $(U_m)_m$ be the sequence in B which obtained by Theorem 1.4 of [18]. By restricting of $(U_m)_m$ to the n^{th} coordinate of this sequence of bound K yields a sequence $(u_m)_m$ in $\mathfrak{A}_n$ with bound at most K, and since $C(\mathfrak{A}_n) \to \infty$ then B can not be character amenable. Define

$$B_k = \{(x_n) \in c_0(\mathfrak{A}_n^\sharp) : x_n = 0 \ for \ n > k\}.$$

For $n \in \mathbb{N}$, let $P_n : c_0(\mathfrak{A}_n^\sharp) \longrightarrow B_n$ be the natural projection of multiplication by $E_n = (e_1, e_2, ..., e_n, 0, 0, ...)$ onto the first n coordinate. Then $P_n(E_n)$ is the identity of B_n, and (E_n) is a central approximate identity for B bounded by 1. Let X be a Banach B-bimodule such that $b.x = \varphi(b)x$, and $x.b = \varphi(b)x$ for all $b \in B$, $x \in X$ $(\varphi \in \Delta(B))$. Now suppose that $D : B \longrightarrow X^*$ is a continuous derivation. By restricting D to some B_n we have a derivation $D_n : B_n \longrightarrow X^*$. Then there exists a bounded sequence (ξ_n) in X^* such that $D_n(b) = b.\xi_n - \xi_n.b$. Then by module actions defined above for $b \in B$ and $x \in X$ we have

$$\langle bE_n.\xi_n, x\rangle = \langle b.\xi_n, x\rangle \qquad and \qquad \langle \xi_n.E_n b, x\rangle = \langle \xi_n.b, x\rangle.$$

Since (E_n) is central, so

$$\begin{aligned} D(b) &= D(\lim_n E_n b) = \lim_n D(E_n b) = \lim_n D_n(E_n b) \\ &= \lim_n bE_n.\xi_n - \xi_n.E_n b = \lim_n b.\xi_n - \xi_n.b. \end{aligned}$$

Then D is approximately inner, and hence B is approximately character amenable. □

One of the famous open problems in the amenability and approximate amenability of Banach algebras is: *Does amenability or approximate amenability of Banach algebra such as $\mathfrak{A}$ implies that amenability or approximate amenability of $\mathfrak{A}^{**}$?* This problem in the case of φ-amenability in Proposition 3.4 of [18] is considered and authors proved that the Banach algebra $\mathfrak{A}$ is φ-amenable if and only if $\mathfrak{A}^{**}$ is $\widetilde{\varphi}$-amenable, where $\widetilde{\varphi}$ is the extension of φ to $\mathfrak{A}^{**}$. Now, we generalize this statement in to approximate φ-amenability of Banach algebras.

Theorem 2.5. *Let $\mathfrak{A}$ be a Banach algebra, let $\varphi \in \Delta(\mathfrak{A})$, and let $\widetilde{\varphi}$ denote the extension of φ to $\mathfrak{A}^{**}$. Then $\mathfrak{A}$ is approximately φ-amenable if and only if $\mathfrak{A}^{**}$ is approximately $\widetilde{\varphi}$-amenable.*

Proof. Let $m(f.a) = \lim_\alpha \varphi(a)m_\alpha(f)$, where $m \in \mathfrak{A}^{**}$ and $(m_\alpha)_\alpha \subseteq \mathfrak{A}^{**}$. Suppose that $\hat{m}$ is the Gelfand transform of m and $\hat{m}_\alpha$ is the Gelfand transform of m_α, for all

α. Let $a^{''} \in \mathfrak{A}^{**}$ and $u \in \mathfrak{A}^{***}$, then there exist nets $(a_\gamma) \subseteq \mathfrak{A}$ and $(u_\beta) \subseteq \mathfrak{A}^*$, such that $a_\gamma \xrightarrow{w^*} a^{''}$ and $u_\beta \xrightarrow{w^*} u$. Therefore

$$\begin{aligned} \langle u.a^{''}, \hat{m}\rangle &= \langle m, u.a^{''}\rangle = \lim_\beta \langle m, u_\beta.a^{''}\rangle = \lim_\beta \lim_\gamma \langle m, u_\beta.a_\gamma\rangle \\ &= \lim_\beta \lim_\gamma \lim_\alpha \varphi(a_\gamma)\langle u_\beta, m_\alpha\rangle = \lim_\beta \lim_\alpha \widetilde{\varphi}(a^{''})\langle u_\beta, m_\alpha\rangle \\ &= \lim_\alpha \widetilde{\varphi}(a^{''})\hat{m}_\alpha(u). \end{aligned}$$

Conversely, let $\mathfrak{A}^{**}$ be approximately $\widetilde{\varphi}$-amenable, then there sexist an element M and net $(M_\alpha)_\alpha$ in $\mathfrak{A}^{****}$, such that $M(f.a) = \lim_\alpha \widetilde{\varphi}(a)M_\alpha(f)$. Now with restriction of M and M_α to $\mathfrak{A}^{**}$, we conclude that $\mathfrak{A}$ is approximately φ-amenable. □

In Theorem 2.3 of [21], Monfared showed that character amenability of Banach algebras similarly to amenability of Banach algebras implies that existence of bounded approximate identity for them. By the following Proposition, we show approximate φ-amenability of Banach algebra similar to approximate amenability implies existence of approximate identity for it's.

Proposition 2.6. *Let $\mathfrak{A}$ be a Banach algebra and $\varphi \in \Delta(\mathfrak{A})$. If $\mathfrak{A}$ is approximately φ-amenable, then $\mathfrak{A}$ have right and left approximate identity.*

Proof. In Theorem 2.1 (ii), take $X = \mathfrak{A}^*$, with right action $a.x = \varphi(a).x$ and zero left action. Remain of proof is exactly similar to proof of Lemma 2.2 of [11]. □

Let $\mathfrak{A}$ be an approximately amenable Banach algebra and

$$\sum : 0 \longrightarrow X^* \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

be an admissible short exact sequence of left $\mathfrak{A}$-modules. Then by Theorem 2.2 of [11], $\sum$ approximately splits. Proof of the following Theorem is similar to proof of Theorem 2.2 of [11].

Theorem 2.7. *Let $\mathfrak{A}$ be a Banach algebra, and let $\varphi \in \Delta(\mathfrak{A})$. Let*

$$\sum : 0 \longrightarrow X^* \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

be an admissible short exact sequence of Banach $\mathfrak{A}$-bimodules such that $a.x = \varphi(a).x$, for all $a \in \mathfrak{A}$ and x is in X, Y and Z. If $\mathfrak{A}$ is approximately φ-amenable, then $\sum$ is approximately splits. That is, there is a net $G_\nu : Z \longrightarrow Y$ of right inverse maps to g such that $\lim_\nu(a.G_\nu - G_\nu.a) = 0$ for $a \in \mathfrak{A}$, and a net $F_\nu : Y \longrightarrow X^*$ of left inverse maps to f such that $\lim_\nu(a.F_\nu - F_\nu.a) = 0$ for $a \in \mathfrak{A}$.

The following Proposition is similar to Proposition 3.3 of [25], but since our proof is different, so we do not remove the proof.

Proposition 2.8. *Let $\mathfrak{A}$ be a Banach algebra, and let J be a closed ideal of $\mathfrak{A}$. Let $\varphi \in \Delta(\mathfrak{A})$ such that $\varphi|_J \neq 0$. If $\mathfrak{A}$ is approximately φ-amenable, then J is approximately $\varphi|_J-$amenable.*

Proof. Let $m \in \mathfrak{A}^{**}$ and net $(m_\alpha)_\alpha$ in $\mathfrak{A}^{**}$ be as in Definition 2.1. Fix α, then there exists a net $(\nu_{\alpha,\beta})$ in $\mathfrak{A}$ such that $\nu_{\alpha,\beta} \longrightarrow m_\alpha$ in w^*-topology. For each $f \in J^\perp$ and any $a \in J$, we have

$$\begin{aligned} m(f.a) &= \lim_\alpha m_\alpha(f.a) = \lim_\beta \lim_\alpha \langle f.a, \nu_{\alpha,\beta}\rangle \\ &= \lim_\beta \lim_\alpha \langle f, a.\nu_{\alpha,\beta}\rangle = 0. \end{aligned}$$

Suppose that $m' \in J^{**}$, such that $m \in \mathfrak{A}^{**}$ is the extending of m'. Without loss of generality, we can suppose that $\varphi(a) = 1$, then

$$m(f.a) = \lim_\alpha m_\alpha(f.a) = \lim_\alpha m_\alpha(f) = 0,$$

for all $f \in J^\perp$. Therefore net $(m_\alpha)_\alpha$ is a net of bounded linear functional on J^*. For each α there exists m'_α in J^{**} such that $m'_\alpha(g) = m_\alpha(f)$, where f is extending of g from J^* into $\mathfrak{A}^*$. By definition of m'_α we have $m'_\alpha(\varphi|_J) = m_\alpha(\varphi) = 1$ and

$$\varphi(a)m'_\alpha(g) = \varphi(a)m_\alpha(f) = m_\alpha(f.a) = m'_\alpha(g.a),$$

for all $g \in J^*$ and $a \in J$. Thus

$$\begin{aligned} m'(g.a) &= m(f.a) = \lim_\alpha m_\alpha(f.a) = \lim_\alpha m'_\alpha(g.a) \\ &= \lim_\alpha \varphi(a)m'_\alpha(g). \end{aligned}$$

□

Proposition 2.9. *Let $\mathfrak{A}$ be a Banach algebra, let J be a closed two-sided ideal in $\mathfrak{A}$, and let $\varphi \in \Delta(\mathfrak{A})$. If $\mathfrak{A}$ is approximately φ-amenable, then $\mathfrak{A}/J$ is approximately $\widetilde{\varphi}$-amenable, where $\widetilde{\varphi}$ is the induction of φ on $\mathfrak{A}/J$.*

Proof. Let $T : \mathfrak{A} \longrightarrow \mathfrak{A}/J$ be a continuous epimorphism. Suppose that X is a Banach $\mathfrak{A}-$bimodule, where $a.x = \varphi(a).x$, for all $a \in \mathfrak{A}, x \in X$. Let $D : \mathfrak{A}/J \longrightarrow X^*$ be a derivation. Then $D \circ T : \mathfrak{A} \longrightarrow X^*$ is a derivation. Define $\widetilde{\varphi} = \varphi \circ T$, thus $\widetilde{\varphi}$ is a homomorphism on $\mathfrak{A}/J$.

Since $\mathfrak{A}$ is approximately φ-amenable, then by Theorem 2.1 (ii), there exists a net $(\xi_\alpha)_\alpha \subset X^*$ such that

$$(D \circ T)(a) = \lim_\alpha T(a).\xi_\alpha - \xi_\alpha.T(a),$$

this shows that D is an approximately inner derivation, by Theorem 2.1, $\mathfrak{A}/J$ is approximately $\widetilde{\varphi}$-amenable. □

Let G be a locally compact group, and let $A(G)$ be the Fourier algebra on G. Ghahramani and Stokke in [13], studied approximate amenability of Fourier algebras on locally compact group G. Character amenability of $A(G)$ is equivalent to amenability of G as a group (Corollary 2.4 of [21]). By Proposition 2.9, we have the following result for $A(G)$.

Corollary 2.10. *Let G be a locally compact group, and H be a closed subgroup of G. If $A(G)$ is approximately character amenable, then $A(H)$ is approximately character amenable.*

Proof. By Lemma 3.8 of [10], $A(H)$ is a quotient of $A(G)$. Then by Proposition 2.9, $A(H)$ is approximately character amenable. $\square$

Proposition 2.11. *Let $\mathfrak{A}$ be a Banach algebra and $\varphi \in \Delta(\mathfrak{A})$. Let J be an ideal in $\mathfrak{A}$ with $J \subseteq \ker\varphi$ and let $\widetilde{\varphi} : \mathfrak{A}/J \longrightarrow \mathbb{C}$ be the homomorphism induced by φ. If J has a right identity and $\mathfrak{A}/J$ is approximately $\widetilde{\varphi}$-amenable then $\mathfrak{A}$ is approximately φ-amenable.*

Proof. Let $T : \mathfrak{A} \longrightarrow \mathfrak{A}/J$ be a epimorphism, such that $\varphi = \widetilde{\varphi} \circ T$. Suppose $m \in \mathfrak{A}^{**}$, such that $T(m)^{**} = n$ and there exists a net $(n_\alpha)_\alpha \subset (J^\perp)^*$, such that $n(a.f) = \lim_\alpha \widetilde{\varphi}(a) n_\alpha(f)$, for all $a \in \mathfrak{A}/J$ and $f \in J^\perp$. For each α there exists $m_\alpha \in \mathfrak{A}^{**}$, such that $T(m_\alpha) = n_\alpha$. For all $x \in \mathfrak{A}$ we have

$$\begin{aligned} T(x)T^{**}(m - me) &= T^{**}(x)T^{**}(m) = T^{**}(x)n = \lim_\alpha \widetilde{\varphi}^{**}(T^{**}(x))n_\alpha \\ &= \lim_\alpha \widetilde{\varphi}(T(x))T^{**}(m_\alpha - m_\alpha e) \\ &= \lim_\alpha \varphi(x)T^{**}(m_\alpha - m_\alpha e), \end{aligned}$$

therefore

$$T^{**}(x(m - me) - \lim_\alpha \varphi(x)(m_\alpha - m_\alpha e)) \longrightarrow 0,$$

Since for each $a \in J$, $(x(m - me) - \lim_\alpha \varphi(x)(m_\alpha - m_\alpha e))a = 0$, thus $x(m - me) - \lim_\alpha \varphi(x)(m_\alpha - m_\alpha e) = 0$. Hence

$$x(m - me) = \lim_\alpha \varphi(x)(m_\alpha - m_\alpha e).$$

$\square$

Let $\mathfrak{A}$ be a non-unital Banach algebra. We denoted the forced unitization of $\mathfrak{A}$ with $\mathfrak{A}^\sharp$ and the adjoined identity element usually be denoted by e unless stated otherwise. Banach algebra $\mathfrak{A}$ is an ideal of $\mathfrak{A}^\sharp$ and $\mathfrak{A}^\sharp = \mathfrak{A} \oplus \mathbb{C}e$. Like as $\mathfrak{A}^\sharp$, for $(\mathfrak{A}^\sharp)^*$ and $(\mathfrak{A}^\sharp)^{**}$, we have $(\mathfrak{A}^\sharp)^* = \mathfrak{A}^* \oplus \mathbb{C}f_0$ and $(\mathfrak{A}^\sharp)^{**} = \mathfrak{A}^{**} \oplus \mathbb{C}m_0$, where $f_0(e) = 1$, $f|_{\mathfrak{A}} = 0$, $m_0(f_0) = 1$ and $m_0|_{\mathfrak{A}^{**}} = 0$.

Proposition 2.12. *Let $\mathfrak{A}$ be a non-unital Banach algebra. Let $\varphi \in \Delta(\mathfrak{A})$ and let φ_e be the unique extension of φ to an element of $\varphi : \mathfrak{A}^\sharp \longrightarrow \mathbb{C}$ be a continuous homomorphism. Then $\mathfrak{A}$ is approximately φ-amenable if and only if $\mathfrak{A}^\sharp$ is approximately φ_e-amenable.*

Proof. See Theorem 3.7 of [25]. $\square$

Proposition 2.13. *Let $\mathfrak{A}$ be a Banach algebra and let $\varphi \in \Delta(\mathfrak{A})$. If $\mathfrak{A}$ is approximately φ-amenable and J is a weakly complemented left ideal of $\mathfrak{A}$. Then J has a right approximate identity and thereupon $\overline{J^2} = J$.*

Proof. Without loss of generality by Proposition 2.12, we can suppose that $\mathfrak{A}$ is unital. Consider the following sequence of left $\mathfrak{A}$-modules

$$\sum : 0 \longrightarrow J \stackrel{i}{\longrightarrow} \mathfrak{A} \longrightarrow \mathfrak{A}/J \longrightarrow 0$$

since $(\mathfrak{A}/J)^* \cong J^\perp$, then we have

$$\sum\nolimits^{**}: 0 \longrightarrow J^{**} \xrightarrow{i^{**}} \mathfrak{A}^{**} \longrightarrow (J^\perp)^* \longrightarrow 0.$$

Since J is a weakly complemented left ideal of $\mathfrak{A}$, then $\sum^{**}$ is admissible, and so by Theorem 2.7, there is a net of maps (Q_α) such that $a.Q_\alpha - Q_\alpha.a \longrightarrow 0$, for all $a \in \mathfrak{A}$.

Since $\mathfrak{A}$ is unital, then $\mathfrak{A}^{**}$ has right identity E, then

$$\langle E, Q_\alpha.a\rangle = \langle i^{**}\hat{a}.E, Q_\alpha\rangle = \langle i^{**}\hat{a}, Q_\alpha\rangle = \hat{a}.$$

for all $a \in J$, and thereby we have

$$\langle E, a.Q_\alpha\rangle = \langle a.Q_\alpha - Q_\alpha.a, E\rangle + \hat{a} \longrightarrow \hat{a},$$

and this show that $\langle E, Q_\alpha.a\rangle \longrightarrow \hat{a}$, for all $a \in J$. Therefore by Proposition 2.6, J has a right approximate identity. □

Example 2.14. In Corollary 2.5 of [21], Monfared showed that the measure algebra $M(G)$ is character amenable if and only if G is a discrete amenable group. Now, by Proposition 2.13, $M(G)$ is approximately character amenable if and only if G is discrete and amenable (see [8]).

Let G be a locally compact group. A linear subspace $S^1(G)$ of $L^1(G)$ is said to be a Segal algebra, if it satisfies the following conditions:

(i) $S^1(G)$ is a dense in $L^1(G)$;

(ii) If $f \in S^1(G)$, then $L_x f \in S^1(G)$, i.e. $S^1(G)$ is left translation invariant;

(iii) $S^1(G)$ is a Banach space under some norm $\|.\|_S$ and $\|L_x f\|_s = \|f\|_s$, for all $f \in S^1(G)$ and $x \in G$;

(iv) Map $x \mapsto L_x f$ from G into $S^1(G)$ is continuous.

For more details about Segal algebras see [22]. Character amenability of Segal algebras studied in [1], and many useful results about approximate amenability of such algebras considered in [6]. For amenable locally compact group G, $S^1(G)$ is φ-amenable for all $\varphi \in \Delta(S^1(G))$, and also $S^1(G)$ is character amenable if and only if $S^1(G) = L^1(G)$ (Proposition 3.1 of [1]). Now, we consider the following Corollary for Segal algebras (for prove, we use technique of proof of Theorem 5.5 of [6]).

Corollary 2.15. *Let G be a locally compact group, and let $S^1(G)$ be a Segal algebra on G. If $S^1(G)$ is approximate character amenable then G is an amenable group.*

Proof. Let G be a non-amenable locally compact group. Then by Theorem 5.2 of [26], There is no finite codimensional, closed, left ideal in $L^1(G)$ has right approximate identity. Let $I_0 = \{f \in L^1(G) : \int_G f(x)dx = 0\}$ be the augmentation ideal in $L^1(G)$. Let J be an ideal of $S^1(G)$ such that $J = S^1(G) \cap I_0$. Ideal J is a 1-codimension two-sided closed ideal in $S^1(G)$. If $S^1(G)$ is approximate character amenable, then by Proposition 2.13, J has a right approximate identity. By Proposition 5.4 of [6], I_0 have a right approximate identity, and this is a contradiction. Therefore G is an amenable group. □

By following Proposition, we show that in which condition the approximate character amenability can be transfer.

Proposition 2.16. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be Banach algebras, and let $h : \mathfrak{A} \longrightarrow \mathfrak{B}$ be a continuous homomorphism with dense range. If $\varphi \in \Delta(\mathfrak{B})$ and $\mathfrak{A}$ is approximately $\varphi \circ h$-amenable, then $\mathfrak{B}$ is approximately φ-amenable.*

Proof. Let $\mathfrak{A}$ be approximately $\varphi\circ h-$amenable, then there exist there exist a bounded linear functional m on $\mathfrak{A}^*$ and a net (m_α) in $\mathfrak{A}^{**}$, such that $m_\alpha(\varphi \circ h) = 1$ and $m(f.a) = \lim_\alpha \varphi(h(a))m_\alpha(f)$ for all $a \in \mathfrak{A}$ and $f \in \mathfrak{A}^*$.

Given $n \in \mathfrak{B}^{**}$ such that $n(g) = m(g.h)$, $g \in \mathfrak{B}^*$. For each $a, a' \in \mathfrak{A}$, we have

$$\begin{aligned} \langle (g.h(a) \circ h, a' \rangle &= \langle g.h(a), h(a') \rangle = \langle g, h(a)h(a') \rangle \\ &= \langle g, h(aa') \rangle = \langle g \circ h, aa' \rangle = \langle (g \circ h).a, a' \rangle. \end{aligned}$$

Therefore $(g.h(a) \circ h = (g \circ h).a$, for all $a \in \mathfrak{A}$. Let $m_\alpha(g \circ h) = n_\alpha(g)$, for $g \in \mathfrak{B}^*$, then

$$\begin{aligned} n(g.h(a)) &= m((g.h(a)) \circ h) = m((g \circ h).a) \\ &= \lim_\alpha \varphi \circ h(a) m_\alpha(g \circ h) = \lim_\alpha \varphi(h(a)) n_\alpha(g), \end{aligned}$$

where $(n_\alpha)_\alpha$ is in $\mathfrak{B}^{**}$. □

References

[1] Alaghmandian, M., Nasr-Isfahani, R., Nemati, M., *Character amenability and contractibility of abstract Segal algebras*, Bull. Aust. Math. Soc., **82**(2010), 274-281.

[2] Azimifard, A., *On character amenability of Banach algebras*, arXiv: 0712.2072.

[3] Bade, W. G., Curtis, P. C., Dales, H. G., *Amenability and weak amenability for Beurling and Lipschitz algebras*, Proc. London Math. Soc., **55**(1987), no. 2, 359-377.

[4] Bonsall, F.F., Duncan, J., *Complete normed algebras*, Springer-Verlag, Berlin, 1973.

[5] Choi, Y., Ghahramani, F., *Approximate amenability of Schatten classes, Lipschitz algebras and second duals of Fourier algebras*, Quart. J. Math., **62**(2011), no. 1, 39-58.

[6] Choi, Y., Ghahramani, F., Zhang, Y., *Approximate and pseudo-amenability of various classes of Banach algebras*, J. Funct. Anal., **256**(2009), 3158-3191.

[7] Dales, H. G., *Banach algebras and automatic continuity*, London Math. Society Monographs, Volume 24, Clarendon Press, Oxford, 2000.

[8] Dales, H.G., Ghahramani, F., Helemskii, A. Ya., *Amenability of measure algebras*, J. London Math. Soc., **66**(2002), no. 2, 213-226.

[9] Dales, H.G., Loy, R.J., Zhang, Y., *Approximate amenability for Banach sequence algebras*, Studia Math., **177**(2006), no. 1, 81-96.

[10] Forrest, B.B., *Amenability and bounded approximate identities in ideals of $A(G)$*, Illinois J. Math., **34**(1990), 1-25.

[11] Ghahramani, F., Loy, R.J., *Genarlized notion of amenability*, J. Funct. Anal., **208**(2004), 229-260.

[12] Ghahramani, F., Loy, R.J., Zhang, Y., *Generalized notion of amenability II*, J. Funct. Anal., **7**(2008), 1776-1810.

[13] Ghahramani, F., Stokke, R., *Approximate and pseudo-amenability of $A(G)$*, Indiana University Mathematics Journal, **56**(2007), 909-930.

[14] Ghahramani, F., Zhang, Y., *Pseudo-amenable and psuedo-contractible Banach algebras*, Math. Proc. Camb. Phil. Soc., **142**(2007), 111-123.

[15] Hewitt, E., Ross, K.A., *Abstract harmonic analysis*, Volume I, Second Edition, Springer-Verlag, Berlin, 1979.

[16] Johnson, B.E., *Cohomology in Banach algebras*, Mem. Amer. Math. Soc., **127**(1972).

[17] Johnson, B.E., *Non-amenability of the Fourier algebra of a compact group*, J. London Math., **50**(1994), 361-374.

[18] Kaniuth, E., Lau, A.T., Pym, J., *On φ-amenability of Banach algebras*, Math. Proc. Camb. Phil. Soc., **144**(2008), 85-96.

[19] Kaniuth, E., Lau, A.T., Pym, J., *On character amenability of Banach algebras*, J. Math. Anal. Appl., **344**(2008), 942-955.

[20] Kelley, J.L., *General topology*, American Book, Van Nostrand, Reinhold, 1969.

[21] Monfared, M.S., *Character amenability of Banach algebras*, Math. Proc. Cambridge Philos. Soc., **144**(2008), 697-706.

[22] Reiter, H., Stegeman, J.D., *Classical harmonic analysis and locally compact groups*, London Math. Soc. Mono. Ser. Volume 22, Oxford, 2000.

[23] Rudin, W., *Functional analysis*, McGraw-Hill, 1973.

[24] Schaefer, H.H., *Topological vector spaces*, Springer-Verlag, 1971.

[25] Shi, L.Y., Wu, Y.J., Ji, Y.Q., *Generalized notion of character amenability*, arXiv:1008.5252v1, 2010.

[26] Willis, G.A., *Approximate units in finite codimensional ideals of group algebras*, J. London Math. Soc., **26**(1982), no. 2, 143-154.

Ali Jabbari
Department of Mathematics, Ardabil Branch
Islamic Azad University, Ardabil, Iran
e-mail: jabbari_al@yahoo.com