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On n-weak amenability of a non-unital Banach algebra and its unitization

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Abstract. In [2] the authors asked if a non-unital Banach Algebra $\mathfrak{A}$ is weakly amenable whenever its unitization $\mathfrak{A}^{\sharp}$ is weakly amenable and whether $\mathfrak{A}^{\sharp}$ is 2-weakly amenable whenever $\mathfrak{A}$ is 2-weakly amenable. In this paper we give a partial solutions to these questions.

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1. Introduction

The notion of n-weak amenability for a Banach algebra was introduced by Dales, Ghahramani and Gronbæk in [2]. The Banach algebra $\mathfrak{A}$ is called n-weakly amenable if $\mathcal{H}^1(\mathfrak{A}, \mathfrak{A}^{(n)}) = (0)$, where $\mathfrak{A}^{(n)}$ refers to the n-th dual of $\mathfrak{A}$. Also $\mathfrak{A}$ is permanently weakly amenable if $\mathfrak{A}$ is n-weakly amenable for each $n \epsilon \mathbb{N}$. In [2] the authors proved the following(Proposition 1.4):
Let $\mathfrak{A}$ be a non-unital Banach algebra, and $n \epsilon \mathbb{N}$.

(i) Suppose $\mathfrak{A}^{\sharp}$ is $2n$-weakly amenable. Then $\mathfrak{A}$ is $2n$-weakly amenable.

(ii) Suppose that $\mathfrak{A}$ is $(2n-1)$-weakly amenable. Then $\mathfrak{A}^{\sharp}$ is $(2n-1)$-weakly amenable.

(iii) Suppose that $\mathfrak{A}$ is commutative. Then $\mathfrak{A}^{\sharp}$ is n-weakly amenable if and only if $\mathfrak{A}$ is n-weakly amenable.

In this paper we consider the converses to (i) and (ii) and give partial solutions to them. Let us recall some definitions.

Definition 1.1. *([6]) A Banach $\mathfrak{A}$-module $\mathbf{X}$ is called neo-unital if for each $x \epsilon \mathbf{X}$ there are $a, a' \epsilon \mathfrak{A}$ and $y, y' \epsilon \mathbf{X}$ with $x = ay = y'a'$.*

Definition 1.2. *([3]) A Banach algebra $\mathfrak{A}$ is called self-induced if $\mathfrak{A}$ and $\mathfrak{A} \hat{\bigotimes}_{\mathfrak{A}} \mathfrak{A}$ are naturally isomorphic.*

Here $\mathfrak{A}\hat{\bigotimes}_{\mathfrak{A}}\mathfrak{A} = \frac{\mathfrak{A}\hat{\bigotimes}\mathfrak{A}}{\mathbf{K}}$ where $\mathbf{K}$ is the closed linear span of $\{ab \otimes c - a \otimes bc : a, b, c\, \epsilon \mathfrak{A}\}$.
Now we proceed to state and prove our theorem.

Theorem 1.3. *Let $\mathfrak{A}$ be a non-unital Banach algebra and suppose that $\mathfrak{A}$ is self-induced.*
(**i**) *If $\mathfrak{A}^\sharp$ is $(2n-1)$-weakly amenable then $\mathfrak{A}$ is $(2n-1)$-weakly amenable.*
(**ii**) *If $\mathfrak{A}$ is $2n$-weakly amenable then $\mathfrak{A}^\sharp$ is $2n$-weakly amenable.*
(**iii**) $\mathcal{H}^2(\mathfrak{A}, \mathfrak{A}^{(2n)}) \cong \mathcal{H}^2(\mathfrak{A}^\sharp, {\mathfrak{A}^\sharp}^{(2n)})$

Proof. Clearly $\mathfrak{A}$ is a closed two-sided ideal in $\mathfrak{A}^\sharp$ with codimension one. We consider the corresponding short exact sequence and its iterated duals. That is,

$$0 \longrightarrow \mathfrak{A} \xrightarrow{\ i\ } \mathfrak{A}^\sharp \xrightarrow{\ \varphi\ } \mathbb{C} \longrightarrow 0$$

where $i : \mathfrak{A} \longrightarrow \mathfrak{A}^\sharp$ defined by $a \mapsto (a, 0)$ and $\varphi : \mathfrak{A}^\sharp \longrightarrow \mathbb{C}$ defined by $(a, \lambda) \mapsto \lambda$.

$$0 \longrightarrow \mathbb{C} \longrightarrow {\mathfrak{A}^\sharp}^{(2n-1)} \longrightarrow \mathfrak{A}^{(2n-1)} \longrightarrow 0 \tag{1.1}$$

$$0 \longrightarrow \mathfrak{A}^{(2n)} \longrightarrow {\mathfrak{A}^\sharp}^{(2n)} \longrightarrow \mathbb{C} \longrightarrow 0 \tag{1.2}$$

It is easy to see that i is an isometric isomorphism and φ is a character on $\mathfrak{A}^\sharp$ with $\ker\varphi = \mathfrak{A}$. Then we make $\mathbb{C}$ a module over $\mathfrak{A}^\sharp$. Indeed,

$$z \cdot (a, \lambda) = (a, \lambda) \cdot z = \varphi(a, \lambda)z = \lambda z$$

where $(a, \lambda)\epsilon\mathfrak{A}^\sharp$ and $z\epsilon\mathbb{C}$.
Now consider the long exact sequence of cohomology groups concerning to (1.1). That is,

$$\begin{aligned} \ldots \longrightarrow \mathcal{H}^m(\mathfrak{A}^\sharp, \mathbb{C}) &\longrightarrow \mathcal{H}^m(\mathfrak{A}^\sharp, {\mathfrak{A}^\sharp}^{(2n-1)}) \\ \longrightarrow \mathcal{H}^m(\mathfrak{A}^\sharp, \mathfrak{A}^{(2n-1)}) &\longrightarrow \mathcal{H}^{(m+1)}(\mathfrak{A}^\sharp, \mathbb{C}) \longrightarrow \ldots . \end{aligned} \tag{1.3}$$

Obviously $\mathfrak{A}$, $\mathfrak{A}^{(n)}$ and $\mathbb{C}$ are unital Banach $\mathfrak{A}^\sharp$-bimodules. So by [4, Theorem 2.3] we have,

$$\mathcal{H}^m(\mathfrak{A}^\sharp, \mathbb{C}) \cong \mathcal{H}^m(\mathfrak{A}, \mathbb{C}) \qquad \textit{and} \qquad \mathcal{H}^m(\mathfrak{A}^\sharp, \mathfrak{A}^{(2n-1)}) \cong \mathcal{H}^m(\mathfrak{A}, \mathfrak{A}^{2n-1}). \tag{1.4}$$

Therefore by substituting (1.4) in (1.3) we get,

$$\begin{aligned} \ldots \longrightarrow \mathcal{H}^m(\mathfrak{A}, \mathbb{C}) &\longrightarrow \mathcal{H}^m(\mathfrak{A}^\sharp, {\mathfrak{A}^\sharp}^{(2n-1)}) \\ \longrightarrow \mathcal{H}^m(\mathfrak{A}, \mathfrak{A}^{(2n-1)}) &\longrightarrow \mathcal{H}^{(m+1)}(\mathfrak{A}, \mathbb{C}) \longrightarrow \ldots . \end{aligned} \tag{1.5}$$

Since $\mathfrak{A}$ is self-induced then $\mathcal{H}^1(\mathfrak{A}, \mathbb{C}) = \mathcal{H}^2(\mathfrak{A}, \mathbb{C}) = (0)$ [4,Lemma 2.5](note that $\mathbb{C}$ is an annihilator $\mathfrak{A}$-bimodule). Hence by sequence (1.5) we obtain,

$$\mathcal{H}^1(\mathfrak{A}^\sharp, {\mathfrak{A}^\sharp}^{(2n-1)}) \cong \mathcal{H}^1(\mathfrak{A}, \mathfrak{A}^{(2n-1)}).$$

Obviously (**i**) holds.
For (**ii**) consider the long exact sequence of cohomology groups corresponding to the short exact sequence (1.2). That is,

$$\begin{aligned} \ldots \longrightarrow \mathcal{H}^m(\mathfrak{A}^\sharp, \mathfrak{A}^{(2n)}) \longrightarrow \mathcal{H}^m(\mathfrak{A}^\sharp, {\mathfrak{A}^\sharp}^{(2n)}) \\ \longrightarrow \mathcal{H}^m(\mathfrak{A}^\sharp, \mathbb{C}) \longrightarrow \mathcal{H}^{m+1}(\mathfrak{A}^\sharp, \mathfrak{A}^{(2n)}) \longrightarrow \ldots . \end{aligned} \tag{1.6}$$

Like before we have,

$$\mathcal{H}^m(\mathfrak{A}^\sharp, \mathbb{C}) \cong \mathcal{H}^m(\mathfrak{A}, \mathbb{C}) \qquad \textit{and} \qquad \mathcal{H}^m(\mathfrak{A}^\sharp, \mathfrak{A}^{(2n)}) \cong \mathcal{H}^m(\mathfrak{A}, \mathfrak{A}^{(2n)}). \tag{1.7}$$

By substituting (1.7) in (1.6) we get,

$$\begin{aligned} \ldots \longrightarrow \mathcal{H}^m(\mathfrak{A}, \mathfrak{A}^{(2n)}) \longrightarrow \mathcal{H}^m(\mathfrak{A}^\sharp, \mathfrak{A}_\sharp^{(2n)}) \longrightarrow \mathcal{H}^m(\mathfrak{A}, \mathbb{C}) \longrightarrow \\ \mathcal{H}^{m+1}(\mathfrak{A}, \mathfrak{A}^{(2n)}) \longrightarrow \mathcal{H}^{m+1}(\mathfrak{A}^\sharp, {\mathfrak{A}^\sharp}^{(2n)}) \longrightarrow \mathcal{H}^{m+1}(\mathfrak{A}, \mathbb{C}) \longrightarrow \ldots . \end{aligned} \tag{1.8}$$

Now if $\mathfrak{A}$ is $2n$-weakly amenable then self-inducement of $\mathfrak{A}$ and (1.8) imply

$$\mathcal{H}^1(\mathfrak{A}^\sharp, {\mathfrak{A}^\sharp}^{(2n)}) = (0).$$

So (**ii**) holds.
For (**iii**) self-inducement of $\mathfrak{A}$ and (1.8) imply

$$\mathcal{H}^2(\mathfrak{A}, \mathfrak{A}^{(2n)}) \cong \mathcal{H}^2(\mathfrak{A}^\sharp, {\mathfrak{A}^\sharp}^{(2n)})$$

□

A special case occurs when the Banach algebra $\mathfrak{A}$ has a left(right) bounded approximate identity. In this case we have the following result.

Proposition 1.4. *If the Banach algebra $\mathfrak{A}$ has a left(right) bounded approximate identity then the theorem holds.*

Proof. By [5, Proposition II.3.13] $\mathfrak{A} \bigotimes_{\mathfrak{A}} \mathfrak{A} \to \mathfrak{A}^2$ given by $a \otimes b \mapsto ab$ is a topological isomorphism. By [1,§11,corollary 11] $\mathfrak{A}^2 = \mathfrak{A}$. So $\mathfrak{A} \bigotimes_{\mathfrak{A}} \mathfrak{A} \cong \mathfrak{A}$. That is $\mathfrak{A}$ is self-induced. Hence the theorem holds. □

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