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Approximation in statistical sense by $n-$multiple sequences of fuzzy positive linear operators

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Abstract. Our primary interest in the present paper is to prove a Korovkin-type approximation theorem for $n-$multiple sequences of fuzzy positive linear operators via statistical convergence. Also, we display an example such that our method of convergence is stronger than the usual convergence.

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1. Introduction

Anastassiou [3] first introduced the fuzzy analogue of the classical Korovkin theory (see also [1], [2], [5], [12]). Recently, some statistical fuzzy approximation theorems have been obtain by using the concept of statistical convergence (see, [6], [8]). The main motivation of this work is the paper introduced by Duman [9]. In this paper, we prove a Korovkin-type approximation theorem in algebraic and trigonometric case for $n-$multiple sequences of fuzzy positive linear operators defined on the space of all real valued n-variate fuzzy continuous functions on a compact subset of the real n-dimensional space via statistical convergence. Also, we display an example such that our method of convergence is stronger than the usual convergence.

We now recall some basic definitions and notations used in the paper.

A fuzzy number is a function $\mu : \mathbb{R} \to [0,1]$, which is normal, convex, upper semi-continuous and the closure of the set $supp(\mu)$ is compact, where

$$supp(\mu) := \{x \in \mathbb{R} : \ \mu(x) > 0\}.$$

The set of all fuzzy numbers are denoted by $\mathbb{R}_{\mathcal{F}}$. Let

$$[\mu]^0 = \overline{\{x \in \mathbb{R} : \ \mu(x) > 0\}} \text{ and } [\mu]^r = \{x \in \mathbb{R} : \ \mu(x) \geq r\}, \ (0 < r \leq 1).$$

Then, it is well-known [13] that, for each $r \in [0,1]$, the set $[\mu]^r$ is a closed and bounded interval of $\mathbb{R}$. For any $u, v \in \mathbb{R}_{\mathcal{F}}$ and $\lambda \in \mathbb{R}$, it is possible to define uniquely the sum

$u \oplus v$ and the product $\lambda \odot u$ as follows:

$$[u \oplus v]^r = [u]^r + [v]^r \text{ and } [\lambda \odot u]^r = \lambda [u]^r, \ (0 \leq r \leq 1).$$

Now denote the interval $[u]^r$ by $\left[u_-^{(r)}, u_+^{(r)}\right]$, where $u_-^{(r)} \leq u_+^{(r)}$ and $u_-^{(r)}, u_+^{(r)} \in \mathbb{R}$ for $r \in [0,1]$. Then, for $u, v \in \mathbb{R}_{\mathcal{F}}$, define

$$u \preceq v \Leftrightarrow u_-^{(r)} \leq v_-^{(r)} \text{ and } u_+^{(r)} \leq v_+^{(r)} \text{ for all } 0 \leq r \leq 1.$$

Define also the following metric $D : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_+$ by

$$D(u,v) = \sup_{r \in [0,1]} \ \max \ \left\{ \left| u_-^{(r)} - v_-^{(r)} \right|, \left| u_+^{(r)} - v_+^{(r)} \right| \right\}$$

(see, for details [3]). Hence, $(\mathbb{R}_{\mathcal{F}}, D)$ is a complete metric space [18].

The concept of statistical convergence was introduced by ([10]). A sequence $x = (x_m)$ of real numbers is said to be statistical convergent to some finite number L, if for every $\varepsilon > 0$,

$$\lim_{k \to \infty} \frac{1}{k} \left| \{ m \leq k : \ |x_m - L| \geq \varepsilon \} \right| = 0,$$

where by $m \leq k$ we mean that $m = 1, 2, ..., k$; and by $|B|$ we mean the cardinality of the set $B \subseteq \mathbb{N}$, the set of natural numbers. We recall ([16], p. 290) that "natural (or asymptotic) density" of a set $B \subseteq \mathbb{N}$ is defined by

$$\delta(B) := \lim_{k \to \infty} \frac{1}{k} \left| \{ m \leq k : m \in B \} \right|,$$

provided that the limit on the right-hand side exists. It is clear that a set $B \subseteq \mathbb{N}$ has natural density 0 if and only if complement $B^c := \mathbb{N} \setminus B$ has natural density 1. Some basic properties of statistical convergence may be found in ([7], [11], [17]). These basic properties of statistical convergence were extended to $n-$multiple sequences by ([14], [15]). Let $\mathbb{N}^n$ be the set of $n-$tuples $\mathbf{m} := (m_1, m_2, ..., m_n)$ with non-negative integers for coordinates m_j, where n is a fixed positive integer. Two tuples $\mathbf{m}$ and $\mathbf{k} := (k_1, k_2, ..., k_n)$ are distinct if and only if $m_j \neq k_j$ for at least one j. $\mathbb{N}^n$ is partially ordered by agreeing that $\mathbf{m} \leq \mathbf{k}$ if and only if $m_j \leq k_j$ for each j.

We say that a $n-$multiple sequence $(x_{\mathbf{m}}) = (x_{m_1, m_2, ..., m_n})$ of real numbers is statistically convergent to some number L if for every $\varepsilon > 0$,

$$\lim_{\min \ k_j \to \infty} \frac{1}{|\mathbf{k}|} \left| \{ \mathbf{m} \leq \mathbf{k} : \ |x_{\mathbf{m}} - L| \geq \varepsilon \} \right| = 0,$$

where $|\mathbf{k}| := \prod_{j=1}^{n} (k_j)$. In this case, we write $st - \lim x_{\mathbf{m}} = L$. The "natural (or asymptotic) density" of a set $B \subseteq \mathbb{N}^n$ can be defined as follows:

$$\delta(B) := \lim_{\min \ k_j \to \infty} \frac{1}{|\mathbf{k}|} \left| \{ \mathbf{m} \leq \mathbf{k} : \ \mathbf{m} \in B \} \ \right|,$$

provided that this limit exists ([14]).

2. Statistical fuzzy Korovkin theory

Let the real numbers $a_i; b_i$ so that $a_i < b_i$, for each $i = \overline{1,n}$ and

$$U := [a_1; b_1] \times [a_2; b_2] \times ... \times [a_n; b_n] .$$

Let $C(U)$ denote the space of all real valued continuous functions on U endowed with the supremum norm

$$\|f\| = \sup_{\mathbf{x}\in U} |f(\mathbf{x})| , \ (f \in C(U)) .$$

Assume that $f : U \to \mathbb{R}_{\mathcal{F}}$ be a fuzzy number valued function. Then f is said to be fuzzy continuous at $\mathbf{x}^0 := (x_1^0, x_2^0, x_3^0, ..., x_n^0) \in U$ whenever $\lim_{\mathbf{m}} x_{\mathbf{m}} = \mathbf{x}^0$, then $\lim_{\mathbf{m}} D(f(x_{\mathbf{m}}), f(\mathbf{x}^0)) = 0$. If it is fuzzy continuous at every point $\mathbf{x} \in U$, we say that f is fuzzy continuous on U. The set of all fuzzy continuous functions on U is denoted by $C_{\mathcal{F}}(U)$. Now let $L : C_{\mathcal{F}}(U) \to C_{\mathcal{F}}(U)$ be an operator. Then L is said to be fuzzy linear if, for every $\lambda_1, \lambda_2 \in \mathbb{R}$ having the same sing and for every $f_1, f_2 \in C_{\mathcal{F}}(U)$, and $\mathbf{x} \in U$,

$$L(\lambda_1 \odot f_1 \oplus \lambda_2 \odot f_2; \mathbf{x}) = \lambda_1 \odot L(f_1; \mathbf{x}) \oplus \lambda_2 \odot L(f_2; \mathbf{x})$$

holds. Also L is called fuzzy positive linear operator if it is fuzzy linear and, the condition $L(f; \mathbf{x}) \preceq L(g; \mathbf{x})$ is satisfied for any $f, g \in C_{\mathcal{F}}(U)$ and all $\mathbf{x} \in U$ with $f(\mathbf{x}) \preceq g(\mathbf{x})$. Also, if $f, g : U \to \mathbb{R}_{\mathcal{F}}$ are fuzzy number valued functions, then the distance between f and g is given by

$$D^*(f, g) = \sup_{\mathbf{x}\in U} \sup_{r\in[0,1]} \max \left\{ \left| f_-^{(r)} - g_-^{(r)} \right| , \left| f_+^{(r)} - g_+^{(r)} \right| \right\}$$

(see for details, [1], [2], [3], [5], [9], [12]). Throughout the paper we use the test functions given by

$$f_0(\mathbf{x}) = 1, \ \ f_i(\mathbf{x}) = x_i, \ f_{n+i}(\mathbf{x}) = x_i^2, \ \ i = \overline{1,n}.$$

Theorem 2.1. *Let $\{L_{\mathbf{m}}\}_{\mathbf{m}\in\mathbb{N}^n}$ be a sequence of fuzzy positive linear operators from $C_{\mathcal{F}}(U)$ into itself. Assume that there exists a corresponding sequence $\left\{ \widetilde{L}_{\mathbf{m}} \right\}_{\mathbf{m}\in\mathbb{N}^n}$ of positive linear operators from $C(U)$ into itself with the property*

$$\{L_{\mathbf{m}}(f; \mathbf{x})\}_{\pm}^{(r)} = \widetilde{L}_{\mathbf{m}} \left(f_{\pm}^{(r)}; \mathbf{x} \right) \tag{2.1}$$

for all $\mathbf{x} \in U$, $r \in [0,1]$, $\mathbf{m} \in \mathbb{N}^n$ and $f \in C_{\mathcal{F}}(U)$. Assume further that

$$st - \lim_{\mathbf{m}} \left\| \widetilde{L}_{\mathbf{m}}(f_i) - f_i \right\| = 0 \ \ \text{for each } i = \overline{0, 2n}. \tag{2.2}$$

Then, for all $f \in C_{\mathcal{F}}(U)$, we have

$$st - \lim_{\mathbf{m}} D^*(L_{\mathbf{m}}(f), f) = 0.$$

Proof. Let $f \in C_{\mathcal{F}}(U)$, $\mathbf{x} = (x_1, ..., x_n) \in U$ and $r \in [0,1]$. By the hypothesis, since $f_{\pm}^{(r)} \in C(U)$, we can write, for every $\varepsilon > 0$, that there exists a number $\delta > 0$ such that $\left| f_{\pm}^{(r)}(\mathbf{u}) - f_{\pm}^{(r)}(\mathbf{x}) \right| < \varepsilon$ holds for every $\mathbf{u} = (u_1 ..., u_n) \in U$ satisfying

$$|\mathbf{u} - \mathbf{x}| := \sqrt{\sum_{i=1}^{n} (u_i - x_i)^2} < \delta.$$

Then we immediately get for all $\mathbf{u} \in U$, that

$$\left| f_{\pm}^{(r)}(\mathbf{u}) - f_{\pm}^{(r)}(\mathbf{x}) \right| \leq \varepsilon + \frac{2M_{\pm}^{(r)}}{\delta^2} \sum_{i=1}^{n} (u_i - x_i)^2,$$

where $M_{\pm}^{(r)} := \left\| f_{\pm}^{(r)} \right\|$. Now, using the linearity and the positivity of the operators $\widetilde{L}_{\mathbf{m}}$, we have, for each $\mathbf{m} \in \mathbb{N}^n$, that

$$\begin{aligned} & \left| \widetilde{L}_{\mathbf{m}}\left(f_{\pm}^{(r)}; \mathbf{x}\right) - f_{\pm}^{(r)}(\mathbf{x}) \right| \\ \leq \; & \varepsilon + \left(\varepsilon + M_{\pm}^{(r)} + \frac{2M_{\pm}^{(r)}}{\delta^2} \sum_{i=1}^{n} x_i^2 \right) \left| \widetilde{L}_{\mathbf{m}}(f_0; \mathbf{x}) - f_0(\mathbf{x}) \right| \\ & + \frac{2M_{\pm}^{(r)}}{\delta^2} \sum_{i=1}^{n} \left\{ \left| \widetilde{L}_{\mathbf{m}}\left(u_i^2; \mathbf{x}\right) - x_i^2 \right| + 2c \left| \widetilde{L}_{\mathbf{m}}(u_i; \mathbf{x}) - x_i \right| \right\} \end{aligned}$$

where $c := \max_{1 \leq i \leq n} \{|a_i|, |b_i|\}$. The last inequality gives that

$$\left| \widetilde{L}_{\mathbf{m}}\left(f_{\pm}^{(r)}; \mathbf{x}\right) - f_{\pm}^{(r)}(\mathbf{x}) \right| \leq \varepsilon + K_{\pm}^{(r)}(\varepsilon) \sum_{i=0}^{2n} \left| \widetilde{L}_{\mathbf{m}}(f_i; \mathbf{x}) - f_i(\mathbf{x}) \right|$$

where $K_{\pm}^{(r)}(\varepsilon) := \max \left\{ \varepsilon + M_{\pm}^{(r)} + \frac{2M_{\pm}^{(r)}}{\delta^2} A, \frac{4M_{\pm}^{(r)}}{\delta^2} c, \frac{2M_{\pm}^{(r)}}{\delta^2} \right\}$ and $A := \sum_{i=1}^{n} x_i^2$ for $x_i \in [a_i, b_i]$, $(i = 1, 2, ...n)$. Also taking supremum over $\mathbf{x} = (x_1 ..., x_n) \in U$, the above inequality implies that

$$\left\| \widetilde{L}_{\mathbf{m}}\left(f_{\pm}^{(r)}\right) - f_{\pm}^{(r)} \right\| \leq \varepsilon + K_{\pm}^{(r)}(\varepsilon) \sum_{i=0}^{2n} \left\| \widetilde{L}_{\mathbf{m}}(f_i) - f_i \right\| \tag{2.3}$$

Now, it follows from (2.1) that

$$\begin{aligned} & D^*(L_{\mathbf{m}}(f), f) \\ = \; & \sup_{\mathbf{x} \in U} \sup_{r \in [0,1]} \max \left\{ \left| \widetilde{L}_{\mathbf{m}}\left(f_{-}^{(r)}; \mathbf{x}\right) - f_{-}^{(r)}(\mathbf{x}) \right|, \left| \widetilde{L}_{\mathbf{m}}\left(f_{+}^{(r)}; \mathbf{x}\right) - f_{+}^{(r)}(\mathbf{x}) \right| \right\} \\ = \; & \sup_{r \in [0,1]} \max \left\{ \left\| \widetilde{L}_{\mathbf{m}}\left(f_{-}^{(r)}\right) - f_{-}^{(r)} \right\|, \left\| \widetilde{L}_{\mathbf{m}}\left(f_{+}^{(r)}\right) - f_{+}^{(r)} \right\| \right\}. \end{aligned}$$

Combining the above equality with (2.3), we have

$$D^{*}\left(L_{\mathbf{m}}\left(f\right),f\right)\leq\varepsilon+K\left(\varepsilon\right)\sum_{i=0}^{2n}\left\Vert \widetilde{L}_{\mathbf{m}}\left(f_{i}\right)-f_{i}\right\Vert \tag{2.4}$$

where $K\left(\varepsilon\right):=\sup\limits_{r\in[0,1]}\max\left\{K_{-}^{(r)}\left(\varepsilon\right),K_{+}^{(r)}\left(\varepsilon\right)\right\}$.

Now, for a given $r>0$, choose $\varepsilon>0$ such that $0<\varepsilon<r$, and also define the following sets:

$$\begin{aligned} G\ &:=\left\{\mathbf{m}\in\mathbb{N}^{n}:D^{*}\left(L_{\mathbf{m}}\left(f\right),f\right)\geq r\right\},\\ G_{i}\ &:=\left\{\mathbf{m}\in\mathbb{N}^{n}:\left\Vert \widetilde{L}_{\mathbf{m}}\left(f_{i}\right)-f_{i}\right\Vert \geq\frac{r-\varepsilon}{\left(2n+1\right)K\left(\varepsilon\right)}\right\},\qquad i=\overline{0,2n}. \end{aligned}$$

Hence, inequality (2.4) yields that

$$G\subset\bigcup_{i=0}^{2n}G_{i}$$

which gives,

$$\begin{aligned} &\lim_{\min\ k_{j}\to\infty}\frac{1}{\left|\mathbf{k}\right|}\left|\left\{\mathbf{m}\leq\mathbf{k}:D^{*}\left(L_{\mathbf{m}}\left(f\right),f\right)\geq r\right\}\right|\\ \leq\ &\lim_{\min\ k_{j}\to\infty}\frac{1}{\left|\mathbf{k}\right|}\left|\left\{\mathbf{m}\leq\mathbf{k}:\ \left\Vert \widetilde{L}_{\mathbf{m}}\left(f_{i}\right)-f_{i}\right\Vert \geq\frac{r-\varepsilon}{\left(2n+1\right)K\left(\varepsilon\right)}\right\}\right|,\qquad i=\overline{0,2n}. \end{aligned}$$

From the hypothesis (2.2), we get

$$\lim_{\min\ k_{j}\to\infty}\frac{1}{\left|\mathbf{k}\right|}\left|\left\{\mathbf{m}\leq\mathbf{k}:D^{*}\left(L_{\mathbf{m}}\left(f\right),f\right)\geq r\right\}\right|=0.$$

So, the proof is completed. □

If $n=1$**,** then Theorem 2.1 reduces to result of [6].

Theorem 2.2. *Let $\left\{L_{m}\right\}_{m\in\mathbb{N}}$ be a sequence of fuzzy positive linear operators from $C_{\mathcal{F}}\left(U\right)$ into itself. Assume that there exists a corresponding sequence $\left\{\widetilde{L}_{m}\right\}_{m\in\mathbb{N}}$ of positive linear operators from $C\left(U\right)$ into itself with the property* (2.1)*. Assume further that*

$$st-\lim_{m}\left\Vert \widetilde{L}_{m}\left(f_{i}\right)-f_{i}\right\Vert =0\ \ \textit{for each } i=0,1,2.$$

Then, for all $f\in C_{\mathcal{F}}\left(U\right)$, we have

$$st-\lim_{m}D^{*}\left(L_{m}\left(f\right),\ f\right)=0.$$

If $n=2$**,** then Theorem 2.1 reduces to new result in classical case.

Theorem 2.3. *Let $\left\{L_{\mathbf{m}}\right\}_{\mathbf{m}\in\mathbb{N}^{2}}$ be a sequence of fuzzy positive linear operators from $C_{\mathcal{F}}\left(U\right)$ into itself. Assume that there exists a corresponding sequence $\left\{\widetilde{L}_{\mathbf{m}}\right\}_{\mathbf{m}\in\mathbb{N}^{2}}$ of* *positive linear operators from $C(U)$ into itself with the property (2.1). Assume further that*

$$\lim_{\mathbf{m}} \left\| \widetilde{L}_{\mathbf{m}}(f_i) - f_i \right\| = 0 \quad \text{for each } i = 0,1,2,3,4.$$

Then, for all $f \in C_{\mathcal{F}}(U)$, we have

$$\lim_{\mathbf{m}} D^{*}\left(L_{\mathbf{m}}(f),\ f\right) = 0.$$

We now show that Theorem 2.1 stronger than Theorem 2.3.

Example 2.4. Let $n = 2$, $U := [0,1] \times [0,1]$ *and define the double sequence* $(u_{\mathbf{m}})$ *by*

$$u_{\mathbf{m}} = \begin{cases} \sqrt{m_1 m_2}, & \text{if } m_1 \text{ and } m_2 \text{ are square,} \\ 0, & \textit{otherwise.} \end{cases}$$

We observe that, $st - \lim_{\mathbf{m}} u_{\mathbf{m}} = 0$ *. But* $(u_{\mathbf{m}})$ *is neither convergent nor bounded. Then consider the Fuzzy Bernstein-type polynomials as follows:*

$$\begin{aligned} B_{\mathbf{m}}^{(\mathcal{F})}(f;\mathbf{x}) &= (1+u_{\mathbf{m}}) \odot \bigoplus_{s=0}^{m_1} \odot \bigoplus_{t=0}^{m_2} \tbinom{m_1}{s}\tbinom{m_2}{t} x_1^s x_2^t (1-x_1)^{m_1-s}(1-x_2)^{m_2-t} \\ &\quad \odot f\left(\tfrac{s}{m_1}, \tfrac{t}{m_2}\right), \end{aligned} \tag{2.5}$$

where $f \in C_{\mathcal{F}}(U)$, $\mathbf{x} = (x_1, x_2) \in U$, $\mathbf{m} \in \mathbb{N}^2$. *In this case, we write*

$$\begin{aligned} \left\{B_{\mathbf{m}}^{(\mathcal{F})}(f;\mathbf{x})\right\}_{\pm}^{(r)} &= \widetilde{B}_{\mathbf{m}}\left(f_{\pm}^{(r)};\mathbf{x}\right) \\ &= (1+u_{\mathbf{m}}) \sum_{s=0}^{m_1}\sum_{t=0}^{m_2} \binom{m_1}{s}\binom{m_2}{t} x_1^s x_2^t (1-x_1)^{m_1-s}(1-x_2)^{m_2-t} \\ &\qquad f_{\pm}^{(r)}\left(\frac{s}{m_1}, \frac{t}{m_2}\right), \end{aligned}$$

where $f_{\pm}^{(r)} \in C(U)$. *Then, we get*

$$\begin{aligned} \widetilde{B}_{\mathbf{m}}(f_0;\mathbf{x}) &= (1+u_{\mathbf{m}}) f_0(\mathbf{x}), \\ \widetilde{B}_{\mathbf{m}}(f_1;\mathbf{x}) &= (1+u_{\mathbf{m}}) f_1(\mathbf{x}), \\ \widetilde{B}_{\mathbf{m}}(f_2;\mathbf{x}) &= (1+u_{\mathbf{m}}) f_2(\mathbf{x}), \\ \widetilde{B}_{\mathbf{m}}(f_3;\mathbf{x}) &= (1+u_{\mathbf{m}}) \left(f_3(\mathbf{x}) + \frac{x_1 - x_1^2}{m_1}\right) \\ \widetilde{B}_{\mathbf{m}}(f_4;\mathbf{x}) &= (1+u_{\mathbf{m}}) \left(f_4(\mathbf{x}) + \frac{x_2 - x_2^2}{m_2}\right). \end{aligned}$$

So we conclude that

$$st - \lim_{\mathbf{m}} \left\| \widetilde{B}_{\mathbf{m}}(f_i) - f_i \right\| = 0 \quad \text{for each } i = 0,1,2,3,4.$$

By Theorem 2.1, we obtain for all $f \in C_{\mathcal{F}}(U)$, that

$$st - \lim_{\mathbf{m}} D^{*}\left(B_{\mathbf{m}}^{(\mathcal{F})}(f),\ f\right) = 0.$$

However, since the sequence $(u_{\mathbf{m}})$ *is not convergent, we conclude that Theorem* 2.3 *do not work for the operators* $\left\{B_{\mathbf{m}}^{(\mathcal{F})}(f;\mathbf{x})\right\}$ *in (*2.5*) while our Theorem* 2.1 *still works.*

Remark 2.5. Let $C_{2\pi}(\mathbb{R}^n)$ denote the space of all real valued continuous and 2π-periodic functions on $\mathbb{R}^n$, $(n \in \mathbb{N})$. By $C_{2\pi}^{\mathcal{F}}(\mathbb{R}^n)$ we denote the space of all fuzzy continuous and 2π-periodic functions on $\mathbb{R}^n$. (see for details [4]). If we use the following test functions

$$f_0(\mathbf{x}) = 1, \;\; f_i(\mathbf{x}) = \cos x_i, \; f_{n+i}(\mathbf{x}) = \sin x_i, \;\; i = \overline{1, n},$$

then the proof of Theorem 2.1 can easily be modified to trigonometric case.

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