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On new Hermite Hadamard Fejér type integral inequalities

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Abstract. In this paper, we establish several weighted inequalities for some differantiable mappings that are connected with the celebrated Hermite-Hadamard Fejér type integral inequality. The results presented here would provide extensions of those given in earlier works.

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1. Introduction

The following inequality is well known in the literature as the Hermite-Hadamard integral inequality (see, [12]):

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

where $f : I \subset \mathbb{R} \to \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$.

The most well-known inequalities related to the integral mean of a convex function f are the Hermite Hadamard inequalities or its weighted versions, the so-called Hermite Hadamard Fejér inequalities. In [4], Fejer gave a weighted generalizatinon of the inequalities (1.1) as the following:

Theorem 1.1. *$f : [a, b] \to \mathbb{R}$, be a convex function, then the inequality*

$$f\left(\frac{a+b}{2}\right)\int_a^b w(x)dx \leq \frac{1}{b-a}\int_a^b f(x)w(x)dx \leq \frac{f(a)+f(b)}{2}\int_a^b w(x)dx \tag{1.2}$$

holds, where $w : [a, b] \to \mathbb{R}$ is nonnegative, integrable, and symmetric about $x = \frac{a+b}{2}$.

For some results which generalize, improve, and extend the inequalities (1.1) and (1.2), (see [1]-[3], [5]-[11], [13], [15] and [16]).

In [2] in order to prove some inequalities related to Hadamard's inequality Dragomir and Agarwal used the following lemma.

Lemma 1.2. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$, be a differentiable mapping on I°, $a, b \in I$ (I° is the interior of I) with $a < b$. If $f' \in L([a, b])$, then we have*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx = \frac{b-a}{2}\int_0^1 (1-2t)\, f'(ta+(1-t)b)dt. \tag{1.3}$$

Theorem 1.3. *([2]) Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$, $f' \in L(a, b)$ and $p > 1$. If the mapping $|f'|^{p/(p-1)}$ is convex on $[a, b]$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{2(p+1)^{1/p}}\left(\frac{|f'(a)|^{p/(p-1)} + |f'(b)|^{p/(p-1)}}{2}\right)^{(p-1)/p}. \tag{1.4}$$

In [9] some inequalities of Hermite-Hadamard type for differentiable convex mappings were proved using the following lemma.

Lemma 1.4. *Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$, be a differentiable mapping on I°, $a, b \in I^\circ$ (I° is the interior of I) with $a < b$. If $f' \in L([a, b])$, then we have*

$$\begin{aligned} &\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) \\ &= (b-a)\left[\int_0^{\frac{1}{2}} tf'(ta+(1-t)b)dt + \int_{\frac{1}{2}}^1 (t-1)\, f'(ta+(1-t)b)dt\right]. \end{aligned} \tag{1.5}$$

One more general result related to (1.5) was established in [10]. The main result in [9] is as follows:

Theorem 1.5. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$, be a differentiable mapping on I°, $a, b \in I$ with $a < b$. If the mapping $|f'|$ is convex on $[a, b]$, then*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \leq \frac{b-a}{4}\left(\frac{|f'(a)| + |f'(b)|}{2}\right). \tag{1.6}$$

In this article, using functions whose derivatives absolute values are convex, we obtained new inequalities of weighted Hermite-Hadamard type. The results presented here would provide extensions of those given in earlier works.

2. Main results

We will establish some new results connected with the left-hand side of (1.2) used the following Lemma. Now, we give the following new Lemma for our results:

Lemma 2.1. *Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$, and $w : [a, b] \to [0, \infty)$ be a differentiable mapping. If $f' \in L[a, b]$, then the following equality holds:*

$$\frac{1}{b-a}\int_a^b f(x)w(x)dx - \frac{1}{b-a}f\left(\frac{a+b}{2}\right)\int_a^b w(x)dx = (b-a)\int_0^1 k(t)f'(ta+(1-t)b)dt \tag{2.1}$$

for each $t \in [0, 1]$, where

$$k(t) = \begin{cases} \int_0^t w(as+(1-s)b)ds, & t \in [0, \frac{1}{2}) \\ -\int_t^1 w(as+(1-s)b)ds, & t \in [\frac{1}{2}, 1]. \end{cases}$$

Proof. It suffices to note that

$$\begin{aligned} I &= \int_0^1 k(t)f'(ta+(1-t)b)dt \\ &= \int_0^{\frac{1}{2}} \left(\int_0^t w(as+(1-s)b)ds\right) f'(ta+(1-t)b)dt \\ &+ \int_{\frac{1}{2}}^1 \left(-\int_t^1 w(as+(1-s)b)ds\right) f'(ta+(1-t)b)dt \\ &= I_1 + I_2. \end{aligned}$$

By integration by parts, we get

$$\begin{aligned} I_1 &= \left(\int_0^t w(as+(1-s)b)ds\right) \frac{f(ta+(1-t)b)}{a-b}\Bigg|_0^{\frac{1}{2}} \\ &- \int_0^{\frac{1}{2}} w(ta+(1-t)b)\frac{f(ta+(1-t)b)}{a-b}dt \\ &= \left(\int_0^{\frac{1}{2}} w(as+(1-s)b)ds\right) \frac{f(\frac{a+b}{2})}{a-b} \\ &- \int_0^{\frac{1}{2}} w(ta+(1-t)b)\frac{f(ta+(1-t)b)}{a-b}dt, \end{aligned}$$

and similarly

$$I_2 = \left(\int_{\frac{1}{2}}^1 w(as+(1-s)b)ds\right) \frac{f(\frac{a+b}{2})}{a-b} - \int_{\frac{1}{2}}^1 w(ta+(1-t)b)\frac{f(ta+(1-t)b)}{a-b}dt.$$

Thus, we can write

$$I = I_1 + I_2 = \left(\int_0^1 w(as+(1-s)b)ds\right) \frac{f(\frac{a+b}{2})}{a-b} - \int_0^1 w(ta+(1-t)b)\frac{f(ta+(1-t)b)}{a-b}dt.$$

Using the change of the variable $x = ta + (1-t)b$ for $t \in [0, 1]$, and multiplying the both sides by $(b-a)$, we obtain (2.1) which completes the proof. □

Remark 2.2. If we take $w(x) = 1$ in Lemma 2.1, then (2.1) reduces to (1.5).

Now, by using the above lemma, we prove our main theorems:

Theorem 2.3. *Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$, and $w : [a, b] \to [0, \infty)$ be a differentiable mapping and symmetric to $\frac{a+b}{2}$. If $|f'|$ is convex on $[a, b]$, then the following inequality holds:*

$$\left|\frac{1}{b-a}\int_a^b f(x)w(x)dx - \frac{1}{b-a}f\left(\frac{a+b}{2}\right)\int_a^b w(x)dx\right|$$
$$\le \left(\frac{1}{(b-a)^2}\int_{\frac{a+b}{2}}^b w(x)\left[(x-a)^2-(b-x)^2\right]dx\right)\left(\frac{|f'(a)|+|f'(b)|}{2}\right) \tag{2.2}$$

Proof. From Lemma 2.1 and the convexity of $|f'|$, it follows that

$$\left|\frac{1}{b-a}\int_a^b f(x)w(x)dx - \frac{1}{b-a}f\left(\frac{a+b}{2}\right)\int_a^b w(x)dx\right|$$
$$\le (b-a)\left\{\int_0^{\frac{1}{2}}\left(\int_0^t w(as+(1-s)b)ds\right)\left[t\left|f'(a)\right|+(1-t)\left|f'(b)\right|\right]dt\right. \tag{2.3}$$
$$\left.+\int_{\frac{1}{2}}^1\left(\int_t^1 w(as+(1-s)b)ds\right)\left[t\left|f'(a)\right|+(1-t)\left|f'(b)\right|\right]dt\right\}$$
$$= Q_1 + Q_2.$$

By change of the order of integration, we have

$$Q_1 = \int_0^{\frac{1}{2}}\int_0^t w(as+(1-s)b)\left(t\left|f'(a)\right|+(1-t)\left|f'(b)\right|\right)dsdt$$
$$= \int_0^{\frac{1}{2}}\int_s^{\frac{1}{2}} w(as+(1-s)b)\left(t\left|f'(a)\right|+(1-t)\left|f'(b)\right|\right)dtds$$
$$= \int_0^{\frac{1}{2}} w(as+(1-s)b)\left[\left(\tfrac{1}{8}-\tfrac{s^2}{2}\right)\left|f'(a)\right|+\left(\tfrac{(1-s)^2}{2}-\tfrac{1}{8}\right)\left|f'(b)\right|\right]ds.$$

and using the change of the variable $x = as + (1-s)b$ for $s \in [0, 1]$,

$$Q_1 = \frac{1}{8(b-a)^3}\int_{\frac{a+b}{2}}^b w(x)\left[\left((b-a)^2-4(b-x)^2\right)\left|f'(a)\right|\right.$$
$$\left.+\left(4(x-a)^2-(b-a)^2\right)\left|f'(b)\right|\right]dx \tag{2.4}$$

Similarly, by change of order of the integration, we obtain

$$Q_2 = \int_{\frac{1}{2}}^1 w(as+(1-s)b)\left[\left(\tfrac{s^2}{2}-\tfrac{1}{8}\right)\left|f'(a)\right|+\left(\tfrac{1}{8}-\tfrac{(1-s)^2}{2}\right)\left|f'(b)\right|\right]ds$$
$$= \frac{1}{8(b-a)^3}\int_a^{\frac{a+b}{2}} w(x)\left[\left(4(b-x)^2-(b-a)^2\right)\left|f'(a)\right|\right.$$
$$\left.+\left((b-a)^2-4(x-a)^2\right)\left|f'(b)\right|\right]dx.$$

Since $w(x)$ is symmetric to $x = \frac{a+b}{2}$, for $w(x) = w(a+b-x)$, we write

$$Q_2 = Q_1 \tag{2.5}$$

A combination of (2.3), (2.4) and (2.5), we get (2.2). This completes the proof. □

Remark 2.4. If we take $w(x) = 1$ in Theorem 2.3, then (2.2) reduces to (1.6).

Theorem 2.5. *Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$, and $w : [a,b] \to [0,\infty)$ be a differentiable mapping and symmetric to $\frac{a+b}{2}$. If $|f'|^q$ is convex on $[a,b]$, $q > 1$, then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)w(x)dx - \frac{1}{b-a}f\left(\frac{a+b}{2}\right)\int_a^b w(x)dx\right| \\
&\le (b-a)\left(\frac{1}{(b-a)^2}\int_{\frac{a+b}{2}}^b \left(x-\frac{a+b}{2}\right)w^p(x)dx\right)^{\frac{1}{p}} \\
&\times\left[\left(\frac{|f'(a)|^q + 2|f'(b)|^q}{24}\right)^{\frac{1}{q}} + \left(\frac{2|f'(a)|^q + |f'(b)|^q}{24}\right)^{\frac{1}{q}}\right].
\end{aligned} \tag{2.6}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. From Lemma 2.1 and using change of the order of integration, we get

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)w(x)dx - \frac{1}{b-a}f\left(\frac{a+b}{2}\right)\int_a^b w(x)dx\right| \\
&\le (b-a)\left\{\left[\int_0^{\frac{1}{2}}\left(\int_0^t w(as+(1-s)b)ds\right)|f'(ta+(1-t)b)|\,dt\right]\right. \\
&\left.+\left[\int_{\frac{1}{2}}^1\left(\int_t^1 w(as+(1-s)b)ds\right)|f'(ta+(1-t)b)|\,dt\right]\right\} \\
&= (b-a)\left\{\left[\int_0^{\frac{1}{2}}\int_s^{\frac{1}{2}} w(as+(1-s)b)\,|f'(ta+(1-t)b)|\,dtds\right]\right. \\
&\left.+\left[\int_{\frac{1}{2}}^1\int_{\frac{1}{2}}^s w(as+(1-s)b)\,|f'(ta+(1-t)b)|\,dtds\right]\right\}.
\end{aligned}$$

By Hölder's inequality, it follows that

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)w(x)dx - \frac{1}{b-a}f\left(\frac{a+b}{2}\right)\int_a^b w(x)dx\right| \\
&\le (b-a)\left\{\left(\int_0^{\frac{1}{2}}\int_s^{\frac{1}{2}} w^p(as+(1-s)b)dtds\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\int_s^{\frac{1}{2}}|f'(ta+(1-t)b)|^q\,dtds\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_{\frac{1}{2}}^1\int_{\frac{1}{2}}^s w^p(as+(1-s)b)dtds\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\int_{\frac{1}{2}}^s|f'(ta+(1-t)b)|^q\,dtds\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

Since $|f'|^q$ is convex on $[a,b]$, we know that for $t\in[0,1]$

$$|f'(ta+(1-t)b)|^q \le t\,|f'(a)|^q + (1-t)\,|f'(b)|^q,$$

hence

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(x)w(x)dx - \frac{1}{b-a}f\left(\frac{a+b}{2}\right)\int_a^b w(x)dx\right|\\ &\le (b-a)\left\{\left(\int_0^{\frac12}\int_s^{\frac12} w^p(as+(1-s)b)dtds\right)^{\frac1p}\left(\int_0^{\frac12}\int_s^{\frac12}\left(t\,|f'(a)|^q+(1-t)\,|f'(b)|^q\right)dtds\right)^{\frac1q}\right.\\ &\left.+\left(\int_{\frac12}^1\int_{\frac12}^s w^p(as+(1-s)b)dtds\right)^{\frac1p}\left(\int_{\frac12}^1\int_{\frac12}^s\left(t\,|f'(a)|^q+(1-t)\,|f'(b)|^q\right)dtds\right)^{\frac1q}\right\}\\ &= R_1+R_2.\end{aligned} \tag{2.7}$$

where $\frac1p+\frac1q=1$. Now, solving the above integrals with the elementary integrals, respectively, we obtain

$$R_1=\left(\frac{1}{2(b-a)^2}\int_{\frac{a+b}{2}}^b (2x-a-b)\,w^p(x)dx\right)^{\frac1p}\left(\frac{|f'(a)|^q+2\,|f'(b)|^q}{24}\right)^{\frac1q} \tag{2.8}$$

and

$$R_2=\left(\frac{1}{2(b-a)^2}\int_a^{\frac{a+b}{2}} (a+b-2x)\,w^p(x)dx\right)^{\frac1p}\left(\frac{2\,|f'(a)|^q+|f'(b)|^q}{24}\right)^{\frac1q}. \tag{2.9}$$

Since $w(x)$ is symmetric to $x=\frac{a+b}{2}$, we write

$$R_2=\left(\frac{1}{2(b-a)^2}\int_a^{\frac{a+b}{2}} (a+b-2x)\,w^p(a+b-x)dx\right)^{\frac1p}=R_1 \tag{2.10}$$

Using (2.8), (2.9) and (2.10), we get (2.6). Hence, the inequality (2.6) is proved. □

Now, we will give some new results connected with the right-hand side of (1.2) used the following Lemma:

Lemma 2.6. *Let $f: I^\circ\subset\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I°, $a,b\in I^\circ$ with $a<b$, and $w:[a,b]\to[0,\infty)$ be a differentiable mapping. If $f'\in L[a,b]$, then the following equality holds:*

$$\frac{1}{b-a}\frac{f(a)+f(b)}{2}\int_a^b w(x)dx-\frac{1}{b-a}\int_a^b f(x)w(x)dx=\frac{(b-a)}{2}\int_0^1 p(t)f'(ta+(1-t)b)dt \tag{2.11}$$

for each $t\in[0,1]$, where

$$p(t)=\int_t^1 w(as+(1-s)b)ds-\int_0^t w(as+(1-s)b)ds.$$

Proof. It suffices to note that

$$\begin{aligned} J &= \int_0^1 p(t) f'(ta+(1-t)b)dt \\ &= \int_0^1 \left(\int_t^1 w(as+(1-s)b)ds \right) f'(ta+(1-t)b)dt \\ &\quad + \int_0^1 \left(-\int_0^t w(as+(1-s)b)ds \right) f'(ta+(1-t)b)dt \\ &= J_1 + J_2. \end{aligned}$$

By integration by parts, we get

$$\begin{aligned} J_1 &= \left(\int_t^1 w(as+(1-s)b)ds \right) \frac{f(ta+(1-t)b)}{a-b} \Bigg|_0^1 \\ &\quad + \int_0^1 w(ta+(1-t)b) \frac{f(ta+(1-t)b)}{a-b} dt \\ &= -\left(\int_0^1 w(as+(1-s)b)ds \right) \frac{f(b)}{a-b} \\ &\quad + \int_0^1 w(ta+(1-t)b) \frac{f(ta+(1-t)b)}{a-b} dt, \end{aligned}$$

and similarly

$$J_2 = -\left(\int_0^1 w(as+(1-s)b)ds \right) \frac{f(a)}{a-b} + \int_0^1 w(ta+(1-t)b) \frac{f(ta+(1-t)b)}{a-b} dt.$$

Thus, we can write

$$\begin{aligned} J &= J_1 + J_2 \\ &= 2\int_0^1 w(ta+(1-t)b) \frac{f(ta+(1-t)b)}{a-b} dt - \left(\int_0^1 w(as+(1-s)b)ds \right) \frac{f(a)+f(b)}{a-b}. \end{aligned}$$

Using the change of the variable $x = ta+(1-t)b$ for $t \in [0,1]$, and multiplying the both sides by $\frac{(b-a)}{2}$, we obtain (2.11), which completes the proof. □

Remark 2.7. If we take $w(x) = 1$ in Lemma 2.6, then (2.11) reduces to (1.3).

Theorem 2.8. *Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$, and $w : [a,b] \to [0,\infty)$ be a differentiable mapping and symmetric to $\frac{a+b}{2}$. If $|f'|$ is convex on $[a,b]$, then the following inequality holds:*

$$\begin{aligned} &\left| \frac{1}{b-a} \frac{f(a)+f(b)}{2} \int_a^b w(x)dx - \frac{1}{b-a} \int_a^b f(x)w(x)dx \right| \\ &\leq \frac{1}{2} \left[\int_0^1 (g(x))^p \, dt \right]^{\frac{1}{p}} \left(\frac{|f'(a)|^q + |f'(b)|^q}{2} \right)^{\frac{1}{q}} \qquad (2.12) \end{aligned}$$

where $g(x) = \left| \int_{a+(b-a)t}^{b-(b-a)t} w(x)dx \right|$ for $t \in [0,1]$.

Proof. From Lemma 2.6, we get

$$\left|\frac{1}{b-a}\frac{f(a)+f(b)}{2}\int_a^b w(x)dx-\frac{1}{b-a}\int_a^b f(x)w(x)dx\right|$$
$$\leq\frac{(b-a)}{2}\left[\int_0^1\left|\int_t^1 w(as+(1-s)b)ds-\int_0^t w(as+(1-s)b)ds\right|\left|f'(ta+(1-t)b)\right|dt\right]$$
$$\leq\frac{1}{2}\left[\int_0^1\left|\int_a^{b-(b-a)t} w(x)dx-\int_{b-(b-a)t}^b w(x)dx\right|\left|f'(ta+(1-t)b)\right|dt\right]. \tag{2.13}$$

Since $w(x)$ is symmetric to $x=\frac{a+b}{2}$, we write

$$\int_a^{b-(b-a)t} w(x)dx-\int_{b-(b-a)t}^b w(x)dx=\int_{a+(b-a)t}^{b-(b-a)t} w(x)dx, \tag{2.14}$$

for $t\in[0,\frac{1}{2}]$ and

$$\int_a^{b-(b-a)t} w(x)dx-\int_{b-(b-a)t}^b w(x)dx=-\int_{b-(b-a)t}^{a+(b-a)t} w(x)dx, \tag{2.15}$$

for $t\in[\frac{1}{2},1]$. If we write (2.14) and (2.15) in (2.13), we have

$$\begin{aligned}&\left|\frac{1}{b-a}\frac{f(a)+f(b)}{2}\int_a^b w(x)dx-\frac{1}{b-a}\int_a^b f(x)w(x)dx\right|\\ \leq\ &\frac{1}{2}\left[\int_0^1 g(x)\left|f'(ta+(1-t)b)\right|dt\right].\end{aligned}$$

where $g(x)=\left|\int_{a+(b-a)t}^{b-(b-a)t} w(x)dx\right|$. By Hölder's inequality, it follows that

$$\begin{aligned}&\left|\frac{1}{b-a}\frac{f(a)+f(b)}{2}\int_a^b w(x)dx-\frac{1}{b-a}\int_a^b f(x)w(x)dx\right|\\ \leq\ &\frac{1}{2}\left[\int_0^1 (g(x))^p dt\right]^{\frac{1}{p}}\left[\int_0^1 \left|f'(ta+(1-t)b)\right|^q dt\right]^{\frac{1}{q}}.\end{aligned}$$

Since $|f'|^q$ is convex on $[a,b]$, we know that for $t\in[0,1]$

$$\left|f'(ta+(1-t)b)\right|^q\leq t\left|f'(a)\right|^q+(1-t)\left|f'(b)\right|^q,$$

hence

$$\left|\frac{1}{b-a}\frac{f(a)+f(b)}{2}\int_a^b w(x)dx-\frac{1}{b-a}\int_a^b f(x)w(x)dx\right|$$
$$\leq\frac{1}{2}\left[\int_0^1 (g(x))^p dt\right]^{\frac{1}{p}}\left(\int_0^1\left(t\left|f'(a)\right|^q+(1-t)\left|f'(b)\right|^q\right)dt\right)^{\frac{1}{q}}$$
$$=\frac{1}{2}\left[\int_0^1 (g(x))^p dt\right]^{\frac{1}{p}}\left(\frac{\left|f'(a)\right|^q+\left|f'(b)\right|^q}{2}\right)^{\frac{1}{q}}$$

which completes the proof. □

Remark 2.9. If we take $w(x) = 1$ in Theorem 2.8, since

$$\int_0^1 \left(\left| \int_{a+(b-a)t}^{b-(b-a)t} dx \right| \right)^p dt = (b-a)^p \int_0^1 |1-2t|^p \, dt = \frac{(b-a)^p}{(p+1)},$$

(2.12) reduces to (1.4).

3. An application

Let d be a division $a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ of the interval $[a, b]$ and $\xi = (\xi_0, ..., \xi_{n-1})$ a sequence of intermediate points, $\xi_i \in [x_i, x_{i+1}]$, $i = \overline{0, n-1}$. Then the following result holds:

Theorem 3.1. *Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$, $f' \in L[a, b]$ and $w : [a, b] \to [0, \infty)$ be a differentiable mapping. If $|f'|$ is convex on $[a, b]$ then we have*

$$\int_a^b f(u)w(u)du = A(f, w, d, \xi) + R(f, w, d, \xi)$$

where

$$A(f, w, d, \xi) := \sum_{i=0}^{n-1} \frac{1}{(x_{i+1} - x_i)} f(\frac{x_i + x_{i+1}}{2}) \left(\int_{x_i}^{x_{i+1}} w(u)du \right).$$

The remainder $R(f, w, d, \xi)$ satisfies the estimation:

$$|R(f, f', d, \xi)|$$

$$\leq \sum_{i=0}^{n-1} \left[\frac{1}{(x_{i+1} - x_i)} \int_{\frac{x_i + x_{i+1}}{2}}^{x_{i+1}} w(u) \left[(u - x_i)^2 - (x_{i+1} - u)^2 \right] du \right] \left(\frac{|f'(x_i)| + |f'(x_{i+1})|}{2} \right) \tag{3.1}$$

for any choice ξ of the intermediate points.

Proof. Apply Theorem 2.3 on the interval $[x_i, x_{i+1}]$, $i = \overline{0, n-1}$ to get

$$\left| \int_{x_i}^{x_{i+1}} f(u)w(u)du - f(\frac{x_i + x_{i+1}}{2}) \left(\int_{x_i}^{x_{i+1}} w(u)du \right) \right|$$

$$\leq \left[\frac{1}{(x_{i+1} - x_i)} \int_{\frac{x_i + x_{i+1}}{2}}^{x_{i+1}} w(u) \left[(u - x_i)^2 - (x_{i+1} - u)^2 \right] du \right] \left(\frac{|f'(x_i)| + |f'(x_{i+1})|}{2} \right).$$

□

Summing the above inequalities over i from 0 to $n-1$ and using the generalized triangle inequality, we get the desired estimation (3.1).

References

[1] Saglam, A., Sarikaya, M.Z., Yıldırım, H., *Some new inequalities of Hermite-Hadamard's type*, Kyungpook Mathematical Journal, **50**(2010), 399-410.

[2] Dragomir, S.S., Agarwal, R.P., *Two inequalities for differentiable mappings and applications to special means of real numbers and trapezoidal formula*, Appl. Math. Lett., **11**(5)(1998), 91–95.

[3] Dragomir, S.S., *Two mappings in connection to Hadamard's inequalities*, J. Math. Anal. Appl., **167**(1992), 42–56.

[4] Fejer, L., *Über die Fourierreihen,* II. Math. Naturwiss. Anz Ungar. Akad. Wiss., **24**(1906), 369–390. (Hungarian).

[5] Hwang, D.-Y., *Some inequalities for differentiable convex mapping with application to weighted trapezoidal formula and higher moments of random variables,* Appl. Math. Comp., **217**(23)(2011), 9598-9605.

[6] Lee, K.C., Tseng, K.L., *On a weighted generalization of Hadamard's inequality for G-convex functions*, Tamsui-Oxford J. Math. Sci., **16**(1)(2000), 91–104.

[7] Lupas, A., *A generalization of Hadamard's inequality for convex functions*, Univ. Beograd. Publ. Elek. Fak. Ser. Mat. Fiz., **544-576**(1976), 115–121.

[8] Pearce, C.E.M., Pečarić, J., *Inequalities for differentiable mappings with application to special means and quadrature formulae*, Appl. Math. Lett., **13**(2)(2000), 51–55.

[9] Kırmacı, U.S., *Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula*, Appl. Math. Comp., **147**(1)(2004), 137-146.

[10] Kırmacı, U.S., Özdemir, M.E., *On some inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula*, Appl. Math. Comp., **153**(2)(2004), 361-368.

[11] Qi, F., Wei, Z.-L., Yang, Q., *Generalizations and refinements of Hermite–Hadamard's inequality*, Rocky Mountain J. Math., **35**(2005), 235–251.

[12] Pečarić, J., Proschan, F., Tong, Y.L., *Convex functions, partial ordering and statistical applications*, Academic Press, New York, 1991.

[13] Sarıkaya, M.Z., Aktan, N., *On the generalization some integral inequalities and their applications* Mathematical and Computer Modelling, **54**(9-10)(2011), 2175-2182.

[14] Sarikaya, M.Z., Avci, M., Kavurmaci, H., *On some inequalities of Hermite-Hadamard type for convex functions*, ICMS International Conference on Mathematical Science, AIP Conference Proceedings 1309, **852**(2010).

[15] Wang, C.-L., Wang, X.-H., *On an extension of Hadamard inequality for convex functions*, Chin. Ann. Math., **3**(1982), 567–570.

[16] Wu, S.-H., *On the weighted generalization of the Hermite-Hadamard inequality and its applications*, The Rocky Mountain J. of Math., **39**(2009), no. 5, 1741–1749.

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