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Looking for an exact difference formula for the Dini-Hadamard-like subdifferential

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Abstract. We use in this paper a new concept of a directional subdifferential, namely the *Dini-Hadamard-like ε-subdifferential*, recently introduced in [29], in order to provide a subdifferential formula for the difference of two directionally approximately starshaped functions (a valuable class of nonsmooth functions, see for instance [32]), under weaker conditions than those presented in [7]. As a consequence, we furnish necessary and sufficient optimality conditions for a nonsmooth optimization problem having the difference of two functions as objective.

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1. Introduction

Since the early 1960's there has been a good amount of interest in generalizations of the pointwise derivative for the purposes of optimization. This has lead to many definitions of *generalized gradients*, *subgradients* and other kind of objects under various names. And all this work in order to solve optimization problems where classical differentiability assumptions are no longer appropriate. One of the most widely used *subdifferential* (set of subgradients) is the one who first appeared for convex functions in the context of convex analysis (see for more details [28, 38, 39] and the references therein). It has found many significant theoretical and practical uses in optimization, economics, mechanics and has proven to be a very interesting mathematical construct. But the attempt to extend this success to functions which are no more convex has proven to be more difficult. We mention here two main approaches.

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The first one uses a *generalized directional derivative* f^∂ of $f : X \to \mathbb{R} \cup \{+\infty\}$ of some type and then defines the subdifferential via the formula

$$\partial f(x) := \{x^* \in X^* : x^* \leq f^\partial(x, \cdot)\}, \tag{1.1}$$

where X^* is the topological dual of X. It is worth mentioning here that any subdifferential construction generated by a *polarity* relation like (1.1) is automatically convex regardless of the convexity of the generating directional derivative. As an example, the *Clarke subdifferential*, who in fact uses a positively homogeneous directional derivative, was the first concept of a subdifferential defined for a general nonconvex function and has been introduced in 1973 by Clarke (see for instance [9, 10]) who performed a real pioneering work in the field of *nonsmooth analysis*, spread far beyond the scope of convexity. But unfortunately, as stated in [4], at some *abnormal points* of certain even Lipschitzian nonsmooth functions, the Clarke subdifferential may include some *extraneous subgradients*. And this because, in general, a convex set often provides a subdifferential that is too large for a lot of optimization problems.

The second approach to define general subdifferentials satisfying useful calculus rules is to take limits of some primitive subdifferential constructions which do not possess such calculus. It is important that limiting constructions depend not only on the choice of the primitive objects but also on the character of the limit: *topological* or *sequential*.

The topological way allows one to develop useful subdifferentials in general infinite dimensional settings, but the biggest drawback is the fact that it may lead to broad constructions and in general they have an intrinsically complicated structure, usually following a three-step procedure. Namely, the definition of ∂f for a Lipschitz function which requires considering restrictions to finite-dimensional (or separable) subspaces with intersections over the collection of all such subspaces, then the definition of a normal cone of a set C at a given point x as the cone generated by the subdifferential of the distance function to C and finally the definition of ∂f for an arbitrary lower semicontinuous function by means of the normal cone to the epigraph of f. In this line of development, many infinite dimensional extensions of the nonconvex constructions in [23, 24] were introduced and strongly developed by Ioffe in a series of many publications starting from 1981 (see [17, 18, 19] for the bibliographies and commentaries therein) on the basis of topological limits of Dini-Hadamard ε-subdifferentials. Such constructions, called also *approximate subdifferentials*, are well defined in more general spaces, but all of them (including also their nuclei) may be broader than the Kruger-Mordukhovich extension, even for Lipshictz functions on Banach spaces with Fréchet differentiable renorms.

The sequential way usually leads to more convenient objects, but it requires some special geometric properties of spaces in question (see for instance [5]). Thus, because the convexity is no longer inherent in the procedure, we are able to define smaller subdifferentials and also to exclude some points from the set of stationary points. The sequential nonconvex subdifferential constructions in Banach spaces were first introduced by Kruger and Mordukhovich [20, 21] on the basis of sequential limits of Fréchet ε-normals and subdifferentials. Such limiting normal cone and subdifferential appeared as infinite dimensional extensions of the corresponding finite dimensional constructions in Mordukhovich [23, 24], motivated by applications to optimization and control. Useful properties of those and related constructions were revealed mainly for Banach spaces with Fréchet differentiable renorms. Let us also emphasize that while the subdifferential theory in finite dimensions has been well developed, there still exist many open questions in infinite dimensional spaces.

While the Fréchet epsilon-subdifferential is as a building block for the Mordukhovich subdifferential in Banach spaces, the Dini-Hadamard one lies at the heart of the so called A-subdifferential introduced by Ioffe. Generated with the help of the *lower Dini* (or *Dini-Hadamard*) *directional derivative*, one of the most attractive construction appeared in the 1970's, the Dini-Hadamard subdifferential and its epsilon enlargement are well known in variational analysis and generalized differentiation but they are not widely used due to the lack of calculus. However, as it has been recently observed in [7], an exact difference formula holds for such subdifferentials under natural assumptions (see also [35]). Moreover, necessary and sufficient optimality conditions for cone-constrained optimization problems having a difference of two functions as objective are established, in case the difference function is *calm* and some additional conditions are fulfilled. Our main goal in this paper is to provide the same formula as mentioned above, but without any calmness assumption. To this end we employ the *Dini-Hadamard-like* ε*-subdifferential* [29], which is defined by the instrumentality of a different kind of *lower limit.* Our analysis relies also on the notion of *spongiously pseuso-dissipativity* of set-valued mappings and involves the notion of a *spongiously local blunt minimizer.*

The reminder of the paper is organized as follows. After introducing in Section 2 some preliminary notions and results especially related to the Dini-Hadamard-like subdifferential, we study in Section 3 some generalized convexity notions in order to provide in the final part of the paper some necessary and sufficient conditions for a point to be a spongiously local blunt minimizer. Finally, we employ the achieved results to the formulation of optimality conditions for a nonsmooth optimization problem having the difference of two functions as objective.

2. Preliminary notions and results

Consider a Banach space X and its topological dual space X^*. We denote the *open ball* with center $\overline{x} \in X$ and radius $\delta > 0$ in X by $B(\overline{x}, \delta)$, while $\overline{B}_X$ and S_X stand for the *closed unit ball* and the *unit sphere* of X, respectively. Having a set $C \subseteq X$, $\delta_C : X \to \mathbb{R} \cup \{+\infty\}$, defined by $\delta_C(x) = 0$ for $x \in C$ and $\delta_C(x) = +\infty$, otherwise, denotes its *indicator function.*

Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a given function. As usual, we denote by $\operatorname{dom} f = \{x \in X : f(x) < +\infty\}$ the *effective domain* of f and by $\operatorname{epi} f = \{(x, \alpha) \in X \times \mathbb{R} : f(x) \leq \alpha\}$ the *epigraph* of f. Dealing with functions that may take infinite values, we adopt the following natural conventions $(+\infty) - (+\infty) = +\infty$ and $0(+\infty) = +\infty$.

For $\varepsilon \geq 0$ the *Fréchet ε-subdifferential* (or the *analytic ε-subdifferential*) of f at $\overline{x} \in \operatorname{dom} f$ is defined by

$$\partial_\varepsilon^F f(\overline{x}) := \left\{ x^* \in X^* : \liminf_{\|h\| \to 0} \frac{f(\overline{x}+h) - f(\overline{x}) - \langle x^*, h \rangle}{\|h\|} \geq -\varepsilon \right\},$$

which means that one has

$$\overline{x}^* \in \partial_\varepsilon^F f(\overline{x}) \Leftrightarrow \quad \begin{array}{l} \text{for all } \alpha > 0 \text{ there exists } \delta > 0 \text{ such that for all } x \in B(\overline{x}, \delta) \\ f(x) - f(\overline{x}) \geq \langle \overline{x}^*, x - \overline{x} \rangle - (\alpha + \varepsilon)\|x - \overline{x}\|. \end{array} \tag{2.1}$$

The following constructions

$$d^- f(\overline{x}; h) := \liminf_{\substack{u \to h \\ t \downarrow 0}} \frac{f(\overline{x} + tu) - f(\overline{x})}{t} = \sup_{\delta > 0} \inf_{\substack{u \in B(h,\delta) \\ t \in (0,\delta)}} \frac{f(\overline{x} + tu) - f(\overline{x})}{t}$$

and (see [17, 18])

$$\partial_\varepsilon^- f(\overline{x}) := \{x^* \in X^* : \langle x^*, h \rangle \leq d^- f(\overline{x}; h) + \varepsilon\|h\| \text{ for all } h \in X\}, \text{ where } \varepsilon \geq 0,$$

are called the *Dini-Hadamard directional derivative* of f at $\overline{x}$ in the direction $h \in X$ and the *Dini-Hadamard ε-subdifferential* of f at $\overline{x}$, respectively.

Similarly, following the two steps procedure of constructing the Dini-Hadamard ε-subdifferential we can define (see [29])

$$D_d^S f(\overline{x}; h) := \sup_{\delta > 0} \inf_{\substack{u \in B(h,\delta) \cap (h + [0,\delta] \cdot B(d,\delta)) \\ t \in (0,\delta)}} \frac{f(\overline{x} + tu) - f(\overline{x})}{t},$$

the *Dini-Hadamard-like directional derivative* of f at $\overline{x}$ in the direction $h \in X$ through $d \in X \setminus \{0\}$ and also, for a given $\varepsilon \geq 0$, the *Dini-Hadamard-like ε-subdifferential* of f at $\overline{x}$

$$\partial_\varepsilon^S f(\overline{x}) := \{x^* \in X^* : \langle x^*, h \rangle \leq D_d^S f(\overline{x}; h) + \varepsilon\|h\| \text{ for all } h \in X \text{ and all } d \in X \setminus \{0\}\}.$$

In case $\varepsilon = 0$, $\partial^- f(\overline{x}) := \partial_0^- f(\overline{x})$ is nothing else than the *Dini-Hadamard subdifferential* of f at $\overline{x}$, while $\partial^S f(\overline{x}) := \partial_0^S f(\overline{x})$ simply denotes the *Dini-Hadamard-like subdifferential* of f at $\overline{x}$. When $\overline{x} \notin \operatorname{dom} f$ we set $\partial_\varepsilon^F f(\overline{x}) = \partial_\varepsilon^- f(\overline{x}) = \partial_\varepsilon^S f(\overline{x}) := \emptyset$ for all $\varepsilon \geq 0$. It is worth emphasizing here that for $\overline{x} \in \operatorname{dom} f$ the following functions $d^- f(\overline{x}; \cdot)$ and $D_d^S f(\overline{x}; \cdot)$ are in general not convex, while $\partial_\varepsilon^- f(\overline{x})$ and $\partial_\varepsilon^S f(\overline{x})$ are always convex sets. Moreover, we notice that $d^- f(\overline{x}; 0)$ is either 0 or $-\infty$ (see [16]).

The function $f : X \to \mathbb{R} \cup \{+\infty\}$ is said to be *calm* at $\overline{x} \in \operatorname{dom} f$ if there exists $c \geq 0$ and $\delta > 0$ such that $f(x) - f(\overline{x}) \geq -c\|x - \overline{x}\|$ for all $x \in B(\overline{x}, \delta)$. As a characterization, for $\overline{x} \in \operatorname{dom} f$ one has (see, for instance, [14]) that f is calm at $\overline{x}$ if and only if $d^- f(\overline{x}; 0) = 0$.

Further, for any $\varepsilon \geq 0$

$$\partial_\varepsilon^F f(\overline{x}) \subseteq \partial_\varepsilon^- f(\overline{x}) \subseteq \partial_\varepsilon^S f(\overline{x}).$$

It is interesting to observe that both inclusions can be even strict (see Example 2.9 belllow and [7] for further remarks and links between the Dini-Hadamard subdifferential and the Fréchet one).

The essential idea behind defining the Dini-Hadamard-like constructions is to employ a directional convergence in place of a usual one. To this aim we say that a sequence (x_n) of X *converges to* $\overline{x}$ *in the direction* $d \in X \setminus \{0\}$ (and we write $(x_n) \underset{d}{\longrightarrow} \overline{x}$) if there exist sequences $(t_n) \to 0$, $t_n \geq 0$ and $(d_n) \to d$ such that $x_n = \overline{x} + t_n d_n$ for each $n \in \mathbb{N}$. Further, a sequence (x_n) is said to *converge directionally* to $\overline{x}$ if there exists $d \in X \setminus \{0\}$ such that $(x_n) \underset{d}{\longrightarrow} \overline{x}$. Our definition, slightly different from the one proposed by Penot in [33], allows us to consider also the constants sequences among the ones which are directionally convergent. Motivated by this observation, we call the *directional lower limit* of f at $\overline{x}$ in the direction $d \in X \setminus \{0\}$ the following limit

$$\liminf_{x \underset{d}{\longrightarrow} \overline{x}} f(x) := \sup_{\delta > 0} \inf_{x \in B(\overline{x},\delta) \cap (\overline{x} + [0,\delta] \cdot B(d,\delta))} f(x).$$

Consequently, since

$$\begin{aligned}
\liminf_{\substack{u \underset{d}{\longrightarrow} h \\ t \underset{1}{\longrightarrow} 0}} \frac{f(\overline{x}+tu) - f(\overline{x})}{t} &:= \sup_{\substack{\delta > 0 \\ \delta' > 0}} \inf_{\substack{u \in B(h,\delta) \cap (h + [0,\delta] \cdot B(d,\delta)) \\ t \in (0,\delta') \cap [0,\delta'] \cdot (1-\delta', 1+\delta')}} \frac{f(\overline{x}+tu) - f(\overline{x})}{t} \\
&= \sup_{\delta > 0} \inf_{\substack{u \in B(h,\delta) \cap (h + [0,\delta] \cdot B(d,\delta)) \\ t \in (0,\delta)}} \frac{f(\overline{x}+tu) - f(\overline{x})}{t},
\end{aligned}$$

we may (formally) write

$$D_d^S f(\overline{x}; h) = \liminf_{\substack{u \underset{d}{\longrightarrow} h \\ t \underset{1}{\longrightarrow} 0}} \frac{f(\overline{x}+tu) - f(\overline{x})}{t} = \liminf_{\substack{u \underset{d}{\longrightarrow} h \\ t \downarrow 0}} \frac{f(\overline{x}+tu) - f(\overline{x})}{t}.$$

Similarly one can define the *directional upper limit* of f at $\overline{x}$ in the direction $d \in X \setminus \{0\}$, since the lower properties symmetrically induce the corresponding upper ones

$$\limsup_{x \underset{d}{\longrightarrow} \overline{x}} f(x) := - \liminf_{x \underset{d}{\longrightarrow} \overline{x}} (-f)(x) = \inf_{\delta > 0} \sup_{x \in B(\overline{x},\delta) \cap (\overline{x} + [0,\delta] \cdot B(d,\delta))} f(x).$$

Moreover, one can easily observe that

$$\liminf_{x \to \overline{x}} f(x) \leq \liminf_{x \underset{d}{\longrightarrow} \overline{x}} f(x) \leq \limsup_{x \underset{d}{\longrightarrow} \overline{x}} f(x) \leq \limsup_{x \to \overline{x}} f(x) \text{ for all } d \in X \setminus \{0\}. \quad (2.2)$$

The next subdifferential notion we need to recall is the one of *G-subdifferential* and we describe in the following the procedure of constructing it (see [19]). To this aim we consider first the *A-subdifferential* of f at $\overline{x} \in \operatorname{dom} f$, which is defined via topological limits as follows

$$\partial^A f(\overline{x}) := \bigcap_{L \in \mathcal{F}(X)} \overline{\operatorname{Limsup}}_{\substack{x \overset{f}{\longrightarrow} \overline{x} \\ \varepsilon > 0}} \partial_\varepsilon^- (f + \delta_{x+L})(x),$$

where $\mathcal{F}(X)$ denotes the collection of all finite dimensional subspaces of X and $\overline{\operatorname{Limsup}}$ stands for the *topological counterpart* of the *sequential Painlevé-Kuratowski upper/outer limit* of a set-valued mapping with sequences replaced by nets and where

$x \xrightarrow{f} \overline{x}$ means $x \longrightarrow \overline{x}$ and $f(x) \longrightarrow f(\overline{x})$. More precisely, for a set-valued mapping $F : X \rightrightarrows X^*$, we say that $x^* \in \overline{\text{Limsup}}_{x\to\overline{x}} F(x)$ if for each weak*-neighborhood $\mathcal{U}$ of the origin of X^* and for each $\delta > 0$ there exists $x \in B(\overline{x}, \delta)$ such that $(x^* + \mathcal{U}) \cap F(x) \neq \emptyset$.

The *G-normal cone* to a set $C \subseteq X$ at $\overline{x} \in C$ is defined as

$$N^G(C, \overline{x}) := \text{cl}^* \left(\bigcup_{\lambda>0} \lambda \partial^A d(\overline{x}, C) \right),$$

where $d(\overline{x}, C) := \inf_{c\in C} \|\overline{x} - c\|$ denotes the *distance* from $\overline{x}$ to C and cl^* stands for the *weak*-closure* of a set in X^*, while the *G-subdifferential* of f at $\overline{x} \in \text{dom} f$ can be defined now as follows

$$\partial^G f(\overline{x}) := \left\{ x^* \in X^* : \ (x^*, -1) \in N^G(\text{epi} f, (\overline{x}, f(\overline{x}))) \right\}.$$

When $\overline{x} \notin \text{dom} f$ we set $\partial^A f(\overline{x}) = \partial^G f(\overline{x}) := \emptyset$. Thus, by taking into account [19, Proposition 4.2] one has the inclusion

$$\partial^F f(x) \subseteq \partial^- f(x) \subseteq \partial^G f(x) \ \text{ for all } x \in X. \tag{2.3}$$

One can notice that when f is a convex function it holds $\partial^F f(x) = \partial^- f(x) = \partial^G f(x) = \partial f(x)$ for all $x \in X$, where $\partial f(\overline{x}) := \{x^* \in X^* : f(x) - f(\overline{x}) \geq \langle x^*, x - \overline{x}\rangle \ \forall x \in X\}$, for $\overline{x} \in \text{dom} f$, and $\partial f(\overline{x}) := \emptyset$, otherwise, denotes the subdifferential of f at $\overline{x}$ in the sense of convex analysis.

It is also worth mentioning that both G- and A-subdifferentials reduce to the basic/limiting/Mordukhovich one whenever X is a finite dimensional space or X is an *Asplund weakly compactly generated* (WCG) space and f is locally Lipschitz at the point in discussion (see [27] and [25, Subsection 3.2.3]).

In what follows, in order to study the behavior of the Dini-Hadamard-like subdifferential we especially need the following result.

Lemma 2.1. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a given function and $\overline{x}, h \in X$. Then the following statements are true:*

(i) $D_d^S f(\overline{x}; h) \leq \liminf_{n\to+\infty} \frac{f(\overline{x}+t_n u_n)-f(\overline{x})}{t_n}$, whenever $(u_n) \underset{d}{\longrightarrow} h$ and $(t_n \downarrow 0)$, with $d \in X \setminus \{0\}$.

(ii) If for some $d \in X \setminus \{0\}$, $D_d^S f(\overline{x}; h) = l \in \mathbb{R} \cup \{-\infty\}$, then there exist sequences $(u_n) \underset{d}{\longrightarrow} h$ and $(t_n \downarrow 0)$ such that $\lim_{n\to+\infty} \frac{f(\overline{x}+t_n u_n)-f(\overline{x})}{t_n} = l$.

Proof. To justify (i), since

$$\liminf_{n\to+\infty} \frac{f(\overline{x} + t_n u_n) - f(\overline{x})}{t_n} := \sup_{n\geq 1} \inf_{k\geq n} \frac{f(\overline{x} + t_k u_k) - f(\overline{x})}{t_k},$$

we only have to show that

$$\sup_{\delta>0} \inf_{\substack{u\in B(h,\delta)\cap(h+[0,\delta]\cdot B(d,\delta)) \\ t\in(0,\delta)}} \frac{f(\overline{x} + tu) - f(\overline{x})}{t} \leq \sup_{n\geq 1} \inf_{k\geq n} \frac{f(\overline{x} + t_k u_k) - f(\overline{x})}{t_k}.$$

Let $\delta > 0$ be fixed. Since $(u_n) \underset{d}{\longrightarrow} h$, there exist sequences $(t_n^{'}) \to 0$, $t_n^{'} \geq 0$ and $(d_n) \to d$ such that $u_n = h + t_n^{'} \cdot d_n$ for all $n \in \mathbb{N}$ and thus there exists $k_0 \in \mathbb{N}$ with the property that for each natural number $k \geq k_0$, $u_k \in B(h,\delta) \cap (h + [0,\delta] \cdot B(d,\delta))$ and $t_k \in (0,\delta)$. Hence

$$D_d^S f(\overline{x};h) \leq \inf_{k \geq k_0} \frac{f(\overline{x}+t_k u_k) - f(\overline{x})}{t_k} \leq \sup_{n \geq 1} \inf_{k \geq n} \frac{f(\overline{x}+t_k u_k) - f(\overline{x})}{t_k}.$$

Taking now the supremum as $\delta > 0$, we obtain the desired conclusion.

(ii) First we study the case $l \in \mathbb{R}$. Using the definition of the directional lower limit, it follows that for any $n \in \mathbb{N}^*$

$$\inf_{\substack{u \in B(h,\delta) \cap (h+[0,\delta] \cdot B(d,\delta)) \\ t \in (0,\delta)}} \frac{f(\overline{x}+tu) - f(\overline{x})}{t} \leq l < l + \frac{1}{n}$$

and consequently, there exists $u_n \in B(h, \frac{1}{n}) \cap (h + [0, \frac{1}{n}] \cdot B(d, \frac{1}{n}))$ and $t_n \in (0, \frac{1}{n})$ with $\frac{f(\overline{x}+t_n u_n)-f(\overline{x})}{t_n} < l + \frac{1}{n}$. Further, for each $n \in \mathbb{N}$ we find $t_n^{'} \in [0, \frac{1}{n}]$ and $d_n \in B(d, \frac{1}{n})$ (with $t_0^{'} := 0$ and $d_0 := d$) such that $u_n = h + t_n^{'} \cdot d_n$, which means nothing else that $(u_n) \underset{d}{\longrightarrow} h$. Moreover, since $\lim_{n \to +\infty} \frac{f(\overline{x}+t_n u_n)-f(\overline{x})}{t_n} \leq l$ and due to assertion (i) we get

$$l = D_d^S f(\overline{x};h) \leq \liminf_{n \to +\infty} \frac{f(\overline{x}+t_n u_n) - f(\overline{x})}{t_n} \leq \lim_{n \to +\infty} \frac{f(\overline{x}+t_n u_n) - f(\overline{x})}{t_n} \leq l.$$

The special case $l = -\infty$ yields for any $n \in \mathbb{N}^*$, $u_n \in B(h, \frac{1}{n}) \cap (h + [0, \frac{1}{n}] \cdot B(d, \frac{1}{n}))$ and $t_n \in (0, \frac{1}{n})$ with the property that $\frac{f(\overline{x}+t_n u_n)-f(\overline{x})}{t_n} < -n$. Thus, we obtain two sequences $(u_n) \underset{d}{\longrightarrow} h$ and $t_n \downarrow 0$ so that $\lim_{n \to +\infty} \frac{f(\overline{x}+t_n u_n)-f(\overline{x})}{t_n} = -\infty$ and finally, the proof of the lemma is complete. □

Remark 2.2. In fact, this result is particularly helpful to conclude that the Dini-Hadamard subdifferential coincide with the Dini-Hadamard-like one in finite dimensions. Indeed, since $d^- f(\overline{x};h) \leq D_d^S f(\overline{x};h)$ for all $d \in X \setminus \{0\}$ and consequently $\partial^- f(\overline{x}) \subseteq \partial^S f(\overline{x})$, we only have to prove that the opposite inclusion holds too. To this end, consider $x^* \in \partial^S f(\overline{x})$, $h \in X$ and let us denote for convenience $d^- f(\overline{x};h) := l$, $l \in \overline{\mathbb{R}}$. Then, in view of Lemma 2.1 above, there exist sequences $(u_n) \to h$ and $(t_n) \downarrow 0$ such that $\lim_{n \to +\infty} \frac{f(\overline{x}+t_n u_n)-f(\overline{x})}{t_n} = l$. Now, due to the finiteness assumption made, we can find $u^{'} \in S_X$ and a subsequence (u_{n_k}) such that

$$(u_{n_k}) \underset{u^{'}}{\longrightarrow} h \quad \text{and} \quad \lim_{k \to +\infty} \frac{f(\overline{x}+t_{n_k} \cdot u_{n_k}) - f(\overline{x})}{t_{n_k}} = l. \tag{2.4}$$

To justify this claim, suppose first that (u_n) has an infinite number of terms not equal to h. Then we can choose a subsequence (u_{n_k}) of (u_n), $u_{n_k} \neq h$ for all $k \in \mathbb{N}$ and we may write $u_{n_k} = h + \|u_{n_k} - h\| \cdot d_{n_k}$ with $d_{n_k} = \frac{u_{n_k}-h}{\|u_{n_k}-h\|}$. Further, since (d_{n_k}) is bounded, there exist $u^{'} \in S_X$ and $(d_{n_{k_l}})$ so that $(d_{n_{k_l}}) \to u^{'}$, and hence $(u_{n_{k_l}}) \underset{u^{'}}{\longrightarrow} h$. If on the contrary u_n has an infinite number of terms equal to h, then we choose a subsequence (u_{n_k}) of (u_n) such that $u_{n_k} = h$ for all $k \in \mathbb{N}$. In this particular case $(u_{n_k}) \underset{u'}{\longrightarrow} h$ for every $u' \in S_X$.

Consequently, relation (2.4) above holds true and hence $l \geq D^S_{u'} f(\overline{x}; h)$, which in turn implies $\langle x^*, h\rangle \leq d^- f(\overline{x}; h)$ and finally $x^* \in \partial^- f(\overline{x})$.

As it was first observed by Penot [33], the concept of a directionally convergent sequence is clearly related to the following notion introduced by Treiman [40].

Definition 2.3. A set $S \subseteq X$ is said to be a *sponge around* $\overline{x} \in X$ if for all $h \in X \setminus \{0\}$ there exist $\lambda > 0$ and $\delta > 0$ such that $\overline{x} + [0, \lambda] \cdot B(h, \delta) \subseteq S$.

Furthermore, the sponges enjoy a nice relationship with the so-called cone-porous sets (see [13, 41] for definition and further remarks). Indeed, accordingly to [11], if S is a sponge around $\overline{x}$ then the complementary set $(X \setminus S) \cup \{\overline{x}\}$ is cone porous in any direction $v \in S_X$. Let us recall also that every neighborhood of a point $\overline{x} \in X$ is also a sponge around $\overline{x}$ and that the converse is not true (see for instance [7, Example 2.2]). However, in case S is a convex set or X is a finite dimensional space (here one can make use of the fact that the unit sphere is compact), then S is also a neighborhood of $\overline{x}$.

Remark 2.4. Trying to answer the question how further can we go with the replacement of a neighborhood by a sponge, it is worth emphasizing that every sponge S around a point $\overline{x} \in X$ has the property (A) and moreover it verifies also (B).

(A) : for all $h \in X \setminus \{0\}$ there exist $\lambda > 0$ and a sponge S' around h such that for all $u \in S'$, $\overline{x} + [0, \lambda] \cdot u \subseteq S$.

(B) : for all $h \in X \setminus \{0\}$ and all $d \in X \setminus \{0\}$ there exists $\delta > 0$ such that for all $u \in B(h, \delta) \cap (h + [0, \delta] \cdot B(d, \delta))$, $\overline{x} + [0, \delta] \cdot u \subseteq S$.

Finally, every set S which satisfies property (B) is a sponge around $\overline{x}$.

Indeed, suppose that S verifies the above property and take an arbitrary $h \in X \setminus \{0\}$. Then there exists $\delta > 0$ such that for all $u \in B(h, \delta) \cap (h + [0, \delta] \cdot B(h, \delta))$, $\overline{x} + [0, \delta] \cdot u \subseteq S$. On the other hand, there exists $\alpha > 0$ ($\alpha < \delta$) so that $h + [0, \alpha] \cdot B(h, \delta) \subseteq B(h, \delta) \cap (h + [0, \delta] \cdot B(h, \delta))$ and therefore $\overline{x} + [0, \delta] \cdot B((\alpha + 1)h, \alpha\delta) \subseteq S$. Consequently, there exist $\alpha' := \delta(\alpha + 1) > 0$ and $\delta' := \frac{\alpha}{\alpha+1} \cdot \delta > 0$ such that $\overline{x} + [0, \alpha'] \cdot B(h, \delta') \subseteq S$ and the conclusion follows easily.

Now we are ready to illustrate the aforementioned relationship between sponges and directionally convergent sequences.

Lemma 2.5. *([40, Lemma 2.1]) A subset S of X is a sponge around $\overline{x}$ if and only if for any sequence (x_n) which converges directionally to $\overline{x}$ there exists $n_0 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, $n \geq n_0$, $x_n \in S$.*

In what follows we mainly focus on the properties of the Dini-Hadamard-like ε-subdifferential. But first, following the lines of the proof of [7, Lemma 2.1] and taking into account relation (2.2), let us remark that Lemma 2.6 bellow holds true not only for the Dini-Hadamard-like subdifferential but also for the Fréchet subdifferential (which is a building block for the *basic/limiting/Mordukhovich subdifferential* in Banach spaces. We refer the reader to the books [25, 26] for a systematic study) and for the Dini-Hadamard one, as well.

Lemma 2.6. *([29, Lemma 2.3]) Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a given function and $\overline{x} \in \operatorname{dom} f$. Then for all $\varepsilon \geq 0$ it holds*

$$\partial_\varepsilon^S f(\overline{x}) = \partial^S (f + \varepsilon\| \cdot - \overline{x}\|)(\overline{x}). \tag{2.5}$$

Thus, using a classical subdifferential formula provided by the convex analysis, one can easily see that, in case f is convex, $\partial_\varepsilon^S f(\overline{x}) = \partial f(\overline{x}) + \varepsilon \overline{B}_{X^*}$ for all $\varepsilon \geq 0$.

The following notion, introduced by Treiman [40], it turns out to be essential also when characterizing the Dini-Hadamard-like subdifferential.

Definition 2.7. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a given function, $\overline{x} \in \operatorname{dom} f$ and $\varepsilon \geq 0$. We say that $x^* \in X^*$ is an H_ε-*subgradient *of f at $\overline{x}$ if there exists a sponge S around $\overline{x}$ such that for all $x \in S$*

$$f(x) - f(\overline{x}) \geq \langle x^*, x - \overline{x} \rangle - \varepsilon \|x - \overline{x}\|.$$

Unlike the one obtained in the case of the Dini-Hadamard subdifferential (we refer to [7, Lemma 2.2] for more details and a similar proof), the following lemma does not require any calmness condition (take into account also here the Remark 2.4 above). As a direct consequence, we mention that for any $\gamma \geq \varepsilon \geq 0$ and $x \in X$

$$\partial_\varepsilon^S f(x) \subseteq \partial_\gamma^S f(x). \tag{2.6}$$

Lemma 2.8. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a given function, $\overline{x} \in \operatorname{dom} f$ and $\varepsilon \geq 0$. The following statements are true:*

(i) If $x^* \in \partial_\varepsilon^S f(\overline{x})$, then x^* is an H_γ-subgradient of f at $\overline{x}$ for all $\gamma > \varepsilon$.
(ii) If x^* is an H_ε-subgradient of f at $\overline{x}$, then $x^* \in \partial_\varepsilon^S f(\overline{x})$.

Moreover, one can even conclude that whenever $\overline{x} \in \operatorname{dom} f$, $\varepsilon \geq 0$ and $\gamma > \varepsilon$ the following set

$$S := \{x \in X : \ f(x) - f(\overline{x}) \geq \langle x^*, x - \overline{x} \rangle - \gamma \|x - \overline{x}\|\} \tag{2.7}$$

is a sponge around $\overline{x}$ not only for those elements $x^* \in \partial_\varepsilon^- f(\overline{x})$ (like in [7, Remark 2.3]) but also for $x^* \in \partial_\varepsilon^S f(\overline{x})$.

Example 2.9. Although in finite dimensions the Dini-Hadamard ε-subdifferential coincide with the corresponding Dini-Hadamard-like one (see for this Remark 2.2, Lemma 2.6 and [7, Lemma 2.1]) this is in general not the case. Indeed, let us consider the function $f : X \to \mathbb{R}$ as being

$$f(x) = \begin{cases} 0, & \text{if } x \in S, \\ a, & \text{otherwise,} \end{cases}$$

where $a < 0$ and S is a sponge around $\overline{x} \in X$ which is not a neighborhood of $\overline{x}$ (for such an example we refer to [7, Example 2.2]). Then taking into account the second assertion of Lemma 2.8, one can easily conclude that for all $\varepsilon \geq 0$, $0 \in \partial_\varepsilon^S f(\overline{x}) \setminus \partial_\varepsilon^- f(\overline{x})$, since 0 is an H_ε-subgradient of f at $\overline{x}$, but f is not calm at $\overline{x}$.

To justify this last assertion, we suppose on the contrary that f is calm at $\overline{x}$. Further, using the aforementioned property of the set S, one can even conclude that for all $n \in \mathbb{N}$ there exists an element $y_n \in B(\overline{x}, \frac{1}{n}) \setminus S$ such that $\|y_n - \overline{x}\| \leq \frac{1}{n}$. But since we may write $y_n = \overline{x} + t_n^{'} \cdot u_n^{'}$ with $t_n^{'} := \sqrt{\|y_n - \overline{x}\|}$ and $u_n^{'} := \frac{y_n - \overline{x}}{\|y_n - \overline{x}\|} \cdot \sqrt{\|y_n - \overline{x}\|}$ and moreover $\lim\limits_{n \to +\infty} \frac{f(y_n) - f(\overline{x})}{t_n^{'}} = -\infty$, we get the following relation $\liminf\limits_{\substack{u \to 0 \\ t \downarrow 0}} \frac{f(\overline{x} + tu) - f(\overline{x})}{t} = -\infty$, and consequently $d^- f(\overline{x}, 0) = -\infty$, a contradiction which completes the proof.

The following result provides a variational description for the Dini-Hadamard-like ε-subdifferential, with no additional calmness assumptions. For the reader convenient we list bellow also the proof.

Theorem 2.10. *([29, Theorem 3.1]) Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be an arbitrary function and $\overline{x} \in \operatorname{dom} f$. Then for all $\varepsilon \geq 0$ one has*

$$x^* \in \partial_\varepsilon^S f(\overline{x}) \Leftrightarrow \begin{array}{l} \forall \alpha > 0 \text{ there exists } S \text{ a sponge around } \overline{x} \text{ such that} \\ \forall x \in S \;\; f(x) - f(\overline{x}) \geq \langle x^*, x - \overline{x} \rangle - (\alpha + \varepsilon)\|x - \overline{x}\|. \end{array} \tag{2.8}$$

Proof. Consider an $\varepsilon \geq 0$ fixed.

In order to justify the inclusion "$\subseteq$", let $x^* \in \partial_\varepsilon^S f(\overline{x})$ and $\alpha > 0$. Now just observe that using Lemma 2.8 above we can easily obtain the existence of a sponge S around $\overline{x}$ such that for all $x \in S$

$$f(x) - f(\overline{x}) \geq \langle x^*, x - \overline{x} \rangle - (\alpha + \varepsilon)\|x - \overline{x}\|.$$

For the second inclusion "$\supseteq$", let us consider an arbitrary element x^* fulfilling the property in the right-hand side of (2.8). Our goal is to show that

$$D_d^S f(\overline{x}; h) \geq \langle x^*, h \rangle - \varepsilon \|h\| \;\forall h \in X, \forall d \in X \setminus \{0\}. \tag{2.9}$$

Let first $h \in X \setminus \{0\}$ and $d \in X \setminus \{0\}$ be fixed. Then for all $k \in \mathbb{N}$, by taking $\alpha_k := \frac{1}{k}$, there exists S_k a sponge around $\overline{x}$ such that for all $x \in S_k$

$$f(x) - f(\overline{x}) \geq \langle x^*, x - \overline{x} \rangle - \left(\frac{1}{k} + \varepsilon\right) \|x - \overline{x}\|.$$

Thus, for all $k \in \mathbb{N}$ there exists $\delta_k > 0$ such that for all $t \in (0, \delta_k)$ and all $u \in B(h, \delta_k) \cap (h + [0, \delta_k] \cdot B(d, \delta_k))$ one has $\overline{x} + tu \in S_k$ and

$$f(\overline{x} + tu) - f(\overline{x}) \geq \langle x^*, tu \rangle - \left(\frac{1}{k} + \varepsilon\right) \|tu\|,$$

which implies in turn that for all $0 < \delta_k^{'} \leq \delta_k$, all $t \in (0, \delta_k^{'})$ and all $u \in B(h, \delta_k^{'}) \cap (h + [0, \delta_k^{'}] \cdot B(d, \delta_k^{'}))$

$$\frac{f(\overline{x} + tu) - f(\overline{x})}{t} \geq \langle x^*, u \rangle - \left(\frac{1}{k} + \varepsilon\right) \|u\|$$

and consequently, for all $k \in \mathbb{N}$ there exists $\delta_k > 0$ such that for all $0 < \delta_k^{'} \leq \delta_k$

$$\begin{aligned} D_d^S f(\overline{x}; h) &\geq \inf_{\substack{u \in B(h,\delta_k^{'}) \cap (h+[0,\delta_k^{'}]\cdot B(d,\delta_k^{'})) \\ t \in (0,\delta_k^{'})}} \frac{f(\overline{x}+tu)-f(\overline{x})}{t} \\ &\geq \inf_{\substack{u \in B(h,\delta_k^{'}) \cap (h+[0,\delta_k^{'}]\cdot B(d,\delta_k^{'})) \\ t \in (0,\delta_k^{'})}} \left[\langle x^*, u\rangle - \left(\frac{1}{k}+\varepsilon\right)\|u\|\right] \\ &\geq \inf_{u \in B(h,\delta_k^{'})} \left[\langle x^*, u\rangle - \left(\frac{1}{k}+\varepsilon\right)\|u\|\right]. \end{aligned}$$

On the other hand, for all $k \in \mathbb{N}$, all $0 < \delta_k^{'} \leq \delta_k$ and all $\delta^{'} \geq \delta_k^{'}$

$$\inf_{u \in B(h,\delta_k^{'})} \left[\langle x^*, u\rangle - \left(\frac{1}{k}+\varepsilon\right)\|u\|\right] \geq \inf_{u \in B(h,\delta^{'})} \left[\langle x^*, u\rangle - \left(\frac{1}{k}+\varepsilon\right)\|u\|\right]$$

and hence, for all $k \in \mathbb{N}$

$$D_d^S f(\overline{x}; h) \geq \liminf_{u \to h} \left[\langle x^*, u\rangle - \left(\frac{1}{k}+\varepsilon\right)\|u\|\right] = \langle x^*, h\rangle - \left(\frac{1}{k}+\varepsilon\right)\|h\|.$$

Finally, passing to the limit as $k \to +\infty$, we obtain

$$D_d^S f(\overline{x}; h) \geq \langle x^*, h\rangle - \varepsilon\|h\|$$

and thus, the relation (2.9) holds true for all $h \in X \setminus \{0\}$ and all $d \in X \setminus \{0\}$.

For the particular case $h = 0$, let $d \in X\setminus\{0\}$ be an arbitrary element. To complete the proof of the theorem we only have to show that $D_d^S f(\overline{x}; 0) \geq 0$. Assuming the contrary, accordingly to Lemma 2.1, one gets two sequences $(u_n) \underset{d}{\longrightarrow} 0$ and $(t_n) \downarrow 0$ such that

$$\lim_{n \to +\infty} \frac{f(\overline{x}+t_n u_n)-f(\overline{x})}{t_n} < 0, \tag{2.10}$$

where $(u_n) \underset{d}{\longrightarrow} 0$ means that there exist sequences $(t_n^{'}) \to 0$ ($t_n^{'} \geq 0$ $\forall n \in \mathbb{N}$) and $(d_n) \to d$ such that $u_n = t_n^{'} \cdot d_n$ for all $n \in N$.

On the other hand, there exists a sponge S around $\overline{x}$ such that

$$f(x) - f(\overline{x}) \geq \langle x^*, x-\overline{x}\rangle - \varepsilon\|x-\overline{x}\|$$

for all $x \in S$ and consequently, we can find a natural number n_0 such that for all $n \in \mathbb{N}$, $n \geq n_0$, $\overline{x} + t_n \cdot u_n \in S$ and hence

$$f(\overline{x}+t_n \cdot u_n) - f(\overline{x}) \geq \langle x^*, t_n \cdot u_n\rangle - \varepsilon\|t_n \cdot u_n\|.$$

Therefore, passing to the limit as $k \to +\infty$, we observe that

$$\lim_{n \to +\infty} \frac{f(\overline{x}+t_n u_n)-f(\overline{x})}{t_n} \geq 0,$$

which in fact contradicts the relation (2.10). □

A similar result to Theorem 2.10, by means of the Dini-Hadamard ε-subdifferential, was given in [7], but in a more restrictive framework.

To a more careful look we can see that also in the case of the Dini-Hadamard-like subdifferential it is a sort of calmness condition that is hiding behind. So, we say that a function $f : X \to \mathbb{R} \cup \{+\infty\}$ is *weakly calm* at $\overline{x} \in \operatorname{dom} f$ if $D^S_d f(\overline{x}; 0) \geq 0$ for all $d \in X \setminus \{0\}$. Actually, unlike the case of the Dini-Hadmamard subdifferential, this last assumption is automatically fulfilled. It is worth mentioning also here that although the weakly calmness assumption is a more general one, it does coincide with the classical calmness condition in finite dimensions.

Proposition 2.11. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a given function and $\overline{x} \in \operatorname{dom} f$. If X is finite dimensional then f is calm at $\overline{x}$ if and only if f is weakly calm at $\overline{x}$.*

Proof. Since one can easily check the "if" part of the proposition, it remains us to show just the "only" if one. Suppose on the contrary that f is not calm at $\overline{x}$. Then $d^- f(\overline{x}; 0) = -\infty$ and hence there exist sequences $(u_n) \to 0$ and $(t_n) \downarrow 0$ such that $\lim\limits_{n \to +\infty} \frac{f(\overline{x}+t_n u_n) - f(\overline{x})}{t_n} = -\infty$. Using now the finiteness property of X, the latter clearly yields an element $s \in S_X$ and a subsequence $(u_{n_k}) \underset{s}{\longrightarrow} 0$ with the property that $\lim\limits_{k \to +\infty} \frac{f(\overline{x}+t_{n_k} \cdot u_{n_k}) - f(\overline{x})}{t_{n_k}} = -\infty$. Consequently $-\infty \geq D^S_s f(\overline{x}; 0)$, which is a contradiction. □

Finally, to conclude this section, let us present a direct consequence of Theorem 2.10 and [7, Theorem 2.3], interesting in itself.

Corollary 2.12. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a given function, $\overline{x} \in \operatorname{dom} f$ and $\varepsilon \geq 0$. If $\partial^S_\varepsilon f(\overline{x}) \neq \emptyset$ then f is calm at $\overline{x}$ if and only if $\partial^-_\varepsilon f(\overline{x}) = \partial^S_\varepsilon f(\overline{x})$.*

3. Some generalized convexity notions

Let us mention in the beginning of this section that the Dini-Hadamard-like subdifferential coincides with the Dini-Hadamard one not only in finite dimensional spaces but also in arbitrarily Banach spaces on some particular classes of functions. Furthermore, these classes of functions, which are in fact larger than the one of convex functions, will reveal themselves to be useful in the sequel. We introduce them now.

Definition 3.1. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a given function and $\overline{x} \in \operatorname{dom} f$. The function f is said to be*

(i) approximately convex *at $\overline{x}$, if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for every $x, y \in B(\overline{x}, \delta)$ and every $t \in [0, 1]$ one has*

$$f((1-t)y + tx) \leq (1-t)f(y) + tf(x) + \varepsilon t(1-t)\|x - y\|. \tag{3.1}$$

(ii) approximately starshaped *at $\overline{x}$, if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for every $x \in B(\overline{x}, \delta)$ and every $t \in [0, 1]$ one has*

$$f((1-t)\overline{x} + tx) \leq (1-t)f(\overline{x}) + tf(x) + \varepsilon t(1-t)\|x - \overline{x}\|. \tag{3.2}$$

(iii) directionally approximately starshaped *at* $\overline{x}$*, if for any* $\varepsilon > 0$ *and any* $u \in S_X$ *there exists* $\delta > 0$ *such that for every* $s \in (0, \delta)$*, every* $v \in B(u, \delta)$ *and every* $t \in [0, 1]$*, when* $x := \overline{x} + sv$*, one has*

$$f((1-t)\overline{x} + tx) \leq (1-t)f(\overline{x}) + tf(x) + \varepsilon t(1-t)\|x - \overline{x}\|. \tag{3.3}$$

While the class of approximately convex functions was initiated and strongly developed by H.V. Ngai, D.T. Luc and M. Théra in [30] (see also [3, 31]), the ones of approximately starshaped and directionally approximately stashaped were introduced and studied in [32]. Actually, they enjoy nice properties and, for instance, the approximate convex functions are stable under finite sums and finite suprema, and moreover the most of the well-known subdifferentials coincide and share several properties of the convex subdifferential (see [30]) on this particular class of functions. Observe also that the class of approximately convex functions is strictly included into the class of approximately starshaped functions, which in turn is contained into the one of directionally approximately starshaped functions (for some examples we refer to [7]). In fact, the last two classes of functions coincide on finite dimensional spaces, as one can easily deduce from the following result.

Proposition 3.2. *(*[7, Proposition 3.1]*) Let* $f : X \to \mathbb{R} \cup \{+\infty\}$ *be a given function and* $\overline{x} \in \operatorname{dom} f$*. Then* f *is directionally approximately starshaped at* $\overline{x}$ *if and only if for any* $\varepsilon > 0$ *there exists a sponge* S *around* $\overline{x}$ *such that for every* $x \in S$ *and* $t \in [0, 1]$ *one has*

$$f((1-t)\overline{x} + tx) \leq (1-t)f(\overline{x}) + tf(x) + \varepsilon t(1-t)\|x - \overline{x}\|. \tag{3.4}$$

It is worth emphasizing here that in view of Remark 2.4, the above characterization via sponges it is also equivalent with the following one.

Proposition 3.3. *Let* $f : X \to \mathbb{R} \cup \{+\infty\}$ *be a given function and* $\overline{x} \in \operatorname{dom} f$*. Then* f *is directionally approximately starshaped at* $\overline{x}$ *if and only if for any* $\varepsilon > 0$*,* $h \in X \setminus \{0\}$ *and any* $d \in X \setminus \{0\}$ *there exists* $\delta > 0$ *such that for every* $s \in (0, \delta)$*,* $v \in B(h, \delta) \cap (h + [0, \delta] \cdot B(d, \delta))$ *and every* $t \in [0, 1]$*, with* $x := \overline{x} + sv$*, the relation (*3.4*) above holds true.*

In fact the class of directionally approximately starshaped functions enjoys also the following property, which is more general then the one obtained in [7, Lemma 3.2], or in [1, Lemma 1].

Lemma 3.4. *Let the function* $f : X \to \mathbb{R} \cup \{+\infty\}$ *be directionally approximately starshaped at* $\overline{x} \in \operatorname{dom} f$*. Then for every* $\alpha > 0$ *and every* $\varepsilon \geq 0$ *there exists a sponge* S *around* $\overline{x}$ *such that for every* $x \in S$ *one has*

$$f(x) - f(\overline{x}) \geq \langle \overline{x}^*, x - \overline{x} \rangle - (\alpha + \varepsilon)\|x - \overline{x}\| \ \forall \overline{x}^* \in \partial_\varepsilon^S f(\overline{x}), \tag{3.5}$$

$$f(\overline{x}) - f(x) \geq \langle x^*, \overline{x} - x \rangle - (\alpha + \varepsilon)\|x - \overline{x}\| \ \forall x^* \in \partial_\varepsilon^S f(x). \tag{3.6}$$

Proof. Fix $\alpha > 0$, $\varepsilon \geq 0$ and consider the set

$$S := \{x \in X : \ f(x) - f(\overline{x}) \geq \langle \overline{x}^*, x - \overline{x} \rangle - (\alpha + \varepsilon)\|x - \overline{x}\| \ \forall \overline{x}^* \in \partial_\varepsilon^S f(\overline{x})\}.$$

In order to complete the proof of the first inequality, our strategy is to show that S is a sponge around $\overline{x}$.

Indeed, let $h \in X \setminus \{0\}$ and $d \in X \setminus \{0\}$ be arbitrary elements and take $\delta > 0$ so that the relation (3.4) above holds true with $x := \overline{x} + sv$, for any $s \in (0, \delta)$, $v \in B(h, \delta) \cap (h + [0, \delta] \cdot B(d, \delta))$ and any $t \in (0, 1]$. Then,

$$f(\overline{x} + tsv) - f(\overline{x}) \leq t[f(\overline{x} + sv) - f(\overline{x})] + \alpha t(1 - t)\|sv\|$$

and hence, after dividing by t, we take the limit inferior as $t \downarrow 0$ and we obtain

$$\liminf_{t \downarrow 0} \frac{f(\overline{x} + tsv) - f(\overline{x})}{t} \leq f(\overline{x} + sv) - f(\overline{x}) + \alpha\|sv\|.$$

But

$$D_d^S f(\overline{x}; sv) \leq \sup_{\delta > 0} \inf_{\substack{u \in \{sv\} \\ t \in (0, \delta)}} \frac{f(\overline{x} + tu) - f(\overline{x})}{t} = \liminf_{t \downarrow 0} \frac{f(\overline{x} + tsv) - f(\overline{x})}{t}$$

and consequently,

$$D_d^S f(\overline{x}; sv) \leq f(\overline{x} + sv) - f(\overline{x}) + \alpha\|sv\|.$$

In other words, for any $h \in X \setminus \{0\}$ and $d \in X \setminus \{0\}$ there exists $\delta > 0$ such that for every $s \in (0, \delta)$ and $v \in B(h, \delta) \cap (h + [0, \delta] \cdot B(d, \delta))$, $\overline{x} + sv \in S$, i.e. S is a sponge around $\overline{x}$, by virtue of Remark 2.4.

Similarly, with $x := \overline{x} + sv$ and $t' := 1 - t$ one has

$$f(x - t'sv) - f(x) \leq t'[f(x - sv) - f(x)] + \alpha t'(1 - t')\|sv\|$$

which implies in turn (following the steps bellow)

$$D_d^S f(x; -sv) \leq f(x - sv) - f(x) + \alpha\|sv\|$$

and finally one obtains that

$$S^{'} := \{x \in X : \ f(\overline{x}) - f(x) \geq \langle x^*, \overline{x} - x\rangle - (\alpha + \varepsilon)\|x - \overline{x}\| \ \forall x^* \in \partial_\varepsilon^S f(x)\}$$

is a sponge around $\overline{x}$, which completes the proof of the second inequality. □

Now we state our main result of this section, thanks to which, the Dini-Hadamard-like subdifferential as well as the Dini-Hadamard one agrees with a great number of well-known subdifferentials such as the Clarke-Rockafellar, the Mordukhovich, the Fréchet and the Ioffe approximate subdifferential on lower semicontinuous and approximately convex functions at a given point of the domain (see for more details [30, Theorem 3.6]).

Proposition 3.5. *Let the function $f : X \to \mathbb{R} \cup \{+\infty\}$ be approximately starshaped at $\overline{x} \in \operatorname{dom} f$. Then for all $\varepsilon \geq 0$ it holds*

$$\partial_\varepsilon^F f(\overline{x}) = \partial_\varepsilon^- f(\overline{x}) = \partial_\varepsilon^S f(\overline{x}).$$

Proof. In view of [32, Lemma 2.6] and [7, Lemma 2.1] the first equality is clearly verified. For the second one, accordingly to Lemma 2.6 above and [7, Lemma 2.1] it is enough to show that it holds true only for $\varepsilon = 0$. To this end we argue why for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $x \in B(\overline{x}, \delta)$ and any $d \in X \setminus \{0\}$

$$D_d^S f(\overline{x}; x - \overline{x}) \leq f(x) - f(\overline{x}) + \varepsilon\|x - \overline{x}\|.$$

This will complete the proof, since given an arbitrary $\overline{x}^* \in \partial^S f(\overline{x})$, the above inequality would provide us the following estimate

$$\langle \overline{x}^*, x - \overline{x} \rangle \leq f(x) - f(\overline{x}) + \varepsilon \|x - \overline{x}\|,$$

i.e. the inclusion $\partial^S f(\overline{x}) \subseteq \partial^F f(\overline{x})$ (due to relation (2.1)) and hence the equality.

So, fix an arbitrary $\varepsilon > 0$. Then, since f is approximately starshaped at $\overline{x}$, we choose $\delta > 0$ so that for any $x \in B(\overline{x}, \delta)$ and any $t \in (0, 1]$

$$f(\overline{x} + t(x - \overline{x})) - f(\overline{x}) \leq t[f(x) - f(\overline{x})] + \varepsilon t(1 - t)\|x - \overline{x}\|.$$

Then, dividing by t and taking limit inferior as $t \downarrow 0$, one obtains

$$\liminf_{t \downarrow 0} \frac{f(\overline{x} + t(x - \overline{x})) - f(\overline{x})}{t} \leq f(x) - f(\overline{x}) + \varepsilon \|x - \overline{x}\|$$

and finally, the desired inequality. □

On the other hand, while [7, Example 3.1] ensures us that the equality $\partial^F f(\overline{x}) = \partial^- f(\overline{x})$ does not hold in case f is only directionally approximately starshaped at $\overline{x} \in \operatorname{dom} f$, Example 2.9 above guarantees the same with the equality $\partial^- f(\overline{x}) = \partial^S f(\overline{x})$, since f is directionally approximately starshaped at $\overline{x}$, but $0 \in \partial^S f(\overline{x}) \setminus \partial^- f(\overline{x})$. Moreover, the function in Example 2.9 shows that in general the class of approximately starshaped functions does not coincide with the one of directionally approximately starshaped functions.

4. Optimality conditions

In what follows we mostly confine ourselves to the study of a subdifferential formula for the difference of two functions. To this end, let us recall first that for two subsets $A, B \subseteq X$ the *star-difference* between them is defined as

$$A \overset{*}{-} B := \{x \in X : \; x + B \subseteq A\} = \bigcap_{b \in B} \{A - b\}.$$

We adopt here the convention $A \overset{*}{-} B := \emptyset$ in case $A = \emptyset, B \neq \emptyset$ and $A \overset{*}{-} B := X$ if $B = \emptyset$. One obviously have $A \overset{*}{-} B + B \subseteq A$ and $A \overset{*}{-} B \subseteq A - B$ if $B \neq \emptyset$. Introduced by Pontrjagin [36] in the context of linear differential games, this notion has been widely used in the field of nonsmooth analysis (see, for instance, [1, 2, 8, 12, 15, 22, 27, 37]).

When dealing with the difference of two functions $g, h : X \to \mathbb{R} \cup \{+\infty\}$ we assume throughout the paper that $\operatorname{dom} g \subseteq \operatorname{dom} h$. This guarantees that the function $f = g - h : X \to \mathbb{R} \cup \{+\infty\}$ is well-defined. Moreover, one can easily observe that $g = f + h$ and $\operatorname{dom} f = \operatorname{dom} g$.

The following simple result yields easily from Theorem 2.10 and due to the fact that the intersection of two sponges around a point is a sponge around that point.

Proposition 4.1. *Let $g, h : X \to \mathbb{R} \cup \{+\infty\}$ be given functions and $f := g - h$. Then for all $\varepsilon, \eta \geq 0$ and all $x \in X$ one has*

$$\partial^S_\varepsilon f(x) \subseteq \partial^S_{\varepsilon+\eta} g(x) \overset{*}{-} \partial^S_\eta h(x). \tag{4.1}$$

In particular, if $\overline{x} \in \operatorname{dom} f$ is a local minimizer of the function $f := g - h$, then

$$0 \in \partial^S g(\overline{x}) \overset{*}{-} \partial^S h(\overline{x})$$

or, equivalently,

$$\partial^S h(\overline{x}) \subseteq \partial^S g(\overline{x}).$$

For similar characterizations to the difference of two functions via the Fréchet subdifferential, by means of the Mordukhovich (basic/limiting) subdifferential and in terms of the Dini-Hadamard one we refer to [27], [25, 26] and [7], respectively.

Although the inclusion (4.1) holds true without no supplementary assumptions on the functions involved, in order to guarantee the reverse one we need to introduce also the following notion.

Definition 4.2. *A set-valued mapping $F : X \rightrightarrows X^*$ is said to be* spongiously pseudo-dissipative *at $\overline{x} \in X$ if for any $\varepsilon > 0$ there exists S a sponge around $\overline{x}$ such that for any $x \in S$ there exist $x^* \in F(x)$ and $\overline{x}^* \in F(\overline{x})$ so that*

$$\langle x^* - \overline{x}^*, x - \overline{x} \rangle \leq \varepsilon \|x - \overline{x}\|$$

or, equivalently, if for any $\varepsilon > 0$ and any $u \in S_X$ there exists $\delta > 0$ such that for any $t \in (0, \delta)$ and $v \in B(u, \delta)$ there exist $x^* \in F(x)$ and $\overline{x}^* \in F(\overline{x})$ so that

$$\langle x^* - \overline{x}^*, v \rangle \leq \varepsilon \|v\|.$$

Actually, there are two ways of extending the *approximately pseudo-dissipativity* introduced by Penot [35]. While the first one was described above by replacing a neighborhood with a sponge, the second one will be presented bellow.

Definition 4.3. *A set-valued mapping $F : X \rightrightarrows X^*$ is said to be* directionally approximately pseudo-dissipative *at $\overline{x} \in X$ if for any $\varepsilon > 0$ and $u \in S_X$ one can find some $\delta > 0$ such that for any $v \in B(u, \delta)$ and any $t \in (0, \delta)$ there exist $x^* \in F(\overline{x} + tv)$ and $\overline{x}^* \in F(\overline{x})$ so that*

$$\langle x^* - \overline{x}^*, x - \overline{x} \rangle \leq \varepsilon.$$

In fact this later conditions are not very restrictive ones, since the following coarse continuity (which has been introduced in [1]) ensures the approximately pseudo-dissipativity and the *spongiously gap-continuity* studied in [7], as well. Let us formulate now this concept.

Definition 4.4. A set-valued mapping $F : X \rightrightarrows Y$ between a topological space X and a metric space Y is said to be *gap-continuous* at $\overline{x} \in X$ if for any $\varepsilon > 0$ one can find some $\delta > 0$ such that for every $x \in B(\overline{x}, \delta)$

$$\operatorname{gap}(F(\overline{x}), F(x)) < \varepsilon,$$

where for two subsets A and B of Y

$$\operatorname{gap}(A, B) := \inf\{d(a, b) : a \in A, b \in B\},$$

with the convention that if one of the sets is empty, then $\operatorname{gap}(A, B) := +\infty$.

When defining a spongiously gap-continuous mapping one only has to replace in the above definition the neighborhood $B(\overline{x}, \delta)$ of $\overline{x}$ with a sponge S around $\overline{x}$. Therefore, every gap-continuous mapping at a point is spongiously gap-continuous and moreover it is also spongiously pseudo-dissipative and directionally approximately pseudo-dissipative at that point, too. Furthermore, every set-valued mapping which is either Hausdorff upper semicontinuous or lower semicontinuous at a given point is gap-continuous at that point (see [34]). Thus, the gap-continuity is a sort of semicontinuity notion which is satisfied in many situations when no other semicontinuity notion holds. Moreover, in case the mapping is single-valued, it coincides with the classical continuity. Clearly, when X is a finite dimensional space then the gap-continuity coincides with the spongiously gap-continuity as well as the approximately pseudo-dissipativity property agrees with the spongiously pseudo-dissipativity and with the directionally approximately pseudo-dissipativity one. It is worth emphasizing also here that the notion of spongiously gap-continuity [7] is equivalent to that of *directionally-gap continuity* introduced later by Penot [35] (we refer the reader to the papers of Penot [35, 34] for more discussions and some criteria ensuring the gap-continuity and also the directionally approximately pseudo-dissipativity). Finally, the following property holds.

Proposition 4.5. *Let $F, G : X \rightrightarrows Y$ be two set-valued mappings. If F is spongiously pseudo-dissipative at $\overline{x} \in X$ and there exists a sponge S around $\overline{x}$ such that $F(x) \subseteq G(x)$ for all $x \in S$, then G is spongiously pseudo-dissipative at $\overline{x}$, too.*

Accordingly to relation (2.6) and the above property, we conclude that for $f : X \to \mathbb{R} \cup \{+\infty\}$ a given function and $\overline{x} \in \operatorname{dom} f$, $\partial^S_\eta f$ is spongiously pseudo-dissipative at $\overline{x}$ for all $\eta > 0$, whenever $\partial^S f$ is spongiously pseudo-dissipative at $\overline{x}$. Hence, following the lines of the proof of [7, Theorem 3.4, Theorem 3.5] we can furnish a formula for the difference of two functions in terms of the Dini-Hadamard-like subdifferential.

Theorem 4.6. *Let $g, h : X \to \mathbb{R} \cup \{+\infty\}$ be two directionally approximately starshaped functions at $\overline{x} \in \operatorname{dom} g$ and $f := g - h$. If for some $\eta \geq 0$ the set-valued mapping $\partial^S_\eta h$ is spongiously pseudo-dissipative at $\overline{x}$, then for all $\varepsilon \geq 0$ it holds*

$$\partial^S_\varepsilon f(\overline{x}) = \partial^S_{\varepsilon+\eta} g(\overline{x}) \overset{*}{-} \partial^S_\eta h(\overline{x}). \tag{4.2}$$

In case the function f is calm at $\overline{x}$ one obtains the result in [7, Theorem 3.5], where the subdifferential in question is the Dini-Hadamard one. For a similar statement in the particular setting $\varepsilon = \eta = 0$, we refer to [35, Theorem 28]. There the function h is assumed to be directionally approximately starshaped, directionally continuous, directionally stable and tangentially convex at $\overline{x}$, a point from $\operatorname{core}(\operatorname{dom} h)$. Similar results expressed by means of the Fréchet subdifferential can be found in [1, Theorem 3] and [35, Theorem 26], where the functions are supposed to be approximately starshaped and a very mild assumption on $\partial^F h$ is required. But taking into account the fact that f may not be calm at $\overline{x}$, or the functions g and h may not be approximately starshaped, or even $\operatorname{core}(\operatorname{dom} h)$ could be empty (for instance, core$(\ell^p_+) = \emptyset$ for any $p \in [1, +\infty)$, see [6]), motivates us to formulate results like Theorem 4.6.

Let us mention now some corollaries whose proofs follows the ideas from [7, Corollary 3.7, Corollary 3.8]. Take also into account Proposition 3.5 above.

Corollary 4.7. *Let $g, h : X \to \mathbb{R} \cup \{+\infty\}$ be two directionally approximately starshaped functions at $\overline{x} \in \operatorname{dom} g$ such that $\partial^S h$ is spongiously pseudo-dissipative at $\overline{x}$ and $f := g - h$. Then the following statements are equivalent:*

(i) there exists $\eta \geq 0$ such that $\partial^S_\eta h(\overline{x}) \subseteq \partial^S_\eta g(\overline{x})$;

(ii) $0 \in \partial^S f(\overline{x})$;

(iii) for all $\eta \geq 0$ $\partial^S_\eta h(\overline{x}) \subseteq \partial^S_\eta g(\overline{x})$.

Corollary 4.8. *Let $g, h : X \to \mathbb{R} \cup \{+\infty\}$ be two given functions, $\overline{x} \in \operatorname{dom} g$ and $f := g - h$. Then the following assertions are true:*

(i) If g and h are convex at $\overline{x}$ and ∂h is spongiously pseudo-dissipative at $\overline{x}$, then it holds

$$\partial^S f(\overline{x}) = \partial g(\overline{x}) \overset{*}{-} \partial h(\overline{x}).$$

(ii) If g is convex, h is directionally approximately starshaped at $\overline{x}$ and $\partial^S h$ is spongiously pseudo-dissipative at $\overline{x}$, then for all $\varepsilon \geq 0$ it holds

$$\partial^S_\varepsilon f(\overline{x}) = (\partial g(\overline{x}) + \varepsilon \overline{B}_{X^*}) \overset{*}{-} \partial^S h(\overline{x}).$$

(iii) If g is lower semicontinuous, approximately convex at $\overline{x}$, h is directionally approximately starshaped at $\overline{x}$ and $\partial^S h$ is spongiously pseudo-dissipative at $\overline{x}$, then for all $\varepsilon \geq 0$ it holds

$$\partial^S_\varepsilon f(\overline{x}) = (\partial^S g(\overline{x}) + \varepsilon \overline{B}_{X^*}) \overset{*}{-} \partial^S h(\overline{x}).$$

The following result, which significantly improves the statement in [7, Corollary 3.6], due to Theorem 4.6 and [35, Theorem 26] (see also Proposition 3.5), is meant to reveal that the Dini-Hadamard-like subdifferential coincides with the Dini-Hadamard subdifferential and with the Fréchet one not only on approximately starshaped functions but also on some particular differences of approximately starshaped functions.

Corollary 4.9. *Let $g, h : X \to \mathbb{R} \cup \{+\infty\}$ be two approximately starshaped functions at $\overline{x} \in \operatorname{dom} g$ with the property that there exists $\eta \geq 0$ such that $\partial^S_\eta h$ is approximately pseudo-dissipative at $\overline{x}$ and $f := g - h$. Then for all $\varepsilon \geq 0$ $\partial^F_\varepsilon f(\overline{x}) = \partial^-_\varepsilon f(\overline{x}) = \partial^S_\varepsilon f(\overline{x})$.*

Moreover, in case $\overline{x} \in \operatorname{core}(\operatorname{dom} h)$ and $\partial^- h$ is only directionally approximately pseudo-dissipative at $\overline{x}$, then one can guarantee that for any $\varepsilon \geq 0$, $\partial^-_\varepsilon f(\overline{x}) = \partial^S_\varepsilon f(\overline{x})$ (see for this [35, Lemma 22, Lemma 24, Lemma 27] and Lemma 4.1).

Finally, we characterize the Dini-Hadamard-like subdifferential by means of the so-called *spongiously local ε-blunt minimizers*. Introduced in [7], they came as a generalization to *local ε-blunt minimizers* studied by Amahroq, Penot and Syam in [1].

Definition 4.10. *Let $C \subseteq X$ be a nonempty set, $f : X \to \mathbb{R} \cup \{+\infty\}$ be a given function, $\overline{x} \in \operatorname{dom} f \cap C$ and $\varepsilon > 0$. We say that $\overline{x}$ is a* spongiously local ε-blunt minimizer of f on the set C *if there exists a sponge S around $\overline{x}$ such that for all $x \in S \cap C$*

$$f(x) \geq f(\overline{x}) - \varepsilon \|x - \overline{x}\|.$$

In case $C = X$, we simply call $\overline{x}$ a spongiously local ε-blunt minimizer of f.

Proposition 4.11. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a given function and $\overline{x} \in \operatorname{dom} f$. Then:*

$$0 \in \partial^S f(\overline{x}) \Leftrightarrow \quad \overline{x} \textit{ is a spongiously local } \varepsilon - \textit{blunt minimizer of } f \textit{ for all } \varepsilon > 0.$$

In the situation when f is calm at $\overline{x}$ one obtains the result in [7, Proposition 3.9], as a particular case. Similarly, in view of the above discussions and results, we can even furnish optimality conditions for the cone-constrained optimization problem $(\mathcal{P})$ studied in [7], by means of the Dini-Hadamard-like subdifferential and without additional calmness assumptions. For the reader convenient we state this result bellow. To this end, let us consider the following optimization problem

$$(\mathcal{P}) \quad \inf_{x \in \mathcal{A}} f(x).$$
$$\mathcal{A} = \{x \in C : k(x) \ \in -K\},$$

where $C \subseteq X$ is a convex and closed set, K, a subset of a Banach space Z, is a nonempty convex and closed *cone* with $K^* := \{z^* \in Z^* : \langle z^*, z\rangle \geq 0 \text{ for all } z \in K\}$ its *dual cone*, $k : X \to Z$, a given function, is assumed to be K-*convex*, meaning that for all $x, y \in X$ and all $t \in [0, 1]$, $(1-t)k(x) + tk(y) - k((1-t)x + ty) \in K$, and K-*epi closed*, meaning that the K-*epigraph* of k, $\operatorname{epi}_K k := \{(x, z) \in X \times Z : z \in k(x) + K\}$, is a closed set and finally $f := g - h$, where $g, h : X \to \mathbb{R} \cup \{+\infty\}$ are two given functions with $\operatorname{dom} g \subseteq \operatorname{dom} h$. For $z^* \in K^*$, by $(z^*k) : X \to \mathbb{R}$ we denote the function defined by $(z^*k)(x) = \langle z^*, k(x)\rangle$ and we also emphasize that in case $Z = \mathbb{R}$ and $K = \mathbb{R}_+$ the notion of K-epi closedness coincide with the classical lower semicontinuity.

Theorem 4.12. *Let be $\overline{x} \in \operatorname{int}(\operatorname{dom} g) \cap \mathcal{A}$. Suppose that g is lower semicontinuous and approximately convex at $\overline{x}$ and that $\bigcup_{\lambda>0} \lambda(k(C) + K)$ is a closed linear subspace of Z. Then the following assertions are true:*

(a) If $\overline{x}$ is a spongiously local ε-blunt minimizer of f on $\mathcal{A}$ for all $\varepsilon > 0$, then the following relation holds

$$\partial^S h(\overline{x}) \subseteq \partial^S g(\overline{x}) + \bigcup_{\substack{z^* \in K^* \\ (z^*k)(\overline{x})=0}} \partial((z^*k) + \delta_C)(\overline{x}). \tag{4.3}$$

(b) Conversely, if h is directionally approximately starshaped at $\overline{x}$, $\partial^S h$ is spongiously pseudo-dissipative at $\overline{x}$ and (4.3) *holds, then $\overline{x}$ is a spongiously local ε-blunt minimizer of f on $\mathcal{A}$ for all $\varepsilon > 0$.*

It is worth mentioning that accordingly to [35, Lemma 22, Lemma 24 and Lemma 27], our final result remains also true in case $\partial^S h$ is directionally approximately pseudo-dissipative at $\overline{x}$. Moreover, in the particular instance when $K = \{0\}$, $k(x) = 0$ for any $x \in X$, g is lower semicontinuous and approximately convex at $\overline{x} \in \operatorname{int}(\operatorname{dom} g) \cap \mathcal{A}$ and h is convex on C and continuous at $\overline{x}$, and hence directionally approximately pseudo-dissipative at $\overline{x}$ (due to the remarkable dissipativity property of the subdifferential in the sense of convex analysis, see [35, Theorem 6]) then $\overline{x}$ is a spongiously local ε-blunt minimizer of f on $\mathcal{A}$ for all $\varepsilon > 0$ if and only if

$$\partial h(\overline{x}) \subseteq \partial^S g(\overline{x}) + N(\mathcal{A}, \overline{x}).$$

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