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Fractional approximation by Cardaliaguet-Euvrard and Squashing neural network operators

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Abstract. This article deals with the determination of the fractional rate of convergence to the unit of some neural network operators, namely, the Cardaliaguet-Euvrard and "squashing" operators. This is given through the moduli of continuity of the involved right and left Caputo fractional derivatives of the approximated function and they appear in the right-hand side of the associated Jackson type inequalities.

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1. Introduction

The Cardaliaguet-Euvrard (3.1) operators were first introduced and studied extensively in [7], where the authors among many other things proved that these operators converge uniformly on compacta, to the unit over continuous and bounded functions. Our "squashing operator" (see [1]) (3.53) was motivated and inspired by the "squashing functions" and related Theorem 6 of [7]. The work in [7] is qualitative where the used bell-shaped function is general. However, our work, though greatly motivated by [7], is quantitative and the used bell-shaped and "squashing" functions are of compact support. We produce a series of Jackson type inequalities giving close upper bounds to the errors in approximating the unit operator by the above neural network induced operators. All involved constants there are well determined. These are pointwise, uniform and L_p, $p \geq 1$, estimates involving the first moduli of continuity of the engaged right and left Caputo fractional derivatives of the function under approximation. We give all necessary background of fractional calculus.

Initial work of the subject was done in [1], where we involved only ordinary derivatives. Article [1] motivated the current work.

2. Background

We need

Definition 2.1. *Let $f \in C(\mathbb{R})$ which is bounded or uniformly continuous, $h > 0$. We define the first modulus of continuity of f at h as follows*

$$\omega_1(f,h) = \sup\{|f(x) - f(y)|; x, y \in \mathbb{R}, \ |x-y| \leq h\} \tag{2.1}$$

Notice that $\omega_1(f,h)$ is finite for any $h > 0$, and

$$\lim_{h \to 0} \omega_1(f,h) = 0.$$

We also need

Definition 2.2. *Let $f : \mathbb{R} \to \mathbb{R}$, $\nu \geq 0$, $n = \lceil \nu \rceil$ ($\lceil \cdot \rceil$ is the ceiling of the number), $f \in AC^n([a,b])$ (space of functions f with $f^{(n-1)} \in AC([a,b])$, absolutely continuous functions), $\forall$ $[a,b] \subset \mathbb{R}$. We call left Caputo fractional derivative (see [8], pp. 49-52) the function*

$$D_{*a}^{\nu} f(x) = \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} f^{(n)}(t) \, dt, \tag{2.2}$$

$\forall$ $x \geq a$, where Γ is the gamma function $\Gamma(\nu) = \int_0^\infty e^{-t} t^{\nu-1} dt$, $\nu > 0$. Notice $D_{*a}^{\nu} f \in L_1([a,b])$ and $D_{*a}^{\nu} f$ exists a.e.on $[a,b]$, $\forall$ $b > a$. We set $D_{*a}^0 f(x) = f(x)$, $\forall$ $x \in [a, \infty)$.

Lemma 2.3. *([5]) Let $\nu > 0$, $\nu \notin \mathbb{N}$, $n = \lceil \nu \rceil$, $f \in C^{n-1}(\mathbb{R})$ and $f^{(n)} \in L_\infty(\mathbb{R})$. Then $D_{*a}^{\nu} f(a) = 0$, $\forall$ $a \in \mathbb{R}$.*

Definition 2.4. *(see also [2], [9], [10]). Let $f : \mathbb{R} \to \mathbb{R}$, such that $f \in AC^m([a,b])$, $\forall$ $[a,b] \subset \mathbb{R}$, $m = \lceil \alpha \rceil$, $\alpha > 0$. The right Caputo fractional derivative of order $\alpha > 0$ is given by*

$$D_{b-}^{\alpha} f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b (J-x)^{m-\alpha-1} f^{(m)}(J) \, dJ, \tag{2.3}$$

$\forall$ $x \leq b$. We set $D_{b-}^0 f(x) = f(x)$, $\forall$ $x \in (-\infty, b]$. Notice that $D_{b-}^{\alpha} f \in L_1([a,b])$ and $D_{b-}^{\alpha} f$ exists a.e.on $[a,b]$, $\forall$ $a < b$.

Lemma 2.5. *([5]) Let $f \in C^{m-1}(\mathbb{R})$, $f^{(m)} \in L_\infty(\mathbb{R})$, $m = \lceil \alpha \rceil$, $\alpha > 0$. Then $D_{b-}^{\alpha} f(b) = 0$, $\forall$ $b \in \mathbb{R}$.*

Convention 2.6. *We assume that*

$$D_{*x_0}^{\alpha} f(x) = 0, \ \text{ for } x < x_0, \tag{2.4}$$

and

$$D_{x_0-}^{\alpha} f(x) = 0, \text{ for } x > x_0, \tag{2.5}$$

for all $x, x_0 \in \mathbb{R}$.

We mention

Proposition 2.7. *(by [3]) Let $f \in C^n(\mathbb{R})$, where $n = \lceil \nu \rceil$, $\nu > 0$. Then $D_{*a}^{\nu} f(x)$ is continuous in $x \in [a, \infty)$.*

Also we have

Proposition 2.8. *(by [3]) Let $f \in C^m(\mathbb{R})$, $m = \lceil \alpha \rceil$, $\alpha > 0$. Then $D^{\alpha}_{b-} f(x)$ is continuous in $x \in (-\infty, b]$.*

We further mention

Proposition 2.9. *(by [3]) Let $f \in C^{m-1}(\mathbb{R})$, $f^{(m)} \in L_\infty(\mathbb{R})$, $m = \lceil \alpha \rceil$, $\alpha > 0$ and*

$$D^{\alpha}_{*x_0} f(x) = \frac{1}{\Gamma(m-\alpha)} \int_{x_0}^{x} (x-t)^{m-\alpha-1} f^{(m)}(t)\, dt, \tag{2.6}$$

for all $x, x_0 \in \mathbb{R} : x \geq x_0$.

Then $D^{\alpha}_{*x_0} f(x)$ is continuous in x_0.

Proposition 2.10. *(by [3]) Let $f \in C^{m-1}(\mathbb{R})$, $f^{(m)} \in L_\infty(\mathbb{R})$, $m = \lceil \alpha \rceil$, $\alpha > 0$ and*

$$D^{\alpha}_{x_0-} f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_{x}^{x_0} (J-x)^{m-\alpha-1} f^{(m)}(J)\, dJ, \tag{2.7}$$

for all $x, x_0 \in \mathbb{R} : x_0 \geq x$.

Then $D^{\alpha}_{x_0-} f(x)$ is continuous in x_0.

Proposition 2.11. *([5]) Let $g \in C_b(\mathbb{R})$ (continuous and bounded), $0 < c < 1$, $x, x_0 \in \mathbb{R}$. Define*

$$L(x, x_0) = \int_{x_0}^{x} (x-t)^{c-1} g(t)\, dt, \text{ for } x \geq x_0, \tag{2.8}$$

and $L(x, x_0) = 0$, for $x < x_0$.

Then L is jointly continuous in $(x, x_0) \in \mathbb{R}^2$.

We mention

Proposition 2.12. *([5]) Let $g \in C_b(\mathbb{R})$, $0 < c < 1$, $x, x_0 \in \mathbb{R}$. Define*

$$K(x, x_0) = \int_{x}^{x_0} (J-x)^{c-1} g(J)\, dJ, \quad \text{for } x \leq x_0, \tag{2.9}$$

and $K(x, x_0) = 0$, for $x > x_0$.

Then $K(x, x_0)$ is jointly continuous in $(x, x_0) \in \mathbb{R}^2$.

Based on Propositions 2.11, 2.12 we derive

Corollary 2.13. *([5]) Let $f \in C^m(\mathbb{R})$, $f^{(m)} \in L_\infty(\mathbb{R})$, $m = \lceil \alpha \rceil$, $\alpha > 0$, $\alpha \notin \mathbb{N}$, $x, x_0 \in \mathbb{R}$. Then $D^{\alpha}_{*x_0} f(x)$, $D^{\alpha}_{x_0-} f(x)$ are jointly continuous functions in (x, x_0) from $\mathbb{R}^2$ into $\mathbb{R}$.*

We need

Proposition 2.14. *([5]) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be jointly continuous. Consider*

$$G(x) = \omega_1 \left(f(\cdot, x), \delta\right)_{[x,+\infty)}, \ \delta > 0, x \in \mathbb{R}. \tag{2.10}$$

(Here ω_1 is defined over $[x, +\infty)$ instead of $\mathbb{R}$.)

Then G is continuous on $\mathbb{R}$.

Proposition 2.15. *([5]) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be jointly continuous. Consider*

$$H(x) = \omega_1 (f(\cdot, x), \delta)_{(-\infty,x]}, \ \delta > 0, x \in \mathbb{R}. \tag{2.11}$$

(Here ω_1 is defined over $(-\infty, x]$ instead of $\mathbb{R}$.)

Then H is continuous on $\mathbb{R}$.

By Propositions 2.14, 2.15 and Corollary 2.13 we derive

Proposition 2.16. *([5]) Let $f \in C^m(\mathbb{R})$, $\left\| f^{(m)} \right\|_\infty < \infty$, $m = \lceil \alpha \rceil$, $\alpha \notin \mathbb{N}$, $\alpha > 0$, $x \in \mathbb{R}$. Then $\omega_1 (D^\alpha_{*x} f, h)_{[x,+\infty)}$, $\omega_1 \left(D^\alpha_{x-} f, h \right)_{(-\infty,x]}$ are continuous functions of $x \in \mathbb{R}$, $h > 0$ fixed.*

We make

Remark 2.17. Let g be continuous and bounded from $\mathbb{R}$ to $\mathbb{R}$. Then

$$\omega_1 (g, t) \le 2 \|g\|_\infty < \infty. \tag{2.12}$$

Assuming that $(D^\alpha_{*x} f)(t)$, $\left(D^\alpha_{x-} f \right)(t)$, are both continuous and bounded in $(x, t) \in \mathbb{R}^2$, i.e.

$$\|D^\alpha_{*x} f\|_\infty \le K_1, \ \forall \ x \in \mathbb{R}; \tag{2.13}$$

$$\left\|D^\alpha_{x-} f\right\|_\infty \le K_2, \ \forall \ x \in \mathbb{R}, \tag{2.14}$$

where $K_1, K_2 > 0$, we get

$$\omega_1 (D^\alpha_{*x} f, \xi)_{[x,+\infty)} \le 2K_1;$$

$$\omega_1 \left(D^\alpha_{x-} f, \xi \right)_{(-\infty,x]} \le 2K_2, \ \forall \ \xi \ge 0, \tag{2.15}$$

for each $x \in \mathbb{R}$.

Therefore, for any $\xi \ge 0$,

$$\sup_{x\in\mathbb{R}} \left[\max \left(\omega_1 (D^\alpha_{*x} f, \xi)_{[x,+\infty)}, \omega_1 \left(D^\alpha_{x-} f, \xi \right)_{(-\infty,x]} \right) \right] \le 2 \max (K_1, K_2) < \infty. \tag{2.16}$$

So in our setting for $f \in C^m(\mathbb{R})$, $\left\| f^{(m)} \right\|_\infty < \infty$, $m = \lceil \alpha \rceil$, $\alpha \notin \mathbb{N}$, $\alpha > 0$, by Corollary 2.13 both $(D^\alpha_{*x} f)(t)$, $\left(D^\alpha_{x-} f \right)(t)$ are jointly continuous in (t, x) on $\mathbb{R}^2$. Assuming further that they are both bounded on $\mathbb{R}^2$ we get (2.16) valid. In particular, each of $\omega_1 (D^\alpha_{*x} f, \xi)_{[x,+\infty)}$, $\omega_1 \left(D^\alpha_{x-} f, \xi \right)_{(-\infty,x]}$ is finite for any $\xi \ge 0$.

Clearly here we have that $\sup\limits_{x\in\mathbb{R}} \omega_1 (D^\alpha_{*x} f, \xi)_{[x,+\infty)} \to 0$, as $\xi \to 0+$, and $\sup\limits_{x\in\mathbb{R}} \omega_1 \left(D^\alpha_{x-} f, \xi \right)_{(-\infty,x]} \to 0$, as $\xi \to 0+$.

Let us now assume only that $f \in C^{m-1}(\mathbb{R})$, $f^{(m)} \in L_\infty(\mathbb{R})$, $m = \lceil \alpha \rceil$, $\alpha > 0$, $\alpha \notin \mathbb{N}$, $x \in \mathbb{R}$. Then, by Proposition 15.114, p. 388 of [4], we find that $D^\alpha_{*x} f \in C([x, +\infty))$, and by [6] we obtain that $D^\alpha_{x-} f \in C((-\infty, x])$.

We make

Remark 2.18. Again let $f \in C^m(\mathbb{R})$, $m = \lceil \alpha \rceil$, $\alpha \notin \mathbb{N}$, $\alpha > 0$; $f^{(m)}(x) = 1$, $\forall \ x \in \mathbb{R}$; $x_0 \in \mathbb{R}$. Notice $0 < m - \alpha < 1$. Then

$$D^\alpha_{*x_0} f(x) = \frac{(x - x_0)^{m-\alpha}}{\Gamma(m - \alpha + 1)}, \ \forall \ x \ge x_0. \tag{2.17}$$

Let us consider $x, y \geq x_0$, then

$$\left|D_{*x_0}^{\alpha} f(x) - D_{*x_0}^{\alpha} f(y)\right| = \frac{1}{\Gamma(m-\alpha+1)} \left|(x-x_0)^{m-\alpha} - (y-x_0)^{m-\alpha}\right|$$

$$\leq \frac{|x-y|^{m-\alpha}}{\Gamma(m-\alpha+1)}. \tag{2.18}$$

So it is not strange to assume that

$$\left|D_{*x_0}^{\alpha} f(x_1) - D_{*x_0}^{\alpha} f(x_2)\right| \leq K |x_1 - x_2|^{\beta}, \tag{2.19}$$

$K > 0$, $0 < \beta \leq 1$, $\forall\ x_1, x_2 \in \mathbb{R}$, $x_1, x_2 \geq x_0 \in \mathbb{R}$, where more generally it is $\left\|f^{(m)}\right\|_{\infty} < \infty$. Thus, one may assume

$$\omega_1\left(D_{x-}^{\alpha} f, \xi\right)_{(-\infty, x]} \leq M_1 \xi^{\beta_1}, \text{ and} \tag{2.20}$$

$$\omega_1\left(D_{*x}^{\alpha} f, \xi\right)_{[x, +\infty)} \leq M_2 \xi^{\beta_2},$$

where $0 < \beta_1, \beta_2 \leq 1$, $\forall\ \xi > 0$, $M_1, M_2 > 0$; any $x \in \mathbb{R}$.

Setting $\beta = \min(\beta_1, \beta_2)$ and $M = \max(M_1, M_2)$, in that case we obtain

$$\sup_{x \in \mathbb{R}} \left\{\max\left(\omega_1\left(D_{x-}^{\alpha} f, \xi\right)_{(-\infty, x]}, \omega_1\left(D_{*x}^{\alpha} f, \xi\right)_{[x, +\infty)}\right)\right\} \leq M \xi^{\beta} \to 0, \text{ as } \xi \to 0+. \tag{2.21}$$

3. Results

3.1. Fractional convergence with rates of the Cardaliaguet-Euvrard neural network operators

We need the following (see [7]).

Definition 3.1. *A function $b : \mathbb{R} \to \mathbb{R}$ is said to be bell-shaped if b belongs to L^1 and its integral is nonzero, if it is nondecreasing on $(-\infty, a)$ and nonincreasing on $[a, +\infty)$, where a belongs to $\mathbb{R}$. In particular $b(x)$ is a nonnegative number and at a b takes a global maximum; it is the center of the bell-shaped function. A bell-shaped function is said to be centered if its center is zero. The function $b(x)$ may have jump discontinuities. In this work we consider only centered bell-shaped functions of compact support $[-T, T]$, $T > 0$. Call $I := \int_{-T}^{T} b(t)\,dt$. Note that $I > 0$.*

We follow [1], [7].

Example 3.2. (1) $b(x)$ can be the characteristic function over $[-1, 1]$.

(2) $b(x)$ can be the hat function over $[-1, 1]$, i.e.,

$$b(x) = \begin{cases} 1 + x, & -1 \leq x \leq 0, \\ 1 - x, & 0 < x \leq 1 \\ 0, & \text{elsewhere.} \end{cases}$$

These are centered bell-shaped functions of compact support.

Here we consider functions $f : \mathbb{R} \to \mathbb{R}$ that are continuous.

In this article we study the fractional convergence with rates over the real line, to the unit operator, of the Cardaliaguet-Euvrard neural network operators (see [7]),

$$(F_n(f))(x) := \sum_{k=-n^2}^{n^2} \frac{f\left(\frac{k}{n}\right)}{I \cdot n^{\alpha}} \cdot b\left(n^{1-\alpha} \cdot \left(x - \frac{k}{n}\right)\right), \tag{3.1}$$

where $0 < \alpha < 1$ and $x \in \mathbb{R}$, $n \in \mathbb{N}$. The terms in the sum (3.1) can be nonzero iff

$$\left|n^{1-\alpha}\left(x - \frac{k}{n}\right)\right| \leq T, \text{ i.e. } \left|x - \frac{k}{n}\right| \leq \frac{T}{n^{1-\alpha}}$$

iff

$$nx - Tn^{\alpha} \leq k \leq nx + Tn^{\alpha}. \tag{3.2}$$

In order to have the desired order of numbers

$$-n^2 \leq nx - Tn^{\alpha} \leq nx + Tn^{\alpha} \leq n^2, \tag{3.3}$$

it is sufficient enough to assume that

$$n \geq T + |x|. \tag{3.4}$$

When $x \in [-T, T]$ it is enough to assume $n \geq 2T$ which implies (3.3).

Proposition 3.3. *Let $a \leq b$, $a, b \in \mathbb{R}$. Let $card(k)$ (≥ 0) be the maximum number of integers contained in $[a, b]$. Then*

$$\max(0, (b-a) - 1) \leq card(k) \leq (b-a) + 1. \tag{3.5}$$

Remark 3.4. We would like to establish a lower bound on $card(k)$ over the interval $[nx - Tn^{\alpha}, nx + Tn^{\alpha}]$. From Proposition 3.3 we get that

$$card(k) \geq \max(2Tn^{\alpha} - 1, 0).$$

We obtain $card(k) \geq 1$, if

$$2Tn^{\alpha} - 1 \geq 1 \text{ iff } n \geq T^{-\frac{1}{\alpha}}.$$

So to have the desired order (3.3) and $card(k) \geq 1$ over $[nx - Tn^{\alpha}, nx + Tn^{\alpha}]$, we need to consider

$$n \geq \max\left(T + |x|, T^{-\frac{1}{\alpha}}\right). \tag{3.6}$$

Also notice that $card(k) \to +\infty$, as $n \to +\infty$. We call $b^* := b(0)$ the maximum of $b(x)$.

Denote by $[\cdot]$ the integral part of a number.

Following [1] we have

$$\sum_{k=\lceil nx - Tn^{\alpha} \rceil}^{[nx + Tn^{\alpha}]} \frac{1}{I \cdot n^{\alpha}} \cdot b\left(n^{1-\alpha} \cdot \left(x - \frac{k}{n}\right)\right)$$

$$\leq \frac{b^*}{I \cdot n^{\alpha}} \cdot \sum_{k=\lceil nx - Tn^{\alpha} \rceil}^{[nx + Tn^{\alpha}]} 1$$

$$\leq \frac{b^*}{I \cdot n^{\alpha}} \cdot (2Tn^{\alpha} + 1) = \frac{b^*}{I} \cdot \left(2T + \frac{1}{n^{\alpha}}\right). \tag{3.7}$$

We will use

Lemma 3.5. *It holds that*

$$S_n(x) := \sum_{k=\lceil nx-Tn^\alpha\rceil}^{[nx+Tn^\alpha]} \frac{1}{I\cdot n^\alpha}\cdot b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)\to 1, \tag{3.8}$$

pointwise, as $n\to+\infty$, where $x\in\mathbb{R}$.

Remark 3.6. Clearly we have that

$$nx-Tn^\alpha \le nx \le nx+Tn^\alpha. \tag{3.9}$$

We prove in general that

$$nx-Tn^\alpha \le [nx] \le nx \le \lceil nx\rceil \le nx+Tn^\alpha. \tag{3.10}$$

Indeed we have that, if $[nx]<nx-Tn^\alpha$, then $[nx]+Tn^\alpha<nx$, and $[nx]+[Tn^\alpha]\le[nx]$, resulting into $[Tn^\alpha]=0$, which for large enough n is not true. Therefore $nx-Tn^\alpha\le[nx]$. Similarly, if $\lceil nx\rceil>nx+Tn^\alpha$, then $nx+Tn^\alpha\ge nx+[Tn^\alpha]$, and $\lceil nx\rceil-[Tn^\alpha]>nx$, thus $\lceil nx\rceil-[Tn^\alpha]\ge\lceil nx\rceil$, resulting into $[Tn^\alpha]=0$, which again for large enough n is not true.

Therefore without loss of generality we may assume that

$$nx-Tn^\alpha \le [nx] \le nx \le \lceil nx\rceil \le nx+Tn^\alpha. \tag{3.11}$$

Hence $\lceil nx-Tn^\alpha\rceil\le[nx]$ and $\lceil nx\rceil\le[nx+Tn^\alpha]$. Also if $[nx]\ne\lceil nx\rceil$, then $\lceil nx\rceil=[nx]+1$. If $[nx]=\lceil nx\rceil$, then $nx\in\mathbb{Z}$; and by assuming $n\ge T^{-\frac{1}{\alpha}}$, we get $Tn^\alpha\ge1$ and $nx+Tn^\alpha\ge nx+1$, so that $[nx+Tn^\alpha]\ge nx+1=[nx]+1$.

We present our first main result

Theorem 3.7. *We consider $f:\mathbb{R}\to\mathbb{R}$. Let $\beta>0$, $N=\lceil\beta\rceil$, $\beta\notin\mathbb{N}$, $f\in AC^N([a,b])$, $\forall\ [a,b]\subset\mathbb{R}$, with $f^{(N)}\in L_\infty(\mathbb{R})$. Let also $x\in\mathbb{R}$, $T>0$, $n\in\mathbb{N}: n\ge\max\left(T+|x|,T^{-\frac{1}{\alpha}}\right)$. We further assume that $D^\beta_{*x}f$, $D^\beta_{x-}f$ are uniformly continuous functions or continuous and bounded on $[x,+\infty)$, $(-\infty,x]$, respectively.*

Then

1)

$$|F_n(f)(x)-f(x)| \le |f(x)|\cdot \tag{3.12}$$

$$\left|\sum_{k=\lceil nx-Tn^\alpha\rceil}^{[nx+Tn^\alpha]}\frac{1}{In^\alpha}b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)-1\right|+$$

$$\frac{b^*}{I}\left(2T+\frac{1}{n^\alpha}\right)\left(\sum_{j=1}^{N-1}\frac{\left|f^{(j)}(x)\right|T^j}{j!n^{(1-\alpha)j}}\right)$$

$$+\frac{b^*}{I}\left(2T+\frac{1}{n^\alpha}\right)\frac{T^\beta}{\Gamma(\beta+1)\,n^{(1-\alpha)\beta}}\cdot$$

$$\left\{\omega_1\left(D^\beta_{*x}f,\frac{T}{n^{1-\alpha}}\right)_{[x,+\infty)}+\omega_1\left(D^\beta_{x-}f,\frac{T}{n^{1-\alpha}}\right)_{(-\infty,x]}\right\},$$

above $\sum_{j=1}^{0} \cdot = 0$,

2)

$$\left| (F_n(f))(x) - \sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!} \left(F_n\left((\cdot - x)^j \right) \right)(x) \right| \leq \tag{3.13}$$

$$\frac{b^*}{I}\left(2T + \frac{1}{n^\alpha}\right)\frac{T^\beta}{\Gamma(\beta+1)\, n^{(1-\alpha)\beta}} \cdot$$

$$\left\{ \omega_1\left(D_{*x}^\beta f, \frac{T}{n^{1-\alpha}} \right)_{[x,+\infty)} + \omega_1\left(D_{x-}^\beta f, \frac{T}{n^{1-\alpha}} \right)_{(-\infty,x]} \right\} =: \lambda_n(x),$$

3) assume further that $f^{(j)}(x) = 0$, *for* $j = 0, 1, ..., N-1$, *we get*

$$|F_n(f)(x)| \leq \lambda_n(x), \tag{3.14}$$

4) in case of $N = 1$, *we obtain*

$$|F_n(f)(x) - f(x)| \leq |f(x)| \cdot \tag{3.15}$$

$$\left| \sum_{k=\lceil nx - Tn^\alpha \rceil}^{[nx+Tn^\alpha]} \frac{1}{In^\alpha} b\left(n^{1-\alpha}\left(x - \frac{k}{n} \right) \right) - 1 \right| +$$

$$\frac{b^*}{I}\left(2T + \frac{1}{n^\alpha}\right)\frac{T^\beta}{\Gamma(\beta+1)\, n^{(1-\alpha)\beta}} \cdot$$

$$\left\{ \omega_1\left(D_{*x}^\beta f, \frac{T}{n^{1-\alpha}} \right)_{[x,+\infty)} + \omega_1\left(D_{x-}^\beta f, \frac{T}{n^{1-\alpha}} \right)_{(-\infty,x]} \right\}.$$

Here we get fractionally with rates the pointwise convergence of $(F_n(f))(x) \to f(x)$, *as* $n \to \infty$, $x \in \mathbb{R}$.

Proof. Let $x \in \mathbb{R}$. We have that

$$D_{x-}^\beta f(x) = D_{*x}^\beta f(x) = 0. \tag{3.16}$$

From [8], p. 54, we get by the left Caputo fractional Taylor formula that

$$f\left(\frac{k}{n}\right) = \sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!}\left(\frac{k}{n} - x\right)^j + \tag{3.17}$$

$$\frac{1}{\Gamma(\beta)} \int_x^{\frac{k}{n}} \left(\frac{k}{n} - J \right)^{\beta-1} \left(D_{*x}^\beta f(J) - D_{*x}^\beta f(x) \right) dJ,$$

for all $x \leq \frac{k}{n} \leq x + Tn^{\alpha-1}$, iff $\lceil nx \rceil \leq k \leq [nx + Tn^\alpha]$, where $k \in \mathbb{Z}$.

Also from [2], using the right Caputo fractional Taylor formula we get

$$f\left(\frac{k}{n}\right) = \sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!}\left(\frac{k}{n} - x\right)^j + \tag{3.18}$$

$$\frac{1}{\Gamma(\beta)} \int_{\frac{k}{n}}^{x} \left(J - \frac{k}{n} \right)^{\beta-1} \left(D_{x-}^\beta f(J) - D_{x-}^\beta f(x) \right) dJ,$$

for all $x - Tn^{\alpha-1} \leq \frac{k}{n} \leq x$, iff $\lceil nx - Tn^{\alpha} \rceil \leq k \leq [nx]$, where $k \in \mathbb{Z}$.

Notice that $\lceil nx \rceil \leq [nx] + 1$.

Hence we have

$$\frac{f\left(\frac{k}{n}\right) b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}{In^{\alpha}} = \sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!} \left(\frac{k}{n} - x\right)^{j} \frac{b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}{In^{\alpha}} + \tag{3.19}$$

$$\frac{b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}{In^{\alpha}\Gamma(\beta)} \int_{x}^{\frac{k}{n}} \left(\frac{k}{n} - J\right)^{\beta-1} \left(D_{*x}^{\beta} f(J) - D_{*x}^{\beta} f(x)\right) dJ,$$

and

$$\frac{f\left(\frac{k}{n}\right) b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}{In^{\alpha}} = \sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!} \left(\frac{k}{n} - x\right)^{j} \frac{b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}{In^{\alpha}} + \tag{3.20}$$

$$\frac{b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}{In^{\alpha}\Gamma(\beta)} \int_{\frac{k}{n}}^{x} \left(J - \frac{k}{n}\right)^{\beta-1} \left(D_{x-}^{\beta} f(J) - D_{x-}^{\beta} f(x)\right) dJ.$$

Therefore we obtain

$$\frac{\sum_{k=[nx]+1}^{[nx+Tn^{\alpha}]} f\left(\frac{k}{n}\right) b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}{In^{\alpha}} = \tag{3.21}$$

$$\sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!} \left(\frac{\sum_{k=[nx]+1}^{[nx+Tn^{\alpha}]} \left(\frac{k}{n} - x\right)^{j} b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}{In^{\alpha}}\right) +$$

$$\sum_{k=[nx]+1}^{[nx+Tn^{\alpha}]} \frac{b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}{In^{\alpha}\Gamma(\beta)} \int_{x}^{\frac{k}{n}} \left(\frac{k}{n} - J\right)^{\beta-1} \left(D_{*x}^{\beta} f(J) - D_{*x}^{\beta} f(x)\right) dJ,$$

and

$$\frac{\sum_{k=\lceil nx - Tn^{\alpha} \rceil}^{[nx]} f\left(\frac{k}{n}\right) b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}{In^{\alpha}} = \tag{3.22}$$

$$\sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!} \frac{\sum_{k=\lceil nx - Tn^{\alpha} \rceil}^{[nx]} \left(\frac{k}{n} - x\right)^{j} b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}{In^{\alpha}} +$$

$$\frac{\sum_{k=\lceil nx - Tn^{\alpha} \rceil}^{[nx]} b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}{In^{\alpha}\Gamma(\beta)} \int_{\frac{k}{n}}^{x} \left(J - \frac{k}{n}\right)^{\beta-1} \left(D_{x-}^{\beta} f(J) - D_{x-}^{\beta} f(x)\right) dJ.$$

We notice here that

$$(F_n(f))(x) := \sum_{k=-n^2}^{n^2} \frac{f\left(\frac{k}{n}\right)}{In^{\alpha}} b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right) = \tag{3.23}$$

$$\sum_{k=\lceil nx - Tn^{\alpha} \rceil}^{[nx+Tn^{\alpha}]} \frac{f\left(\frac{k}{n}\right)}{In^{\alpha}} b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right).$$

Adding the two equalities (3.21) and (3.22) we obtain

$$(F_n(f))(x) =$$

$$\sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!} \left(\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \left(\frac{k}{n}-x\right)^{j} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{In^{\alpha}} \right) + \theta_n(x), \tag{3.24}$$

where

$$\theta_n(x) := \frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx]} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{In^{\alpha}\Gamma(\beta)} \cdot$$

$$\int_{\frac{k}{n}}^{x} \left(J-\frac{k}{n}\right)^{\beta-1} \left(D_{x-}^{\beta} f(J) - D_{x-}^{\beta} f(x)\right) dJ +$$

$$\sum_{k=[nx]+1}^{[nx+Tn^{\alpha}]} \frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{In^{\alpha}\Gamma(\beta)} \int_{x}^{\frac{k}{n}} \left(\frac{k}{n}-J\right)^{\beta-1} \left(D_{*x}^{\beta} f(J) - D_{*x}^{\beta} f(x)\right) dJ. \tag{3.25}$$

We call

$$\theta_{1n}(x) := \frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx]} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{In^{\alpha}\Gamma(\beta)} \cdot$$

$$\int_{\frac{k}{n}}^{x} \left(J-\frac{k}{n}\right)^{\beta-1} \left(D_{x-}^{\beta} f(J) - D_{x-}^{\beta} f(x)\right) dJ, \tag{3.26}$$

and

$$\theta_{2n}(x) := \sum_{k=[nx]+1}^{[nx+Tn^{\alpha}]} \frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{In^{\alpha}\Gamma(\beta)} \cdot$$

$$\int_{x}^{\frac{k}{n}} \left(\frac{k}{n}-J\right)^{\beta-1} \left(D_{*x}^{\beta} f(J) - D_{*x}^{\beta} f(x)\right) dJ. \tag{3.27}$$

I.e.

$$\theta_n(x) = \theta_{1n}(x) + \theta_{2n}(x). \tag{3.28}$$

We further have

$$(F_n(f))(x) - f(x) = f(x) \left(\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{In^{\alpha}} - 1 \right) + \tag{3.29}$$

$$\sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!} \left(\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \left(\frac{k}{n}-x\right)^{j} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{In^{\alpha}} \right) + \theta_n(x),$$

and

$$|(F_n(f))(x) - f(x)| \le |f(x)| \left| \sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \frac{1}{In^{\alpha}} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right) - 1 \right| +$$

$$\sum_{j=1}^{N-1} \frac{\left|f^{(j)}(x)\right|}{j!} \left(\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \left|x-\frac{k}{n}\right|^{j} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{In^{\alpha}} \right) + |\theta_n(x)| \le$$

$$|f(x)| \left| \sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \frac{1}{In^{\alpha}} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right) - 1 \right| + \tag{3.30}$$

$$\sum_{j=1}^{N-1} \frac{\left|f^{(j)}(x)\right|}{j!} \frac{T^j}{n^{(1-\alpha)j}} \left(\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{In^{\alpha}} \right) + |\theta_n(x)| =: (*).$$

But we have

$$\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \frac{1}{In^{\alpha}} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right) \leq \frac{b^*}{I}\left(2T+\frac{1}{n^{\alpha}}\right), \tag{3.31}$$

by (3.7).

Therefore we obtain

$$|(F_n(f))(x) - f(x)| \leq |f(x)| \left| \sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \frac{1}{In^{\alpha}} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right) - 1 \right| +$$

$$\frac{b^*}{I}\left(2T+\frac{1}{n^{\alpha}}\right)\left(\sum_{j=1}^{N-1} \frac{\left|f^{(j)}(x)\right| T^j}{j! n^{(1-\alpha)j}}\right) + |\theta_n(x)|. \tag{3.32}$$

Next we see that

$$\gamma_{1n} := \frac{1}{\Gamma(\beta)} \left| \int_{\frac{k}{n}}^{x} \left(J-\frac{k}{n}\right)^{\beta-1} \left(D_{x-}^{\beta} f(J) - D_{x-}^{\beta} f(x)\right) dJ \right| \leq \tag{3.33}$$

$$\frac{1}{\Gamma(\beta)} \int_{\frac{k}{n}}^{x} \left(J-\frac{k}{n}\right)^{\beta-1} \left| D_{x-}^{\beta} f(J) - D_{x-}^{\beta} f(x)\, dJ \right| \leq$$

$$\frac{1}{\Gamma(\beta)} \int_{\frac{k}{n}}^{x} \left(J-\frac{k}{n}\right)^{\beta-1} \omega_1\left(D_{x-}^{\beta} f, |J-x|\right)_{(-\infty,x]} dJ \leq$$

$$\frac{1}{\Gamma(\beta)} \omega_1\left(D_{x-}^{\beta} f, \left|x-\frac{k}{n}\right|\right)_{(-\infty,x]} \int_{\frac{k}{n}}^{x} \left(J-\frac{k}{n}\right)^{\beta-1} dJ \leq$$

$$\frac{1}{\Gamma(\beta)} \omega_1\left(D_{x-}^{\beta} f, \frac{T}{n^{1-\alpha}}\right)_{(-\infty,x]} \frac{\left(x-\frac{k}{n}\right)^{\beta}}{\beta} \leq$$

$$\frac{1}{\Gamma(\beta+1)} \omega_1\left(D_{x-}^{\beta} f, \frac{T}{n^{1-\alpha}}\right)_{(-\infty,x]} \frac{T^{\beta}}{n^{(1-\alpha)\beta}}.$$

That is

$$\gamma_{1n} \leq \frac{T^{\beta}}{\Gamma(\beta+1) n^{(1-\alpha)\beta}} \omega_1\left(D_{x-}^{\beta} f, \frac{T}{n^{1-\alpha}}\right)_{(-\infty,x]}. \tag{3.34}$$

Furthermore

$$|\theta_{1n}(x)| \leq \sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx]} \frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{In^{\alpha}} \gamma_{1n} \leq \tag{3.35}$$

$$\left(\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx]} \frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{In^{\alpha}}\right) \frac{T^{\beta}}{\Gamma(\beta+1) n^{(1-\alpha)\beta}} \omega_1\left(D_{x-}^{\beta} f, \frac{T}{n^{1-\alpha}}\right)_{(-\infty,x]} \leq$$

$$\left(\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{In^{\alpha}}\right)\frac{T^{\beta}}{\Gamma(\beta+1)\,n^{(1-\alpha)\beta}}\omega_1\left(D_{x-}^{\beta}f,\frac{T}{n^{1-\alpha}}\right)_{(-\infty,x]}\leq$$

$$\frac{b^*}{I}\left(2T+\frac{1}{n^{\alpha}}\right)\frac{T^{\beta}}{\Gamma(\beta+1)\,n^{(1-\alpha)\beta}}\omega_1\left(D_{x-}^{\beta}f,\frac{T}{n^{1-\alpha}}\right)_{(-\infty,x]}.$$

So that

$$|\theta_{1n}(x)|\leq\frac{b^*}{I}\left(2T+\frac{1}{n^{\alpha}}\right)\frac{T^{\beta}}{\Gamma(\beta+1)\,n^{(1-\alpha)\beta}}\omega_1\left(D_{x-}^{\beta}f,\frac{T}{n^{1-\alpha}}\right)_{(-\infty,x]}. \tag{3.36}$$

Similarly we derive

$$\gamma_{2n}:=\frac{1}{\Gamma(\beta)}\left|\int_x^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\beta-1}\left(D_{*x}^{\beta}f(J)-D_{*x}^{\beta}f(x)\right)dJ\right|\leq \tag{3.37}$$

$$\frac{1}{\Gamma(\beta)}\int_x^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\beta-1}\left|D_{*x}^{\beta}f(J)-D_{*x}^{\beta}f(x)\right|dJ\leq$$

$$\frac{\omega_1\left(D_{*x}^{\beta}f,\frac{T}{n^{1-\alpha}}\right)_{[x,+\infty)}}{\Gamma(\beta+1)}\left(\frac{k}{n}-x\right)^{\beta}\leq$$

$$\frac{\omega_1\left(D_{*x}^{\beta}f,\frac{T}{n^{1-\alpha}}\right)_{[x,+\infty)}}{\Gamma(\beta+1)}\frac{T^{\beta}}{n^{(1-\alpha)\beta}}.$$

That is

$$\gamma_{2n}\leq\frac{T^{\beta}}{\Gamma(\beta+1)\,n^{(1-\alpha)\beta}}\omega_1\left(D_{*x}^{\beta}f,\frac{T}{n^{1-\alpha}}\right)_{[x,+\infty)}. \tag{3.38}$$

Consequently we find

$$|\theta_{2n}(x)|\leq\left(\sum_{k=[nx]+1}^{[nx+Tn^{\alpha}]}\frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{In^{\alpha}}\right)\cdot$$

$$\frac{T^{\beta}}{\Gamma(\beta+1)\,n^{(1-\alpha)\beta}}\omega_1\left(D_{*x}^{\beta}f,\frac{T}{n^{1-\alpha}}\right)_{[x,+\infty)}\leq \tag{3.39}$$

$$\frac{b^*}{I}\left(2T+\frac{1}{n^{\alpha}}\right)\frac{T^{\beta}}{\Gamma(\beta+1)\,n^{(1-\alpha)\beta}}\omega_1\left(D_{*x}^{\beta}f,\frac{T}{n^{1-\alpha}}\right)_{[x,+\infty)}.$$

So we have proved that

$$|\theta_n(x)|\leq\frac{b^*}{I}\left(2T+\frac{1}{n^{\alpha}}\right)\frac{T^{\beta}}{\Gamma(\beta+1)\,n^{(1-\alpha)\beta}}\cdot \tag{3.40}$$

$$\left\{\omega_1\left(D_{*x}^{\beta}f,\frac{T}{n^{1-\alpha}}\right)_{[x,+\infty)}+\omega_1\left(D_{x-}^{\beta}f,\frac{T}{n^{1-\alpha}}\right)_{(-\infty,x]}\right\}.$$

Combining (3.32) and (3.40) we have (3.12). □

As an application of Theorem 3.7 we give

Theorem 3.8. *Let $\beta > 0$, $N = \lceil \beta \rceil$, $\beta \notin \mathbb{N}$, $f \in C^N(\mathbb{R})$, with $f^{(N)} \in L_\infty(\mathbb{R})$. Let also $T > 0$, $n \in \mathbb{N} : n \geq \max\left(2T, T^{-\frac{1}{\alpha}}\right)$. We further assume that $D_{*x}^{\beta} f(t)$, $D_{x-}^{\beta} f(t)$ are both bounded in $(x,t) \in \mathbb{R}^2$. Then*

1)

$$\|F_n(f) - f\|_{\infty,[-T,T]} \leq \|f\|_{\infty,[-T,T]} \cdot \tag{3.41}$$

$$\left\| \sum_{k=\lceil nx-Tn^{\alpha} \rceil}^{[nx+Tn^{\alpha}]} \frac{1}{In^{\alpha}} b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right) - 1 \right\|_{\infty,[-T,T]} +$$

$$\frac{b^*}{I}\left(2T + \frac{1}{n^{\alpha}}\right)\left(\sum_{j=1}^{N-1} \frac{\|f^{(j)}\|_{\infty,[-T,T]} T^j}{j! n^{(1-\alpha)j}}\right) + \frac{b^*}{I}\left(2T + \frac{1}{n^{\alpha}}\right) \frac{T^{\beta}}{\Gamma(\beta+1) n^{(1-\alpha)\beta}} \cdot$$

$$\left\{ \sup_{x \in [-T,T]} \omega_1\left(D_{*x}^{\beta} f, \frac{T}{n^{1-\alpha}}\right)_{[x,+\infty)} + \sup_{x \in [-T,T]} \omega_1\left(D_{x-}^{\beta} f, \frac{T}{n^{1-\alpha}}\right)_{(-\infty,x]} \right\},$$

2) in case of $N = 1$, we obtain

$$\|F_n(f) - f\|_{\infty,[-T,T]} \leq \|f\|_{\infty,[-T,T]} \cdot \tag{3.42}$$

$$\left\| \sum_{k=\lceil nx-Tn^{\alpha} \rceil}^{[nx+Tn^{\alpha}]} \frac{1}{In^{\alpha}} b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right) - 1 \right\|_{\infty,[-T,T]} +$$

$$\frac{b^*}{I}\left(2T + \frac{1}{n^{\alpha}}\right) \frac{T^{\beta}}{\Gamma(\beta+1) n^{(1-\alpha)\beta}} \cdot$$

$$\left\{ \sup_{x \in [-T,T]} \omega_1\left(D_{*x}^{\beta} f, \frac{T}{n^{1-\alpha}}\right)_{[x,+\infty)} + \sup_{x \in [-T,T]} \omega_1\left(D_{x-}^{\beta} f, \frac{T}{n^{1-\alpha}}\right)_{(-\infty,x]} \right\}.$$

An interesting case is when $\beta = \frac{1}{2}$.

Assuming further that $\left\| \sum_{k=\lceil nx-Tn^{\alpha} \rceil}^{[nx+Tn^{\alpha}]} \frac{1}{In^{\alpha}} b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right) - 1 \right\|_{\infty,[-T,T]} \to 0$, as $n \to \infty$, we get fractionally with rates the uniform convergence of $F_n(f) \to f$, as $n \to \infty$.

Proof. From (3.12), (3.15) of Theorem 3.7, and by Remark 2.17.

Also by

$$\sum_{k=\lceil nx-Tn^{\alpha} \rceil}^{[nx+Tn^{\alpha}]} \frac{1}{In^{\alpha}} b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right) \leq \frac{b^*}{I}(2T+1), \tag{3.43}$$

we get that

$$\left\| \sum_{k=\lceil nx-Tn^{\alpha} \rceil}^{[nx+Tn^{\alpha}]} \frac{1}{In^{\alpha}} b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right) - 1 \right\|_{\infty,[-T,T]} \leq \left(\frac{b^*}{I}(2T+1) + 1\right). \tag{3.44}$$

□

One can also apply Remark 2.18 to the last Theorem 3.8, to get interesting and simplified results.

We make

Remark 3.9. Let $b(x)$ be a centered bell-shaped continuous function on $\mathbb{R}$ of compact support $[-T, T]$, $T > 0$. Let $x \in [-T^*, T^*]$, $T^* > 0$, and $n \in \mathbb{N} : n \geq \max\left(T + T^*, T^{-\frac{1}{\alpha}}\right)$, $0 < \alpha < 1$. Consider $p \geq 1$.

Clearly we get here that

$$\left|\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \frac{1}{In^{\alpha}} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)-1\right|^{p} \leq \left(\frac{b^*}{I}(2T+1)+1\right)^{p}, \tag{3.45}$$

for all $x \in [-T^*, T^*]$, for any $n \geq \max\left(T + T^*, T^{-\frac{1}{\alpha}}\right)$.

By Lemma 3.5, we obtain that

$$\lim_{n\to\infty}\left|\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \frac{1}{In^{\alpha}} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)-1\right|^{p} = 0, \tag{3.46}$$

all $x \in [-T^*, T^*]$.

Now it is clear, by the bounded convergence theorem, that

$$\lim_{n\to\infty}\left\|\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \frac{1}{In^{\alpha}} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)-1\right\|_{p,[-T^*,T^*]} = 0. \tag{3.47}$$

Let $\beta > 0$, $N = \lceil\beta\rceil$, $\beta \notin \mathbb{N}$, $f \in C^N(\mathbb{R})$, $f^{(N)} \in L_\infty(\mathbb{R})$. Here both $D^{\alpha}_{*x} f(t)$, $D^{\alpha}_{x-} f(t)$ are bounded in $(x,t) \in \mathbb{R}^2$.

By Theorem 3.7 we have

$$|F_n(f)(x) - f(x)| \leq \|f\|_{\infty,[-T^*,T^*]} \cdot \tag{3.48}$$

$$\left|\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \frac{1}{In^{\alpha}} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)-1\right| +$$

$$\frac{b^*}{I}\left(2T+\frac{1}{n^{\alpha}}\right)\left(\sum_{j=1}^{N-1}\frac{\left|f^{(j)}(x)\right| T^j}{j! n^{(1-\alpha)j}}\right)$$

$$+\frac{b^*}{I}\left(2T+\frac{1}{n^{\alpha}}\right)\frac{T^{\beta}}{\Gamma(\beta+1)\, n^{(1-\alpha)\beta}} \cdot$$

$$\left\{\sup_{x\in[-T^*,T^*]}\omega_1\left(D^{\beta}_{*x}f, \frac{T}{n^{1-\alpha}}\right)_{[x,+\infty)} + \sup_{x\in[-T^*,T^*]}\omega_1\left(D^{\beta}_{x-}f, \frac{T}{n^{1-\alpha}}\right)_{(-\infty,x]}\right\}.$$

Applying to the last inequality (3.48) the monotonicity and subadditive property of $\|\cdot\|_p$, we derive the following L_p, $p \geq 1$, interesting result.

Theorem 3.10. *Let $b(x)$ be a centered bell-shaped continuous function on $\mathbb{R}$ of compact support $[-T,T]$, $T>0$. Let $x\in[-T^*,T^*]$, $T^*>0$, and $n\in\mathbb{N}:n\geq\max\left(T+T^*,T^{-\frac{1}{\alpha}}\right)$, $0<\alpha<1$, $p\geq 1$. Let $\beta>0$, $N=\lceil\beta\rceil$, $\beta\notin\mathbb{N}$, $f\in C^N(\mathbb{R})$, with $f^{(N)}\in L_\infty(\mathbb{R})$. Here both $D^\beta_{*x}f(t)$, $D^\beta_{x-}f(t)$ are bounded in $(x,t)\in\mathbb{R}^2$. Then*

1)

$$\|F_nf-f\|_{p,[-T^*,T^*]}\leq\|f\|_{\infty,[-T^*,T^*]}\cdot \tag{3.49}$$

$$\left\|\sum_{k=\lceil nx-Tn^\alpha\rceil}^{[nx+Tn^\alpha]}\frac{1}{In^\alpha}b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)-1\right\|_{p,[-T^*,T^*]}+$$

$$\frac{b^*}{I}\left(2T+\frac{1}{n^\alpha}\right)\left(\sum_{j=1}^{N-1}\frac{\|f^{(j)}\|_{p,[-T^*,T^*]}T^j}{j!n^{(1-\alpha)j}}\right)+$$

$$\frac{2^{\frac{1}{p}}T^{*\frac{1}{p}}b^*}{I}\left(2T+\frac{1}{n^\alpha}\right)\frac{T^\beta}{\Gamma(\beta+1)n^{(1-\alpha)\beta}}\cdot$$

$$\left\{\sup_{x\in[-T^*,T^*]}\omega_1\left(D^\beta_{*x}f,\frac{T}{n^{1-\alpha}}\right)_{[x,+\infty)}+\sup_{x\in[-T^*,T^*]}\omega_1\left(D^\beta_{x-}f,\frac{T}{n^{1-\alpha}}\right)_{(-\infty,x]}\right\},$$

2) When $N=1$, we derive

$$\|F_nf-f\|_{p,[-T^*,T^*]}\leq\|f\|_{\infty,[-T^*,T^*]}\cdot \tag{3.50}$$

$$\left\|\sum_{k=\lceil nx-Tn^\alpha\rceil}^{[nx+Tn^\alpha]}\frac{1}{In^\alpha}b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)-1\right\|_{p,[-T^*,T^*]}+$$

$$\frac{2^{\frac{1}{p}}T^{*\frac{1}{p}}b^*}{I}\left(2T+\frac{1}{n^\alpha}\right)\frac{T^\beta}{\Gamma(\beta+1)n^{(1-\alpha)\beta}}\cdot$$

$$\left\{\sup_{x\in[-T^*,T^*]}\omega_1\left(D^\beta_{*x}f,\frac{T}{n^{1-\alpha}}\right)_{[x,+\infty)}+\sup_{x\in[-T^*,T^*]}\omega_1\left(D^\beta_{x-}f,\frac{T}{n^{1-\alpha}}\right)_{(-\infty,x]}\right\}.$$

By (3.49), (3.50) we derive the fractional L_p, $p\geq 1$, convergence with rates of F_nf to f.

3.2. The "Squashing operators" and their fractional convergence to the unit with rates

We need (see also [1], [7]).

Definition 3.11. *Let the nonnegative function $S:\mathbb{R}\to\mathbb{R}$, S has compact support $[-T,T]$, $T>0$, and is nondecreasing there and it can be continuous only on either $(-\infty,T]$ or $[-T,T]$. S can have jump discontinuities. We call S the "squashing function".*

Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Assume that

$$I^* := \int_{-T}^{T} S(t)\, dt > 0. \tag{3.51}$$

Obviously

$$\max_{x \in [-T,T]} S(x) = S(T). \tag{3.52}$$

For $x \in \mathbb{R}$ we define the "squashing operator" ([1])

$$(G_n(f))(x) := \sum_{k=-n^2}^{n^2} \frac{f\left(\frac{k}{n}\right)}{I^* \cdot n^\alpha} \cdot S\left(n^{1-\alpha} \cdot \left(x - \frac{k}{n}\right)\right), \tag{3.53}$$

$0 < \alpha < 1$ and $n \in \mathbb{N} : n \geq \max\left(T + |x|, T^{-\frac{1}{\alpha}}\right)$. It is clear that

$$(G_n(f))(x) = \sum_{k=\lceil nx - Tn^\alpha \rceil}^{[nx + Tn^\alpha]} \frac{f\left(\frac{k}{n}\right)}{I^* \cdot n^\alpha} \cdot S\left(n^{1-\alpha} \cdot \left(x - \frac{k}{n}\right)\right). \tag{3.54}$$

Here we study the fractional convergence with rates of $(G_n f)(x) \to f(x)$, as $n \to +\infty$, $x \in \mathbb{R}$.

Notice that

$$\sum_{k=\lceil nx - Tn^\alpha \rceil}^{[nx + Tn^\alpha]} 1 \leq (2Tn^\alpha + 1). \tag{3.55}$$

From [1] we need

Lemma 3.12. *It holds that*

$$D_n(x) := \sum_{k=\lceil nx - Tn^\alpha \rceil}^{[nx + Tn^\alpha]} \frac{1}{I^* \cdot n^\alpha} \cdot S\left(n^{1-\alpha} \cdot \left(x - \frac{k}{n}\right)\right) \to 1, \tag{3.56}$$

pointwise, as $n \to +\infty$, where $x \in \mathbb{R}$.

We present our second main result

Theorem 3.13. *We consider $f : \mathbb{R} \to \mathbb{R}$. Let $\beta > 0$, $N = \lceil \beta \rceil$, $\beta \notin \mathbb{N}$, $f \in AC^N([a,b])$, $\forall\ [a,b] \subset \mathbb{R}$, with $f^{(N)} \in L_\infty(\mathbb{R})$. Let also $x \in \mathbb{R}, T > 0, n \in \mathbb{N} : n \geq \max\left(T + |x|, T^{-\frac{1}{\alpha}}\right)$. We further assume that $D_{*x}^\beta f$, $D_{x-}^\beta f$ are uniformly continuous functions or continuous and bounded on $[x, +\infty)$, $(-\infty, x]$, respectively.*

Then

1)

$$|G_n(f)(x) - f(x)| \leq |f(x)| \cdot \tag{3.57}$$

$$\left| \sum_{k=\lceil nx - Tn^\alpha \rceil}^{[nx + Tn^\alpha]} \frac{1}{I^* n^\alpha} S\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right) - 1 \right| +$$

$$\frac{S(T)}{I^*}\left(2T + \frac{1}{n^\alpha}\right)\left(\sum_{j=1}^{N-1} \frac{\left|f^{(j)}(x)\right| T^j}{j! n^{(1-\alpha)j}}\right)$$

$$+\frac{S(T)}{I^*}\left(2T+\frac{1}{n^\alpha}\right)\frac{T^\beta}{\Gamma(\beta+1)\,n^{(1-\alpha)\beta}}\cdot$$

$$\left\{\omega_1\left(D_{*x}^\beta f,\frac{T}{n^{1-\alpha}}\right)_{[x,+\infty)}+\omega_1\left(D_{x-}^\beta f,\frac{T}{n^{1-\alpha}}\right)_{(-\infty,x]}\right\},$$

above $\sum_{j=1}^{0}\cdot=0$,

2)

$$\left|(G_n(f))(x)-\sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\left(G_n\left((\cdot-x)^j\right)\right)(x)\right|\leq \tag{3.58}$$

$$\frac{S(T)}{I^*}\left(2T+\frac{1}{n^\alpha}\right)\frac{T^\beta}{\Gamma(\beta+1)\,n^{(1-\alpha)\beta}}\cdot$$

$$\left\{\omega_1\left(D_{*x}^\beta f,\frac{T}{n^{1-\alpha}}\right)_{[x,+\infty)}+\omega_1\left(D_{x-}^\beta f,\frac{T}{n^{1-\alpha}}\right)_{(-\infty,x]}\right\}=:\lambda_n^*(x),$$

3) assume further that $f^{(j)}(x)=0$*, for* $j=0,1,...,N-1$*, we get*

$$|G_n(f)(x)|\leq\lambda_n^*(x), \tag{3.59}$$

4) in case of $N=1$*, we obtain*

$$|G_n(f)(x)-f(x)|\leq|f(x)|\cdot \tag{3.60}$$

$$\left|\sum_{k=\lceil nx-Tn^\alpha\rceil}^{[nx+Tn^\alpha]}\frac{1}{I^*n^\alpha}S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)-1\right|+$$

$$\frac{S(T)}{I^*}\left(2T+\frac{1}{n^\alpha}\right)\frac{T^\beta}{\Gamma(\beta+1)\,n^{(1-\alpha)\beta}}\cdot$$

$$\left\{\omega_1\left(D_{*x}^\beta f,\frac{T}{n^{1-\alpha}}\right)_{[x,+\infty)}+\omega_1\left(D_{x-}^\beta f,\frac{T}{n^{1-\alpha}}\right)_{(-\infty,x]}\right\}.$$

Here we get fractionally with rates the pointwise convergence of $(G_n(f))(x)\to f(x)$*, as* $n\to\infty$*,* $x\in\mathbb{R}$*.*

Proof. Let $x\in\mathbb{R}$. We have that

$$D_{x-}^\beta f(x)=D_{*x}^\beta f(x)=0.$$

From [8], p. 54, we get by the left Caputo fractional Taylor formula that

$$f\left(\frac{k}{n}\right)=\sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\left(\frac{k}{n}-x\right)^j+ \tag{3.61}$$

$$\frac{1}{\Gamma(\beta)}\int_x^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\beta-1}\left(D_{*x}^\beta f(J)-D_{*x}^\beta f(x)\right)dJ,$$

for all $x\leq\frac{k}{n}\leq x+Tn^{\alpha-1}$, iff $\lceil nx\rceil\leq k\leq[nx+Tn^\alpha]$, where $k\in\mathbb{Z}$.

Also from [2], using the right Caputo fractional Taylor formula we get

$$f\left(\frac{k}{n}\right)=\sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\left(\frac{k}{n}-x\right)^{j}+ \tag{3.62}$$

$$\frac{1}{\Gamma(\beta)}\int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\beta-1}\left(D_{x-}^{\beta}f(J)-D_{x-}^{\beta}f(x)\right)dJ,$$

for all $x-Tn^{\alpha-1}\leq\frac{k}{n}\leq x$, iff $\lceil nx-Tn^{\alpha}\rceil\leq k\leq[nx]$, where $k\in\mathbb{Z}$.

Hence we have

$$\frac{f\left(\frac{k}{n}\right)S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{I^{*}n^{\alpha}}=\sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\left(\frac{k}{n}-x\right)^{j}\frac{S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{I^{*}n^{\alpha}}+ \tag{3.63}$$

$$\frac{S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{I^{*}n^{\alpha}\Gamma(\beta)}\int_{x}^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\beta-1}\left(D_{*x}^{\beta}f(J)-D_{*x}^{\beta}f(x)\right)dJ,$$

and

$$\frac{f\left(\frac{k}{n}\right)S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{I^{*}n^{\alpha}}=\sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\left(\frac{k}{n}-x\right)^{j}\frac{S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{I^{*}n^{\alpha}}+ \tag{3.64}$$

$$\frac{S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{I^{*}n^{\alpha}\Gamma(\beta)}\int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\beta-1}\left(D_{x-}^{\beta}f(J)-D_{x-}^{\beta}f(x)\right)dJ.$$

Therefore we obtain

$$\frac{\sum_{k=[nx]+1}^{[nx+Tn^{\alpha}]}f\left(\frac{k}{n}\right)S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{I^{*}n^{\alpha}}= \tag{3.65}$$

$$\sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\left(\frac{\sum_{k=[nx]+1}^{[nx+Tn^{\alpha}]}\left(\frac{k}{n}-x\right)^{j}S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{I^{*}n^{\alpha}}\right)+$$

$$\sum_{k=[nx]+1}^{[nx+Tn^{\alpha}]}\frac{S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{I^{*}n^{\alpha}\Gamma(\beta)}\int_{x}^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\beta-1}\left(D_{*x}^{\beta}f(J)-D_{*x}^{\beta}f(x)\right)dJ,$$

and

$$\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx]}f\left(\frac{k}{n}\right)S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{I^{*}n^{\alpha}}= \tag{3.66}$$

$$\sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx]}\left(\frac{k}{n}-x\right)^{j}S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{I^{*}n^{\alpha}}+$$

$$\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx]}S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{I^{*}n^{\alpha}\Gamma(\beta)}\int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\beta-1}\left(D_{x-}^{\beta}f(J)-D_{x-}^{\beta}f(x)\right)dJ.$$

Adding the two equalities (3.65) and (3.66) we obtain

$$(G_{n}(f))(x)=$$

$$\sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!} \left(\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \left(\frac{k}{n}-x\right)^j S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{I^* n^{\alpha}} \right) + M_n(x), \tag{3.67}$$

where

$$M_n(x) := \frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx]} S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{I^* n^{\alpha}\Gamma(\beta)} \cdot$$

$$\int_{\frac{k}{n}}^{x} \left(J-\frac{k}{n}\right)^{\beta-1} \left(D_{x-}^{\beta} f(J) - D_{x-}^{\beta} f(x)\right) dJ +$$

$$\sum_{k=[nx]+1}^{[nx+Tn^{\alpha}]} \frac{S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{I^* n^{\alpha}\Gamma(\beta)} \int_{x}^{\frac{k}{n}} \left(\frac{k}{n}-J\right)^{\beta-1} \left(D_{*x}^{\beta} f(J) - D_{*x}^{\beta} f(x)\right) dJ. \tag{3.68}$$

We call

$$M_{1n}(x) := \sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx]} \frac{S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{I^* n^{\alpha}\Gamma(\beta)} \cdot$$

$$\int_{\frac{k}{n}}^{x} \left(J-\frac{k}{n}\right)^{\beta-1} \left(D_{x-}^{\beta} f(J) - D_{x-}^{\beta} f(x)\right) dJ, \tag{3.69}$$

and

$$M_{2n}(x) := \sum_{k=[nx]+1}^{[nx+Tn^{\alpha}]} \frac{S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{I^* n^{\alpha}\Gamma(\beta)} \cdot$$

$$\int_{x}^{\frac{k}{n}} \left(\frac{k}{n}-J\right)^{\beta-1} \left(D_{*x}^{\beta} f(J) - D_{*x}^{\beta} f(x)\right) dJ. \tag{3.70}$$

I.e.

$$M_n(x) = M_{1n}(x) + M_{2n}(x). \tag{3.71}$$

We further have

$$(G_n(f))(x) - f(x) = f(x) \left(\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{I^* n^{\alpha}} - 1 \right) + \tag{3.72}$$

$$\sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!} \left(\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \left(\frac{k}{n}-x\right)^j S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{I^* n^{\alpha}} \right) + M_n(x),$$

and

$$|(G_n(f))(x) - f(x)| \le |f(x)| \left| \sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \frac{1}{I^* n^{\alpha}} S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right) - 1 \right| + \tag{3.73}$$

$$\sum_{j=1}^{N-1} \frac{\left|f^{(j)}(x)\right|}{j!} \left(\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \left|x-\frac{k}{n}\right|^j S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{I^* n^{\alpha}} \right) + |M_n(x)| \le$$

$$|f(x)|\left|\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\frac{1}{I^{*}n^{\alpha}}S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)-1\right|+$$

$$\sum_{j=1}^{N-1}\frac{\left|f^{(j)}(x)\right|}{j!}\frac{T^{j}}{n^{(1-\alpha)j}}\left(\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{I^{*}n^{\alpha}}\right)+|M_{n}(x)|=:(*). \quad (3.74)$$

Therefore we obtain

$$|(G_{n}(f))(x)-f(x)|\leq|f(x)|\left|\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\frac{1}{I^{*}n^{\alpha}}S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)-1\right|+$$

$$\frac{S(T)}{I^{*}}\left(2T+\frac{1}{n^{\alpha}}\right)\left(\sum_{j=1}^{N-1}\frac{\left|f^{(j)}(x)\right|T^{j}}{j!n^{(1-\alpha)j}}\right)+|M_{n}(x)|. \quad (3.75)$$

We call

$$\gamma_{1n}:=\frac{1}{\Gamma(\beta)}\left|\int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\beta-1}\left(D_{x-}^{\beta}f(J)-D_{x-}^{\beta}f(x)\right)dJ\right|. \quad (3.76)$$

As in the proof of Theorem 3.7 we have

$$\gamma_{1n}\leq\frac{T^{\beta}}{\Gamma(\beta+1)n^{(1-\alpha)\beta}}\omega_{1}\left(D_{x-}^{\beta}f,\frac{T}{n^{1-\alpha}}\right)_{(-\infty,x]}. \quad (3.77)$$

Furthermore

$$|M_{1n}(x)|\leq\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx]}\frac{S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{I^{*}n^{\alpha}}\gamma_{1n}\leq \quad (3.78)$$

$$\left(\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx]}\frac{S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{I^{*}n^{\alpha}}\right)\frac{T^{\beta}}{\Gamma(\beta+1)n^{(1-\alpha)\beta}}\omega_{1}\left(D_{x-}^{\beta}f,\frac{T}{n^{1-\alpha}}\right)_{(-\infty,x]}\leq$$

$$\left(\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\frac{S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{I^{*}n^{\alpha}}\right)\frac{T^{\beta}}{\Gamma(\beta+1)n^{(1-\alpha)\beta}}\omega_{1}\left(D_{x-}^{\beta}f,\frac{T}{n^{1-\alpha}}\right)_{(-\infty,x]}\leq$$

$$\frac{S(T)}{I^{*}}\left(2T+\frac{1}{n^{\alpha}}\right)\frac{T^{\beta}}{\Gamma(\beta+1)n^{(1-\alpha)\beta}}\omega_{1}\left(D_{x-}^{\beta}f,\frac{T}{n^{1-\alpha}}\right)_{(-\infty,x]}.$$

So that

$$|M_{1n}(x)|\leq\frac{S(T)}{I^{*}}\left(2T+\frac{1}{n^{\alpha}}\right)\frac{T^{\beta}}{\Gamma(\beta+1)n^{(1-\alpha)\beta}}\omega_{1}\left(D_{x-}^{\beta}f,\frac{T}{n^{1-\alpha}}\right)_{(-\infty,x]}. \quad (3.79)$$

We also call

$$\gamma_{2n}:=\frac{1}{\Gamma(\beta)}\left|\int_{x}^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\beta-1}\left(D_{*x}^{\beta}f(J)-D_{*x}^{\beta}f(x)\right)dJ\right|. \quad (3.80)$$

As in the proof of Theorem 3.7 we get

$$\gamma_{2n} \leq \frac{T^{\beta}}{\Gamma(\beta+1) n^{(1-\alpha)\beta}} \omega_1 \left(D_{*x}^{\beta} f, \frac{T}{n^{1-\alpha}} \right)_{[x,+\infty)}. \tag{3.81}$$

Consequently we find

$$|M_{2n}(x)| \leq \left(\sum_{k=[nx]+1}^{[nx+Tn^{\alpha}]} \frac{S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{I^{*} n^{\alpha}} \right) \cdot$$

$$\frac{T^{\beta}}{\Gamma(\beta+1) n^{(1-\alpha)\beta}} \omega_1 \left(D_{*x}^{\beta} f, \frac{T}{n^{1-\alpha}} \right)_{[x,+\infty)} \leq \tag{3.82}$$

$$\frac{S(T)}{I^{*}} \left(2T + \frac{1}{n^{\alpha}} \right) \frac{T^{\beta}}{\Gamma(\beta+1) n^{(1-\alpha)\beta}} \omega_1 \left(D_{*x}^{\beta} f, \frac{T}{n^{1-\alpha}} \right)_{[x,+\infty)}.$$

So we have proved that

$$|M_n(x)| \leq \frac{S(T)}{I^{*}} \left(2T + \frac{1}{n^{\alpha}} \right) \frac{T^{\beta}}{\Gamma(\beta+1) n^{(1-\alpha)\beta}} \cdot \tag{3.83}$$

$$\left\{ \omega_1 \left(D_{*x}^{\beta} f, \frac{T}{n^{1-\alpha}} \right)_{[x,+\infty)} + \omega_1 \left(D_{x-}^{\beta} f, \frac{T}{n^{1-\alpha}} \right)_{(-\infty,x]} \right\}.$$

Combining (3.75) and (3.83) we have (3.57). □

As an application of Theorem 3.13 we give

Theorem 3.14. *Let $\beta > 0$, $N = \lceil \beta \rceil$, $\beta \notin \mathbb{N}$, $f \in C^{N}(\mathbb{R})$, with $f^{(N)} \in L_{\infty}(\mathbb{R})$. Let also $T > 0$, $n \in \mathbb{N} : n \geq \max\left(2T, T^{-\frac{1}{\alpha}}\right)$. We further assume that $D_{*x}^{\beta} f(t)$, $D_{x-}^{\beta} f(t)$ are both bounded in $(x,t) \in \mathbb{R}^2$. Then*

1)

$$\|G_n(f) - f\|_{\infty,[-T,T]} \leq \|f\|_{\infty,[-T,T]} \cdot \tag{3.84}$$

$$\left\| \sum_{k=\lceil nx-Tn^{\alpha} \rceil}^{[nx+Tn^{\alpha}]} \frac{1}{I^{*} n^{\alpha}} S\left(n^{1-\alpha} \left(x - \frac{k}{n} \right) \right) - 1 \right\|_{\infty,[-T,T]} +$$

$$\frac{S(T)}{I^{*}} \left(2T + \frac{1}{n^{\alpha}} \right) \left(\sum_{j=1}^{N-1} \frac{\left\| f^{(j)} \right\|_{\infty,[-T,T]} T^{j}}{j! n^{(1-\alpha)j}} \right) +$$

$$\frac{S(T)}{I^{*}} \left(2T + \frac{1}{n^{\alpha}} \right) \frac{T^{\beta}}{\Gamma(\beta+1) n^{(1-\alpha)\beta}} \cdot$$

$$\left\{ \sup_{x\in[-T,T]} \omega_1 \left(D_{*x}^{\beta} f, \frac{T}{n^{1-\alpha}} \right)_{[x,+\infty)} + \sup_{x\in[-T,T]} \omega_1 \left(D_{x-}^{\beta} f, \frac{T}{n^{1-\alpha}} \right)_{(-\infty,x]} \right\},$$

2) in case of $N = 1$, we obtain

$$\|G_n(f) - f\|_{\infty,[-T,T]} \leq \|f\|_{\infty,[-T,T]} \cdot \tag{3.85}$$

$$\left\| \sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \frac{1}{I^{*}n^{\alpha}} S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right) - 1 \right\|_{\infty,[-T,T]} +$$
$$\frac{S(T)}{I^{*}}\left(2T+\frac{1}{n^{\alpha}}\right)\frac{T^{\beta}}{\Gamma(\beta+1)\,n^{(1-\alpha)\beta}}\cdot$$
$$\left\{ \sup_{x\in[-T,T]} \omega_1\left(D_{*x}^{\beta}f, \frac{T}{n^{1-\alpha}}\right)_{[x,+\infty)} + \sup_{x\in[-T,T]} \omega_1\left(D_{x-}^{\beta}f, \frac{T}{n^{1-\alpha}}\right)_{(-\infty,x]} \right\}.$$

An interesting case is when $\beta = \frac{1}{2}$.

Assuming further that $\left\| \sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \frac{1}{I^{*}n^{\alpha}} S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right) - 1 \right\|_{\infty,[-T,T]} \to 0$, *as* $n \to \infty$, *we get fractionally with rates the uniform convergence of* $G_n(f) \to f$, *as* $n \to \infty$.

Proof. From (3.57), (3.60) of Theorem 3.13, and by Remark 2.17.

Also by

$$\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \frac{1}{I^{*}n^{\alpha}} S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right) \le \frac{S(T)}{I^{*}}(2T+1), \tag{3.86}$$

we get that

$$\left\| \sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \frac{1}{I^{*}n^{\alpha}} S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right) - 1 \right\|_{\infty,[-T,T]} \le \left(\frac{S(T)}{I^{*}}(2T+1)+1\right). \tag{3.87}$$

□

One can also apply Remark 2.18 to the last Theorem 3.14, to get interesting and simplified results.

Note 3.15. *The maps* F_n, G_n, $n \in \mathbb{N}$, *are positive linear operators.*

We finish with

Remark 3.16. The condition of Theorem 3.8 that

$$\left\| \sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \frac{1}{In^{\alpha}} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right) - 1 \right\|_{\infty,[-T,T]} \to 0, \tag{3.88}$$

as $n \to \infty$, is not uncommon.

We give an example related to that.

We take as $b(x)$ the characteristic function over $[-1,1]$, that is $\chi_{[-1,1]}(x)$. Here $T = 1$ and $I = 2$, $n \ge 2$, $x \in [-1,1]$.

We get that

$$\sum_{k=\lceil nx-n^{\alpha}\rceil}^{[nx+n^{\alpha}]} \frac{1}{2n^{\alpha}} \chi_{[-1,1]}\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right) \overset{(3.2)}{=} \sum_{k=\lceil nx-n^{\alpha}\rceil}^{[nx+n^{\alpha}]} \frac{1}{2n^{\alpha}} =$$

$$\frac{1}{2n^{\alpha}}\left(\sum_{k=\lceil nx-n^{\alpha}\rceil}^{[nx+n^{\alpha}]} 1\right) = \frac{([nx+n^{\alpha}] - \lceil nx-n^{\alpha}\rceil + 1)}{2n^{\alpha}}. \tag{3.89}$$

But we have

$$[nx+n^{\alpha}] - \lceil nx-n^{\alpha}\rceil + 1 \leq 2n^{\alpha} + 1,$$

hence

$$\frac{([nx+n^{\alpha}] - \lceil nx-n^{\alpha}\rceil + 1)}{2n^{\alpha}} \leq 1 + \frac{1}{2n^{\alpha}}. \tag{3.90}$$

Also it holds

$$[nx+n^{\alpha}] - \lceil nx-n^{\alpha}\rceil + 1 \geq 2n^{\alpha} - 2 + 1 = 2n^{\alpha} - 1,$$

and

$$\frac{([nx+n^{\alpha}] - \lceil nx-n^{\alpha}\rceil + 1)}{2n^{\alpha}} \geq 1 - \frac{1}{2n^{\alpha}}. \tag{3.91}$$

Consequently we derive that

$$-\frac{1}{2n^{\alpha}} \leq \left(\sum_{k=\lceil nx-n^{\alpha}\rceil}^{[nx+n^{\alpha}]} \frac{1}{2n^{\alpha}}\chi_{[-1,1]}\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right) - 1\right) \leq \frac{1}{2n^{\alpha}}, \tag{3.92}$$

for any $x \in [-1, 1]$ and for any $n \geq 2$.

Hence we get

$$\left\|\sum_{k=\lceil nx-n^{\alpha}\rceil}^{[nx+n^{\alpha}]} \frac{1}{2n^{\alpha}}\chi_{[-1,1]}\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right) - 1\right\|_{\infty,[-1,1]} \to 0, \text{ as } n \to \infty. \tag{3.93}$$

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