Stud. Univ. Babeş-Bolyai Math. 57(2012), No. 3, 325–329

A remark on the proof of Cobzaş-Mustăţa theorem concerning norm preserving extension of convex Lipschitz functions

Iulian Cîmpean

Abstract. In this paper we present an alternative proof of a result concerning norm preserving extension of convex Lipschitz functions due to Ştefan Cobzaş and Costică Mustăţa (see Norm preserving extension of convex Lipschitz functions, Journal of Approximation Theory,24(3)(1987),236-244). Our proof is based on the Choquet Topological lemma, (see J.L.Doob, Classical potential theory and its probabilistic counterpart, Springer Verlag 2001).

Mathematics Subject Classification (2010): 26A16, 26A51.

Keywords: Extension of Lipschitz functions, convex functions, Choquet topological lemma.

1. Introduction

Taking into account a famous result due to Rademacher which states that a Lipschitz function $f : U = \mathring{U} \subseteq \mathbb{R}^m \to \mathbb{R}^n$ is differentiable outside of a Lebesgue null subset of U, one can say that, from the point of view of real analysis the condition of being Lipschitz should be viewed as a weakened version of differentiability. Therefore, the class of Lipschitz functions has been intensively studied. The paper [9] is a very good introduction to the study of Lipschitz topology. One can also consult [16] and [22] for further details about Lipschitz functions.

The problem of the extension of a Lipschitz function is a central one in the theory of Lipschitz functions. Let us mention here just a phrase due to Earl Mickle (see [11]) which sustains our statement: "In a problem on surface area the writer and Helsel were confronted with the following question: Can a Lipschitz function be extended to a Lipschitz transformation defined in the whole space?" Consequently, there is no surprise that there exist a lot of results in this direction (see for example [1]-[5], [7], [8], [10]-[15], [17]-[21]).

2. Preliminaries

Let (X, d) be a metric space. A function $f : X \to \mathbb{R}$ is called Lipschitz if there exist a constant number $M \geq 0$ such that

$$|f(x) - f(y)| \leq Md(x, y) \tag{2.1}$$

for all $x, y \in X$.

The smallest constant M verifying (2.1) is called the norm of f and is denoted by $\|f\|_X$.

Denote by Lip$X$ the linear space of all Lipschitz functions on X.

Now let Y be a nonvoid subset of X. A norm preserving extension of a function $f \in$ Lip$Y$ to X is a function $F \in$ Lip$X$ such that

$$F|_Y = f$$

and

$$\|f\|_Y = \|F\|_X .$$

By a result of McShane [10], every $f \in$ Lip$Y$ has a norm preserving extension $F \in$ Lip$X$. Two of these extensions are given by:

$$F_1(x) = \sup\{f(y) - \|f\|_Y d(x, y) \mid y \in Y\} \tag{2.2}$$

$$F_2(x) = \inf\{f(y) + \|f\|_Y d(x, y) \mid y \in Y\} \tag{2.3}$$

Every norm preserving extension F of f satisfies:

$$F_1(x) \leq F(x) \leq F_2(x),$$

for all $x \in X$ (see [4]).

It turns out that these results remain true for convex norm preserving extensions.

More precisely, given a normed linear space X and a nonvoid convex subset Y of X, Ş. Cobzaş and C. Mustăţa proved the following two results:

Theorem 2.1. (see [4]) *Every convex function $f \in LipY$ has a convex norm preserving extension F in $LipX$.*

Theorem 2.2. (see [4]) *For every convex function f in $LipY$, there exist two convex functions F_1, F_2, which are norm preserving extensions of f, such that:*

$$F_1(x) \leq F(x) \leq F_2(x),$$

for all $x \in X$ and for every convex norm preserving extension F.

The proof for the last theorem focuses on the existence of F_1, the existence of F_2 following from the fact that the function defined in (3) is also convex.

We will present an alternative proof for the existence of F_1, which is based on the Choquet topological lemma.

3. The result

Lemma 3.1. (Choquet topological lemma) (see [6], Appendix VIII) *Let* $U = \{u_\beta, \beta \in I\}$ *be a family of functions from a second countable Hausdorff space into* $\overline{\mathbb{R}}$*, and if* $J \subseteq I$*, define*

$$u^J = \inf \{u_\beta \mid \beta \in J\}.$$

Then there is a countable subset J *of* I *such that*

$$u_+^J = u_+^I.$$

In particular, if U *is directed downward, then there is a decreasing sequence* $(u_{\beta_n})_{n\geq 1} \subseteq U$ *with limit* v *such that*

$$v_+ = u_+^I.$$

By f_+, where f is a function from a Hausdorff space into $\overline{\mathbb{R}}$, we denote the lower semicontinuous minorant of f, which majorizes every lower semicontinuous minorant of f. That is

$$f_+(x_0) = f(x_0) \wedge \liminf_{x\to x_0} f(x).$$

Proof. The first assertion of the lemma is proved in [6] (Appendix VIII), so we will prove only the last assertion:

The first conclusion of the lemma assures us of the existence of a countable subset J of I such that

$$u_+^J = u_+^I, \tag{3.1}$$

which allows us to rewrite the family $\{u_\beta \mid \beta \in J\}$ as a sequence $(u_n)_{n\geq 1}$.

In order to complete the proof, we construct a decreasing sequence $(u_{\alpha_n})_{n\geq 1} \subseteq U$ with limit v such that $v_+ = u_+^I$, as follows:

Let $u_{\alpha_1} = u_1$. For each $n \geq 2$, let u_{α_n} be a function from U such that

$$u_{\alpha_n} \leq \min\left(u_{\alpha_{n-1}}, u_n\right).$$

This construction is possible because U is supposed downward directed. Let v be the limit of this decreasing sequence. Since $u_{\alpha_n} \leq u_n$, we have that $v \leq \inf_{n\geq 1} u_n = u^J$, so that:

$$v_+ \leq u_+^J. \tag{3.2}$$

On the orher hand, $u_{\alpha_n} \geq u^I$, for all $n \geq 1$, so that

$$v_+ \geq u_+^I. \tag{3.3}$$

Now, from (3.1), (3.2) and (3.3) it follows that

$$v_+ = u_+^J = u_+^I.$$ □

We need another lemma, also used and proved by Ş. Cobzaş and C. Mustăţa:

Lemma 3.2. (see [4]) *The set* $E_Y^c(f)$ *of all convex norm preserving extensions of* f *is downward directed (with respect to the pointwise ordering).*

Now, to prove the existence of F_1, combine the two lemmas as follows:
In Lemma 3.1 take $I = E_Y^c(f)$ and $u_\beta = \beta$, for each $\beta \in I$. Define

$$F_1 = u_+^I.$$

According to the same lemma, there is a decreasing sequence $(u_{\beta_n})_{n\geq 1}$ with limit v, such that $v_+ = u_+^I$. Since $u_{\beta_n} \in E_Y^c(f)$, then v is also in $E_Y^c(f)$, so that

$$v = v_+ = u_+^I = F_1 \in E_Y^c(f).$$

Clearly F_1 minimizes any other $F \in E_Y^c(f)$, which ends the proof.

References

[1] Assouad, P., *Remarques sur un article de Israel Aharoni sur les prolongements Lipschitziens dans* c_0, Israel Journal of Mathematics, **1**(1987), 97-100.

[2] Assouad, P., *Prolongements Lipschitziens dans* R^n, Bulletin de la Société Mathématique de France, **111**(1983), 429-448.

[3] Bressan, A., Cortesi, A., *Lipschitz extensions of convex-valued maps*, Atti della Accademia Nazionale dei Lincei, Rendiconti, Classe di Scenze Fische, Mathematice e Naturali, Serie VIII, **80**(1986), 530-532.

[4] Cobzaş, Ş., Mustăţa, C., *Norm preserving extensions of convex Lipschitz functions*, Journal of Approximation Theory, **24**(1978), 236-244.

[5] Czipszer, J., Gehér, L., *Extension of functions satisfying a Lipschitz condition*, Acta Mathematica Academica Scientiarum Hungarica, **6**(1955), 213-220.

[6] Doob J. L., *Classical potential theory and its probabilistic counterpart*, Springer Verlag, 2001.

[7] Flett, T.M., *Extensions of Lipschitz functions*, Journal of London Mathematical Society, **7**(1974), 604-608.

[8] Kirszbraun, M.D., *Über die zusammenziehenden und Lipschitzschen Transformationen*, Fundamenta Mathematicae, **22**(1934), 77-108.

[9] Luukkainen, J., Väisälä, J., *Elements of Lipschitz topology*, Annales Academiae Scientiarum Fennicae, Series A I. Mathematica, **3**(1977), 85-122.

[10] McShane, E.J., *Extension of range of functions*, Bulletin of the American Mathematical Society, **40**(1934), 837-842.

[11] Mickle, E.J., *On the extension of a transformation*, Bulletin of the American Mathematical Society, **55**(1949), 160-164.

[12] Miculescu, R., *Extensions of some locally Lipschitz maps*, Bulletin Mathématique de la Société des Sciences Mathématiques de Roumanie, **41**(89)(1998), 197-203.

[13] Miculescu, R., *Approximations by Lipschitz functions generated by extensions*, Real Analysis Exchange, **28**(2002/2003), 33-40.

[14] Miculescu, R., *Über die Erweiterung einer Metrik*, Mathematical Reports, **56**(2004), 451-457.

[15] Miculescu, R., *Some observations on generalized Lipschitz functions*, The Rocky Mountain Journal Of Mathematics, **37**(2007), 893-903.

[16] Miculescu, R., Mortici, C., *Functii Lipschitz*, Editura Academiei Romane, 2004.

[17] Mustăţa, C., *Norm preserving extension of starshaped Lipschitz functions*, Mathematica(Cluj), **19**(42)(1978), 183-187.

[18] Roşoiu, A., Frăţilă, D., *On the Lipschitz extension constant for a complex valued Lipschitz function*, Studia Universitatis Babeş-Bolyai, Mathematica, **53**(2008), 101-108.

[19] Schoenberg, I.J., *On a theorem of Kirszbraun and Valentine*, American Mathematical Monthly, **60**(1953), 620-622.

[20] Schonbeck, S.O., *On the extension of Lipschitz maps*, Arkiv für Mathematik, **7**(1967).

[21] Valentine, F.A., *A Lipschitz condition preserving extension for a vector function*, American Journal of Mathematics, **67**(1945), 83-93.

[22] Weaver, N., *Lipschitz Algebras*, World Scientific, Singapore, 1999.

Iulian Cîmpean
"Simion Stoilow" Institute of Mathematics of the Romanian Academy
Calea Griviţei Street, No. 21
010702 Bucharest, Romania
e-mail: cimpean_iulian2005@yahoo.com