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Some extensions on Fan Ky's inequality

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Abstract. In this paper we study the inequalities of the determinants of the positive definite matrices and the invertible matrices by applying the integral method and matrix theory such that extensions of Fan Ky's inequality are established. And then an improvement of Fan Ky's inequality is given by using the positive definiteness of Gram matrix.

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1. Introduction

In view of the importance of the inequality in theory and applications (see [1], [2]), it has been absorbing much interest of mathematicians. The various ways of proving inequalities appear in a great deal of papers. In particular, Kuang enumerated more than 50 methods in the paper [3]. It is obvious that these methods have the characteristic of themselves, technique, theory and applications. The purpose of the present paper is to study the discrete inequalities by applying a thought way on the proof of the inequality of the continuous function, and to try for a new path and to play to throw out a minnow to catch whale role in research and development. Explicitly, the extensions and improvement on the famous Fan Ky inequality are established by applying this method.

For convenience, we introduce some notations and functions.

The determinant of matrix X of order n is denoted by $|X|$ and a unit-matrix of order n is denoted by I. Let $x = (x_1, \ x_1, \ \cdots, x_n\ )$ be an n-dimension vector, $f(x)$ and $g(x)$ be functions with n variables. Let E be an inner product space, f and g be elements of E. Then the inner product of f and g is defined by the following n-ple integral:

$$(f,\ g) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} f(x)g(x)\mathrm{dx},$$

where $\mathrm{dx} = \mathrm{dx}_1\mathrm{dx}_2 \cdots \mathrm{dx}_n$. And the norm of f is given by $\|f\| = \sqrt{(f, f)}$.

Let $f(x),\ g(x) > 0$ and $r,\ s > 0$. We stipulate that

$$(f^r, g^s) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} f^r(x) g^s(x) \mathrm{dx}, \quad \|f\|_r = \left(\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} f^r(x) \mathrm{dx} \right)^{\frac{1}{r}}, \quad \|f\|_2 = \|f\|,$$

$S_r(f,h) = \left(f^{\frac{r}{2}}, h\right) \|f\|_r^{-\frac{1}{r}}$.

where h is a variable unit-vector with n variables, i.e. $\|h\| = 1$, and it can be chosen in accordance with our requirements. In particular, $S_r(f,h) = 0$ if h is orthogonal to $f^{\frac{r}{2}}$.

Throughout this paper, we shall frequently use these notations.

2. Statement of main results

Let A, B be two positive definite matrices of order n, $0 \le \lambda \le 1$. Then

$$|A|^{\lambda} |B|^{1-\lambda} \le |\lambda A + (1-\lambda)B|. \tag{2.1}$$

This is the famous Fan Fy's inequality (see [3]). Recently, this inequality has been studied in some papers (such as [4, 5] etc.) Below we will build some extensions and a refinement of (2.1) by using the integral method and matrix theory.

First, we establish some extensions of (2.1).

Theorem 2.1. *Let m be a positive integer greater than 1, $A_i (i = 1, 2, \cdots, m)$ be positive definite matrix of order n, $\sum_{i=1}^{m} \frac{1}{p_i} = 1$ and $p_i > 1$. Then*

$$\prod_{i=1}^{m} |A_i|^{\frac{1}{p_i}} \le \left| \sum_{i=1}^{m} \frac{1}{p_i} A_i \right|. \tag{2.2}$$

In particular, for case $m = 2$, we have

$$|A|^{\frac{1}{p}} |B|^{\frac{1}{q}} \le \left| \frac{1}{p} A + \frac{1}{q} B \right|,$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $p > 1$. Clearly, it is the inequality (2.1). It follows that the inequality (2.2) is an extension of (2.1).

Remark 2.2. Inequality (2.1) shows that the function $f : PD \to (0, \infty)$ defined by $f(A) = |A|$, where PD is the set of positive defined matrices of order n is log-concave. So, Theorem 2.1 is Jensen 's inequality for f.

If $p < 1$, applying the reverse Hölder inequality, then the following reverse Fan Ky inequality is obtained:

$$|A|^{\frac{1}{p}} |B|^{\frac{1}{q}} > |\frac{1}{p} A + \frac{1}{q} B|.$$

If A_i $(i = 1, 2, \cdots, m)$ is invertible matrix of order n and A_i' is a transform of A_i, then $A_i A_i'$ is a positive definite matrix of order n and $|A_i A_i'| = |A_i|^2$. Based on Theorem 2.1, the following result is obtained.

Corollary 2.3. *If* $A_i(i = 1, 2, \cdots, m)$ *is a invertible matrix of order* n,*then*

$$\prod_{i=1}^{m} |A_i^2|^{\frac{1}{p_i}} \leq \left| \sum_{i=1}^{m} \frac{1}{p_i} A_i A_i' \right|. \tag{2.3}$$

Let $A_i(i = 1, 2, \cdots, m)$ is a symmetrical matrix of order n. Then there exists a sufficiently big k_i such that $k_i I + A_i$ is a positive definite matrix. Let $k = \max\{k_1, k_2, \cdots k_m\}$. Then we have the following result.

Corollary 2.4. *With the assumptions as the above-mentioned, then*

$$\prod_{i=1}^{m} |kI + A_i|^{\frac{1}{p_i}} \leq \left| \sum_{i=1}^{m} \frac{1}{p_i} (kI + A_i) \right|. \tag{2.4}$$

Next, we shall establish a refinement of (2.1).

Theorem 2.5. *Let* A, B *be two positive definite matrices of order* n. *If* $\frac{1}{p} + \frac{1}{q} = 1$ *and* $p > 1$, *then*

$$|A|^{\frac{1}{p}} |B|^{\frac{1}{q}} \leq \left| \frac{1}{p} A + \frac{1}{q} B \right| (1 - R)^{\frac{2}{r}}, \tag{2.5}$$

where

$$R = (4\pi)^{\frac{n}{2}} \left(\left(\frac{|A|^{\frac{1}{2}}}{| A + \pi I|} \right)^{\frac{1}{2}} - \left(\frac{|B|^{\frac{1}{2}}}{|B + \pi I|} \right)^{\frac{1}{2}} \right)^2, \quad r = \max\{p, q\}.$$

Remark 2.6. In fact, Theorem 2.5 establishes a refinement of Fan Ky inequality.

If A and B are two invertible matrices of order n, A' and B' are respectively transforms of A and B, then AA' and BB' are positive definite matrices of order n. And notice that $|AA'| = |A|^2$ and $|BB'| = |B|^2$. Based on Theorem 2.5, the following result is obtained.

Corollary 2.7. *With the assumptions as the above-mentioned, then*

$$|A|^{\frac{1}{p}} |B|^{\frac{1}{q}} \leq \left| \frac{1}{p} AA' + \frac{1}{q} BB' \right|^{\frac{1}{2}} \left(1 - \tilde{R}\right)^{\frac{1}{r}}, \tag{2.6}$$

where

$$\tilde{R} = (4\pi)^{\frac{n}{2}} \left(\left(\frac{|A\ |}{|AA' + \pi I|} \right)^{\frac{1}{2}} - \left(\frac{|B\ |}{|BB' + \pi I|} \right)^{\frac{1}{2}} \right)^2, \quad r = \max\{p, q\}$$

3. Proofs of main results

In order to apply the integral method and matrix theory to prove our assertions, we need the following lemmas.

Lemma 3.1. *Let D be a positive definite matrix of order n. Then*

$$\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\exp\left(-xDx'\right)\mathrm{dx}=\left(\frac{\pi^n}{|D|}\right)^{\frac{1}{2}},\tag{3.1}$$

where the vector $x=(x_1,\ x_2,\ \cdots,\ x_n)$,$x'$ is transform of x and $\mathrm{dx}=\mathrm{dx}_1\mathrm{dx}_2\cdots\mathrm{dx}_n$. *This result is the well known. Its proof is omitted here.*

Lemma 3.2. *Let $\frac{1}{p}+\frac{1}{q}=1$ and $p>\ 1$. If $0<\|f\|_p<+\infty$ and $0<\|g\|_q<+\infty$, then*

$$(f,\ g)\leq\|f\|_p\|g\|_q(1-R)^{\frac{1}{r}},\tag{3.2}$$

where $R=(S_p(f,h)-S_q(g,h))^2, r=\max\{p,\ q\}$, $\|h\|=1$ and $(f^{p/2},h)(g^{q/2},h)\geq 0$.

And the equality in (3.2) *holds if and only if $f^{p/2}$ and $g^{q/2}$ are linearly dependent; or h is a linear combination of $f^{p/2}$ and $g^{q/2}$, and $(f^{\frac{p}{2}},h)(g^{\frac{q}{2}},h)=0$ but h is not simultaneously orthogonal to $f^{\frac{p}{2}}$ and $g^{\frac{q}{2}}$.*

Proof. First, we consider the case $p=2$. Let f,g and h be three arbitrary functions with n variables. If $\|h\|=1$, then

$$(f,\ g)^2\leq\|f\|^2\|g\|^2-(\|f\|u-\|g\|v)^2,\tag{3.3}$$

where $u=(g,h),\quad v=(f,h),\ uv\geq 0$. And the equality in (3.3) holds if and only if f,g and h are linearly dependent; or h is a linear combination of f and g, and $uv=0$ but h is not simultaneously orthogonal to f and g. In fact, consider the Gram determinant constructed by the functions f,g and h:

$$G(f,g,h)=\begin{vmatrix}(f,f) & (f,g) & (f,h)\\ (g,f) & (g,g) & (g,h)\\ (h,f) & (h,g) & (h,h)\end{vmatrix}.$$

According to the positive definiteness of the Gram matrix, we have $G(f,g,h)\geq 0$, and $G(f,g,h)=0$ if and only if f,g and h are linearly dependent.

Expanding this determinant and using the condition $\|h\|=1$, we obtain

$$\begin{aligned}G(f,g,h)&=\|f\|^2\|g\|^2-(f,g)^2-\{\|f\|^2u^2-2(f,g))uv+\|g\|^2v^2\}\\&\leq\|f\|^2\|g\|^2-(f,g)^2-\{\|f\|^2u^2-2(f,g))|uv|+\|g\|^2v^2\}\\&\leq\|f\|^2\|g\|^2-(f,g)^2-(\|f\||u|-\|g\||v|)^2\\&\leq\|f\|^2\|g\|^2-(f,g)^2-(\|f\|u-\|g\|v)^2\end{aligned}$$

where $u=(g,h)$, $v=(f,h)$ and $uv\geq 0$. And the equality holds if and only if f,g and h are linearly dependent; or h is a linear combination of f and g, and $uv=0$ but h is not simultaneously orthogonal to f and g.

The inequality (3.3) can be written in the following form:

$$(f,g)^2\leq\|f\|^2\|g\|^2(1-r_2),\tag{3.4}$$

where $r_2=(S_2(f,h)-S_2(g,h))^2$. Namely, when $p=2$, the inequality (3.2) is valid. It is obvious that the inequality (3.4) is a refinement of the Cauchy inequality and that it is also extensions of the corresponding results of the papers [3, 6, 7].

Next, consider the case $p \neq 2$. Not loss generality, let $p > q > 1$. Since $\frac{1}{p} + \frac{1}{q} = 1$, we have $p > 2$. Let $\alpha = \frac{p}{2}, \beta = \frac{p}{p-2}$. Then $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. By applying Hölder's inequality, we have

$$(f, g) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} f(x)g(x)\mathrm{dx} = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \left\{f(x)(g(x))^{q/p}\right\}(g(x))^{1-q/p}\mathrm{dx}$$

$$\leq \left\{\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \left(f(x)(g(x))^{q/p}\right)^{\alpha}\mathrm{dx}\right\}^{1/\alpha} \left\{\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \left\{(g(x))^{1-q/p}\right\}^{\beta}\mathrm{dx}\right\}^{1/\beta}$$

$$= \left(f^{p/2}, g^{q/2}\right)^{2/p} \|g\|_q^{q(1-2/p)}. \tag{3.5}$$

And the equality in (3.5) holds if and only if $f^{p/2}$ and $g^{q/2}$ are linearly dependent. In fact, The equality in (3.5) holds if and only if for any a positive integer k, there exists a positive number c_1, such that $(fg^{q/p})^{\alpha} = c_1(g^{1-q/p})^{\beta}$. After simplifications, we obtain $f^{p/2} = c_1 g^{q/2}$.

If f and g in (3.4) are replaced by $f^{\frac{p}{2}}$ and $g^{\frac{q}{2}}$ respectively, then we have

$$(f^{p/2}, g^{q/2})^2 \leq \|f\|_p^p \|g\|_q^q (1 - R), \tag{3.6}$$

where $R = (S_p(f, h) - S_q(g, h))^2$. Substituting (3.6) into(3.5), we obtain after simplifications

$$(f, g) \leq \|f\|_p \|g\|_q (1 - R)^{\frac{1}{p}}. \tag{3.7}$$

It is known from (3.4) that the equality in (3.7) holds if and only if $f^{p/2}$ and $g^{q/2}$ are linearly dependent; or h is a linear combination of $f^{p/2}$ and $g^{q/2}$, and $(f^{\frac{p}{2}}, h)(g^{\frac{q}{2}}, h) = 0$ but h is not simultaneously orthogonal to $f^{\frac{p}{2}}$ and $g^{\frac{q}{2}}$. Notice that the symmetry of p and q, it follows that the inequality (3.2) is valid.

It is very easy to prove Theorem 2.1, it is omitted here.

Proof of Theorem 2.5. Let $f(x) = \exp(-\frac{1}{p}(xAx'))$ and $g(x) = \exp(-\frac{1}{q}(xBx')$, where $\frac{1}{p} + \frac{1}{q} = 1$ and $p > 1$. Based on (3.2) and (3.1), we have

$$\frac{\pi^{\frac{n}{2}}}{\left|\frac{1}{p}A + \frac{1}{q}B\right|^{\frac{1}{2}}} = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} f(x)g(x)\mathrm{dx}$$

$$\leq \left\{\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} f^p(x)\mathrm{dx}\right\}^{\frac{1}{p}} \left\{\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} g^q(x)\mathrm{dx}\right\}^{\frac{1}{q}} (1 - R)^{\frac{1}{r}}$$

$$= \frac{\pi^{\frac{n}{2}}}{\left(|A|^{\frac{1}{p}}|B|^{\frac{1}{q}}\right)^{\frac{1}{2}}}(1 - R)^{\frac{1}{r}}. \tag{3.8}$$

We attain from (3.8) after simplifications

$$|A|^{\frac{1}{p}}|B|^{\frac{1}{q}} \leq |\frac{1}{p}A + \frac{1}{q}B|(1 - R)^{\frac{1}{r}} \tag{3.9}$$

where $r = \max\{p,\ q\}$.
We only need to compute R in (3.9). It is known from (3.2) that

$$R = (S_p(f,\ h) - S_q(g,\ h))^2 = \left(\frac{(f^{p/2},\ h)}{\|f\|_p^{p/2}} - \frac{(g^{q/2},\ h\ )}{\|g\|_q^{q/2}} \right)^2,$$

where $h = \exp\left(-\frac{1}{2}xCx'\right)$, Let $C = \pi I$. Then $|C| = \pi^n$, Based on (3.1),we have

$$\|h\| = \left\{ \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} h^2(x)\mathrm{dx} \right\}^{1/2} = 1. \tag{3.10}$$

It is easy to deduce that

$$\begin{aligned}
\left(f^{\frac{p}{2}},\ h\right) &= \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} f^{\frac{p}{2}}(x)h\,(x)\,\mathrm{dx} \\
&= \frac{\pi^{\frac{n}{2}}}{\left|\ \frac{1}{2}\ (A\ +\ \pi I)\right|^{\frac{n}{2}}} = (2\pi)^{\frac{n}{2}} \left(\frac{1}{|A + \pi I|} \right)^{\frac{1}{2}}, \\
\|f\|_p^{\frac{p}{2}} &= \left\{ \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} f^p(x)\mathrm{dx} \right\}^{\frac{1}{2}} = \left\{ \frac{\pi^n}{|A|} \right\}^{\frac{1}{4}}, \\
S_p\,(f,\ h) &= \ \left(f^{\frac{p}{2}},\ \ h\right) \|f\|_p^{-\frac{1}{p}} = (2\pi)^{\frac{n}{2}} \left(\frac{1}{|A + \pi I|} \right)^{\frac{1}{2}} \left\{ \frac{|A|}{\pi^n} \right\}^{\frac{1}{4}} \\
&= (4\pi)^{\frac{n}{4}} \left(\frac{|A|^{\frac{1}{2}}}{|\ A\ +\ \pi I|} \right)^{\frac{1}{2}}.
\end{aligned}$$

Similarly, we have $S_q(g,\ h) = (4\pi)^{\frac{n}{4}} \left(\frac{|B|^{\frac{1}{2}}}{|\ B + \pi I|} \right)^{\frac{1}{2}}$.
It follows that

$$\begin{aligned}
R &= (S_p(f,h) - S_q(g,h))^2 \\
&= (4\pi)^{\frac{n}{2}} \left(\left(\frac{|A|^{\frac{1}{2}}}{|\ A + \pi I|} \right)^{\frac{1}{2}} - \left(\frac{|B|^{\frac{1}{2}}}{|\ B + \pi I|} \right)^{\frac{1}{2}} \right)^2.
\end{aligned}$$

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