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Distortion theorems for certain subclasses of typically real functions

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Abstract. In this paper we discuss the class $\mathcal{T}(\frac{1}{2})$ of typically real functions in the unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ which are given by the formula

$$f(z) = \int_{-1}^{1} \frac{z}{\sqrt{1-2zt+z^2}} d\mu(t).$$

Three other classes: $\mathcal{K}_R$, $\mathcal{S}^*_R(\frac{1}{2})$ and $\mathcal{K}_R(i)$, consisting of convex functions, starlike functions of order 1/2 and convex in the direction of the imaginary axis, all with real coefficients, are contained in $\mathcal{T}(\frac{1}{2})$. The main idea of the paper is to obtain some distorsion results concerning $\mathcal{T}(\frac{1}{2})$ and apply them in solving analogous problems in $\mathcal{K}_R$, $\mathcal{S}^*_R(\frac{1}{2})$, $\mathcal{K}_R(i)$.

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1. Preliminaries

Let $\mathcal{T}(\frac{1}{2})$ denote a subclass of typically real functions $\mathcal{T}$ consisting of functions given by the formula

$$f(z) = \int_{-1}^{1} f_t(z) d\mu(t) \ , z \in \Delta \ , \tag{1.1}$$

where $\Delta = \{z \in \mathbb{C} : |z| < 1\}$,

$$f_t(z) = \frac{z}{\sqrt{1-2zt+z^2}} \ , \tag{1.2}$$

μ is a probability measure on $[-1, 1]$ and the square root is chosen that $\sqrt{1} = 1$. This class was introduced and discussed by Szynal in [6], [3].

Observe that the kernel functions $f_t \in \mathcal{T}(\frac{1}{2})$ and analogous functions

$$k_t(z) = \frac{z}{1-2zt+z^2}$$

in the class $\mathcal{T}$ are connected by a simple relation

$$(f_t(z))^2 = zk_t(z)\ . \tag{1.3}$$

All functions of the class $\mathcal{T}(\frac{1}{2})$ have real coefficients given by

$$a_n = \int_{-1}^{1} P_{n-1}(t)d\mu(t)\ , \tag{1.4}$$

where P_n is the n-th Legendre polynomial. From the properties of the Legendre polynomials we know that $|P_n| \le 1$ for every $n \in \mathbb{N}$. Hence all coefficients of every function $f \in \mathcal{T}(\frac{1}{2})$ are bounded by 1.

Similar property holds also for the following subclasses of univalent functions: $\mathcal{K}$ - convex functions, $\mathcal{S}^*(\frac{1}{2})$ - starlike functions of order 1/2, $\mathcal{K}(i)$ - functions that are convex in the direction of the imaginary axis and $\mathcal{K}_R$, $\mathcal{S}^*_R(\frac{1}{2})$, $\mathcal{K}_R(i)$ consisting of functions with real coefficients.

For the classes with real coefficients the integral formulae are known. They are a consequence of known relations between $\mathcal{K}_R$, $\mathcal{K}_R(i)$, $\mathcal{S}^*_R(\frac{1}{2})$ and $\mathcal{T}$, $\mathcal{P}_R$, where $\mathcal{P}_R$ is the class of functions having real coefficients and a positive real part.

Namely, for functions normalized by the equalities $f(0) = f'(0) - 1 = 0$ and $z \in \Delta$ we have

$$f \in \mathcal{K}_R \quad \text{iff} \quad 1 + z\tfrac{f''(z)}{f'(z)} \in \mathcal{P}_R\ , \tag{1.5}$$

$$f \in \mathcal{K}_R(i) \quad \text{iff} \quad zf'(z) \in \mathcal{T}\ , \tag{1.6}$$

$$f \in \mathcal{S}^*_R(\tfrac{1}{2}) \quad \text{iff} \quad \operatorname{Re}\tfrac{zf'(z)}{f(z)} > \tfrac{1}{2}\ . \tag{1.7}$$

The relation (1.7) can be written as follows

$$f \in \mathcal{S}^*_R(\tfrac{1}{2}) \quad \text{iff} \quad 2\tfrac{zf'(z)}{f(z)} - 1 \in \mathcal{P}_R\ . \tag{1.8}$$

From (1.5), (1.6), (1.8) we obtain

$$f \in \mathcal{K}_R, \quad \text{iff} \quad f'(z) = \exp\left(-\int_0^\pi \ln(1 - 2z\cos\varphi + z^2)\, d\mu(\varphi)\right),$$

$$f \in \mathcal{K}_R(i) \quad \text{iff} \quad f(z) = \int_0^\pi h_\varphi(z)d\mu(\varphi)\ ,$$

$$\text{where } h_\varphi(z) = \begin{cases} \frac{z}{1-z} & \text{for} \quad \varphi = 0\,, \\ \frac{1}{2i\sin\varphi}\ln\frac{1-ze^{-i\varphi}}{1-ze^{i\varphi}} & \text{for} \quad \varphi \in (0,\pi)\,, \\ \frac{z}{1+z} & \text{for} \quad \varphi = \pi\,. \end{cases}$$

$$f \in \mathcal{S}^*_R(\tfrac{1}{2}) \quad \text{iff} \quad f(z) = z\exp\int_0^\pi \ln\frac{1}{\sqrt{1 - 2z\cos\varphi + z^2}}\, d\mu(\varphi)\ ,$$

where $\mu \in P_{[0,\pi]}$.

Investigating classes $\mathcal{K}_R$, $\mathcal{K}_R(i)$, $\mathcal{S}^*_R(\frac{1}{2})$ with the use of their integral formulae is rather difficult. It is easier, in some cases, to obtain results in $\mathcal{T}(\frac{1}{2})$ and than to transfer them onto the classes mentioned above.

Remark 1.1. It is easy to check that

a) $f_t(z) = \frac{z}{\sqrt{1-2zt+z^2}} \in \mathcal{S}_R^*\left(\frac{1}{2}\right)$ for all $t \in [-1,1]$,
b) $f_1(z) = \frac{z}{1-z}$ and $f_{-1}(z) = \frac{z}{1+z}$ belong to $\mathcal{K}_R$,
c) $g_\alpha(z) = \alpha\frac{z}{1-z} + (1-\alpha)\frac{z}{1+z} \in \mathcal{K}_R(i)$ for all $\alpha \in [0,1]$,
d) $g_{1/2}(z) = \frac{z}{1-z^2} \in \mathcal{S}_R^*$.

In [4] it was proved that

$$\mathcal{K}_R \subset \mathcal{S}_R^*\left(\tfrac{1}{2}\right) \subset \mathcal{T}\left(\tfrac{1}{2}\right) \tag{1.9}$$

and

$$\mathcal{K}_R \subset \mathcal{K}_R(i) \subset \mathcal{T}(\tfrac{1}{2}) \ . \tag{1.10}$$

Moreover, there exist functions in $\mathcal{T}(\frac{1}{2})$ that are not univalent. For example, for every fixed $t \in (0,1)$ the function

$$f(z) = \frac{1}{2}\left[\frac{z}{\sqrt{1-2tz+z^2}} + \frac{z}{\sqrt{1+2tz+z^2}}\right] \quad , \quad z \in \Delta$$

is locally univalent in the disk Δ_{r_t}, where $r_t \in (0,1)$, and is not univalent in any disk Δ_r with $r \geq r_t$. Discussing these functions, one can prove that the radius of univalence for $\mathcal{T}\left(\frac{1}{2}\right)$ is not greater then $\sqrt{7}/3 = 0.881\ldots$. From this reason, we conclude that $\mathcal{K}_R$, $\mathcal{K}_R(i)$ and $\mathcal{S}_R^*(\frac{1}{2})$ are proper subclasses of $\mathcal{T}(\frac{1}{2})$.

2. Main results for $\mathcal{T}\left(\frac{1}{2}\right)$

At the beginning observe that for any function f with real coefficients, if D is symmetric with respect to the real axis then $f(D)$ also has the same property. It is a reason why in problems involving sets $f(D)$ one can discuss only a set $f(D^+)$ and than apply the reflection of this set with respect to the real axis. Throughout the paper we write $D^+ = D \cap \{z : \operatorname{Im} z > 0\}$.

Let $D_z(\mathcal{T}(\frac{1}{2}))$ denote the region of values $f(z)$ for a fixed $z \in \Delta$ while f varies the class $\mathcal{T}(\frac{1}{2})$. The important property of functions of the class $\mathcal{T}\left(\frac{1}{2}\right)$ is established in the following lemma.

Lemma 2.1. [4] *For a fixed $x \in (-1,1)$ the set $D_x(\mathcal{T}(\frac{1}{2}))$ coincides with the segment $[\frac{x}{1+x}, \frac{x}{1-x}]$. For a fixed $z \in \Delta^+$ the set $D_z(\mathcal{T}(\frac{1}{2}))$ is a convex set whose boundary consists of a curve $\{f_t(z) :\ t \in [-1,1]\}$ and a line segment $\{\alpha\frac{z}{1-z} + (1-\alpha)\frac{z}{1+z} :\ \alpha \in [0,1]\}$.*

We also need the distortion theorem for $\mathcal{T}$ obtained by Goluzin.

Theorem 2.2. [1] *For $f \in \mathcal{T}$ and $z \in \Delta \setminus \{0\}$ we have*

a)

$$|f(z)| \leq \begin{cases} \left|\frac{z}{(1-z)^2}\right| & \operatorname{Re}\frac{1+z^2}{z} \geq 2 \\ \frac{1}{\left|\operatorname{Im}\frac{1+z^2}{z}\right|} & \left|\operatorname{Re}\frac{1+z^2}{z}\right| < 2 \\ \left|\frac{z}{(1+z)^2}\right| & \operatorname{Re}\frac{1+z^2}{z} \leq -2 \ , \end{cases} \tag{2.1}$$

b)

$$\arg\frac{z}{(1+z)^2} \leq \arg f(z) \leq \arg\frac{z}{(1-z)^2} \quad \mathit{for} \quad z\in\Delta^+ , \tag{2.2}$$

$$\arg\frac{z}{(1-z)^2} \leq \arg f(z) \leq \arg\frac{z}{(1+z)^2} \quad \mathit{for} \quad z\in\Delta^- . \tag{2.3}$$

Basing on the above facts we obtain some distortion results for $\mathcal{T}\left(\frac{1}{2}\right)$.

Theorem 2.3. *For* $f\in\mathcal{T}\left(\frac{1}{2}\right)$ *and* $z\in\Delta\setminus\{0\}$ *we have*

$$|f(z)| \leq \begin{cases} \left|\frac{z}{1-z}\right| & \mathrm{Re}\,\frac{1+z^2}{z} \geq 2 \\ |z|\sqrt{\frac{|z|}{(1-|z|^2)|\,\mathrm{Im}\,z|}} & \left|\mathrm{Re}\,\frac{1+z^2}{z}\right| < 2 \\ \left|\frac{z}{1+z}\right| & \mathrm{Re}\,\frac{1+z^2}{z} \leq -2 . \end{cases} \tag{2.4}$$

The extremal functions in this theorem are: $f(z)=\frac{z}{1-z}$, $f(z)=\frac{z}{\sqrt{1-2t_0z+z^2}}$, where $t_0=\frac{1}{2}\,\mathrm{Re}\,\frac{1+z^2}{z}$, and $f(z)=\frac{z}{1+z}$.
The sets which appear in Theorem 2.2 and in Theorem 2.3 are presented in Fig. 1.
Proof. Assume that $f\in\mathcal{T}\left(\frac{1}{2}\right)$ and $z\in\Delta^+$. The relation (1.3) leads to

$$\max\left\{|f(z)|^2 : f\in\mathcal{T}(1/2)\right\} = \max\left\{|w|^2 : w\in D_z(\mathcal{T}(1/2))\right\} =$$
$$\max\left\{|f_t(z)|^2 : t\in[-1,1]\right\} = \max\{|zk_t(z)| : t\in[-1,1]\} =$$
$$|z|\max\{|w| : w\in D_z(\mathcal{T})\} = |z|\max\{|h(z)| : h\in\mathcal{T}\} . \tag{2.5}$$

The inequality (2.4) is a simple consequence of (2.5) and Theorem 2.2. □

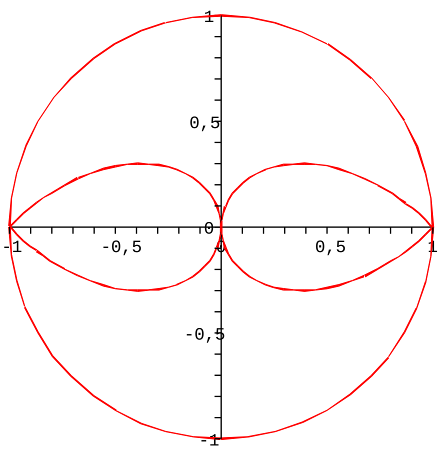

FIGURE 1. Subsets of Δ described in Theorems 2.2 and 2.3.

Theorem 2.4. *For* $f \in \mathcal{T}\left(\frac{1}{2}\right)$ *and* $z \in \Delta$ *we have*

$$|f(z)| \geq \begin{cases} \left|\frac{z}{1+z}\right| & \operatorname{Re}\frac{1}{z} \geq 1 \\ \left|\frac{\operatorname{Im} z}{1-z^2}\right| & \left|\operatorname{Re}\frac{1}{z}\right| < 1 \\ \left|\frac{z}{1-z}\right| & \operatorname{Re}\frac{1}{z} \leq -1\ . \end{cases} \tag{2.6}$$

The extremal functions in this theorem are: $f(z) = \frac{z}{1+z}$, $f(z) = \frac{z(1+(2\alpha-1)z)}{1-z^2}$, where $2\alpha - 1 = -\operatorname{Re}\frac{1}{z}$, and $f(z) = \frac{z}{1-z}$.

It can be easily observed that both sets given by the inequalities: $\operatorname{Re}\frac{1}{z} \geq 1$ and $\operatorname{Re}\frac{1}{z} \leq -1$ are disks with the same radius 1 and centered in 1 and -1 respectively.

To prove Theorem 2.4 it is enough to write

$$\min\{|f(z)| : f \in \mathcal{T}(1/2)\} = \min\{|g_\alpha(z)| : \alpha \in [-1,1]\} =$$
$$\left|\frac{z^2}{1-z^2}\right| \min\left\{\left|\frac{1}{z} + (2\alpha - 1)\right| : \alpha \in [-1,1]\right\} \tag{2.7}$$

and observe that for $z \neq 0$

$$\min\left\{\left|\frac{1}{z} + p\right| : p \in [-1,1]\right\} = \begin{cases} \left|\frac{1}{z} - 1\right| & \text{if} \quad \operatorname{Re}\left(\frac{1}{z} - 1\right) \geq 0 \\ \left|\operatorname{Im}\frac{1}{z}\right| & \text{if} \quad \left|\operatorname{Re}\frac{1}{z}\right| < 1 \\ \left|\frac{1}{z} + 1\right| & \text{if} \quad \operatorname{Re}\left(\frac{1}{z} + 1\right) \leq 0\ . \end{cases}$$

From Theorem 2.3 and Theorem 2.4 we get

Corollary 2.5. *For* $f \in \mathcal{T}\left(\frac{1}{2}\right)$ *and* $|z| = r$ *we have*

$$\frac{r}{1+r} \leq |f(z)| \leq \frac{r}{1-r}. \tag{2.8}$$

Equalities hold for $f(z) = \frac{z}{1-z}$ at points $z = \pm r$ and for $f(z) = \frac{z}{1+z}$ at points $z = \mp r$.

Proof. We shall prove only the upper estimate. The proof of the lower one is similar and will be omitted.

Assume that $z = re^{i\varphi}$, where r is a fixed number from $(0,1)$, and φ varies in $[0,\pi]$. According to Theorem 2.3 we shall estimate $|f(z)|$ in three sets separately.

I. If $\operatorname{Re}\frac{1+z^2}{z} \geq 2$ (and $|z| = r$) then $\cos\varphi \geq \frac{2r}{1+r^2}$ and

$$\left|\frac{z}{1-z}\right| = \frac{r}{\sqrt{1 - 2r\cos\varphi + r^2}} \leq \frac{r}{1-r}\ .$$

II. If $|\operatorname{Re}\frac{1+z^2}{z}| < 2$ (and $|z| = r$) then $\varphi \in (\varphi_0, \pi - \varphi_0)$, where $\varphi_0 = \arccos\frac{2r}{1+r^2}$ and

$$|z|\sqrt{\frac{|z|}{(1-|z|^2)\,|\operatorname{Im} z|}} = \frac{r}{\sqrt{(1-r^2)\sin\varphi}} \leq \frac{r\sqrt{1+r^2}}{1-r^2}\ .$$

III. If $\operatorname{Re}\frac{1+z^2}{z} \le -2$ (and $|z| = r$) then $\cos\varphi \le -\frac{2r}{1+r^2}$ and

$$\left|\frac{z}{1+z}\right| = \frac{r}{\sqrt{1+2r\cos\varphi + r^2}} \le \frac{r}{1-r} \ .$$

Since $\frac{r\sqrt{1+r^2}}{1-r^2} \le \frac{r}{1-r}$ the proof of the right hand side inequality in (2.8) is complete. □

Similarly as in Theorems 2.3 and 2.4 we can estimate the argument of $f(z)$. Taking into account Lemma 2.1, the relation (1.3) and Theorem 2.2 we obtain for $z \in \Delta^+$

$$\max\{\arg f(z) : f \in \mathcal{T}(1/2)\} = \max\{\arg f_t(z) : t \in [-1,1]\} = \frac{1}{2}\max\left\{\arg\left(f_t(z)\right)^2 : t \in [-1,1]\right\} = \frac{1}{2}\max\{\arg z k_t(z) : t \in [-1,1]\} = \frac{1}{2}\left(\arg z + \max\{\arg k_t(z) : t \in [-1,1]\}\right) = \frac{1}{2}\left(\arg z + \arg k_1(z)\right) = \frac{1}{2}\arg\left(f_1(z)\right)^2 = \arg f_1(z) \ . \quad (2.9)$$

The above equalities hold if minimum is taken instead of maximum and the functions k_{-1} and f_{-1} instead of k_1 and f_1. In the same way we can estimate $\arg f(z)$ for $z \in \Delta^-$. This argument leads us to the following theorem.

Theorem 2.6. *For $f \in \mathcal{T}\left(\frac{1}{2}\right)$ we have*

$$\arg\frac{z}{1+z} \le \arg f(z) \le \arg\frac{z}{1-z} \quad \textit{for} \quad z \in \Delta^+ \quad (2.10)$$

$$\arg\frac{z}{1-z} \le \arg f(z) \le \arg\frac{z}{1+z} \quad \textit{for} \quad z \in \Delta^- \ . \quad (2.11)$$

The extremal functions are: $f(z) = \frac{z}{1+z}$ and $f(z) = \frac{z}{1-z}$, respectively.
In the paper [4] the following theorem was proved.

Theorem 2.7. *For every $r \in (0,1)$*

$$\bigcup_{f \in \mathcal{T}\left(\frac{1}{2}\right)} f(\Delta_r) = f_{-1}(\Delta_r) \cup f_1(\Delta_r) \ .$$

In other words it means that each set $f(\Delta_r)$ for $f \in \mathcal{T}\left(\frac{1}{2}\right)$ is included in $f_{-1}(\Delta_r) \cup f_1(\Delta_r)$. Both sets $f_{-1}(\Delta_r)$, $f_1(\Delta_r)$ are disks, centered in $-\frac{r^2}{1-r^2}$ and $\frac{r^2}{1-r^2}$ respectively and having the same radius $\frac{r}{1-r^2}$. Therefore

Corollary 2.8. *If $f \in \mathcal{T}\left(\frac{1}{2}\right)$ then for $|z| = r$*

$$a)\ |\operatorname{Re} f(z)| \le \frac{r}{1-r} \ ,$$

$$b)\ |\operatorname{Im} f(z)| \le \frac{r}{1-r^2} \ .$$

Remark 2.9. The result of Corollary 2.5 can be obtained considering the class $\mathcal{R}$ consisting of functions satisfying in Δ the condition

$$\operatorname{Re}\frac{f(z)}{z} > \frac{1}{2}\ .$$

A function $f \in \mathcal{R}$ can be associated with a function $p \in \mathcal{P}_R$ as follows

$$f(z) = \frac{1}{2}\, z\,(p(z)+1)\ .$$

From estimates valid for $\mathcal{P}_R$ we get for $|z| = r$ that

$$|f(z)| \le \frac{1}{2}\, r\left(\frac{1+r}{1-r}+1\right) = \frac{r}{1-r}\ . \tag{2.12}$$

Analogously,

$$f'(z) = \frac{1}{2}\ \left(zp'(z)+p(z)+1\right)$$

results in

$$|f'(z)| \le \frac{1}{2}\ \left(\frac{2r}{1-r^2}+\frac{1+r}{1-r}+1\right) = \frac{1}{(1-r)^2}\ . \tag{2.13}$$

Equalities in the above estimates hold if $p(z) = \frac{1+z}{1-z}$ or $p(z) = \frac{1-z}{1+z}$, and consequently, if $f(z) = \frac{z}{1-z}$ or $f(z) = \frac{z}{1+z}$.

On the other hand, it was proved (see,[5]) that $\mathcal{T}\left(\frac{1}{2}\right) \subset \mathcal{R}$. Moreover, the extremal functions in (2.12) and (2.13) belong to $\mathcal{T}\left(\frac{1}{2}\right)$. Therefore,

Corollary 2.10. *For $f \in \mathcal{T}\left(\frac{1}{2}\right)$ and $|z| = r$ we have*

$$|f'(z)| \le \frac{1}{(1-r)^2}\ . \tag{2.14}$$

3. Conclusions for other classes

Taking into account extremal functions in the results stated above we obtain the conclusions concerning subclasses of $\mathcal{T}\left(\frac{1}{2}\right)$ mentioned in Section 1.
From Theorems 2.3 and 2.4 we get

Corollary 3.1. *For $f \in \mathcal{S}^*_R\left(\frac{1}{2}\right)$ and $z \in \Delta \setminus \{0\}$ we have*

$$|f(z)| \le \begin{cases} \left|\frac{z}{1-z}\right| & \operatorname{Re}\frac{1+z^2}{z} \ge 2 \\ |z|\sqrt{\frac{|z|}{(1-|z|^2)|\operatorname{Im} z|}} & \left|\operatorname{Re}\frac{1+z^2}{z}\right| < 2 \\ \left|\frac{z}{1+z}\right| & \operatorname{Re}\frac{1+z^2}{z} \le -2\ . \end{cases} \tag{3.1}$$

Corollary 3.2. *For $f \in \mathcal{K}_R(i)$ and $z \in \Delta \setminus \{0\}$ we have*

$$|f(z)| \geq \begin{cases} \left|\frac{z}{1+z}\right| & \mathrm{Re}\,\frac{1}{z} \geq 1 \\ \left|\frac{\mathrm{Im}\,z}{1-z^2}\right| & \left|\mathrm{Re}\,\frac{1}{z}\right| < 1 \\ \left|\frac{z}{1-z}\right| & \mathrm{Re}\,\frac{1}{z} \leq -1 \ . \end{cases} \tag{3.2}$$

From the results stated above and from Corollaries 2.8 and 2.10 we conclude

Corollary 3.3. *If A is one of the following classes $\mathcal{K}_R$, $\mathcal{K}_R(i)$, $\mathcal{S}^*_R(\frac{1}{2})$, then for $|z| = r$ and $f \in A$*

$$a)\ |\,\mathrm{Re}\, f(z)| \leq \frac{r}{1-r}\ ,$$

$$b)\ |\,\mathrm{Im}\, f(z)| \leq \frac{r}{1-r^2}\ ,$$

$$c)\ \frac{r}{1+r} \leq |f(z)| \leq \frac{r}{1-r}\ ,$$

$$d)\ |f'(z)| \leq \frac{1}{(1-r)^2}\ .$$

The estimates in c) and d) are well-known. They were obtained by Gronwall and Loewner (for $\mathcal{K}$) and by Robertson (for $\mathcal{K}(i)$, $\mathcal{S}^*(\frac{1}{2})$), see for example [2]. Of course, they are true also for functions with real coefficients.

Corollary 3.4. *If A is one of the following classes $\mathcal{K}_R$, $\mathcal{K}_R(i)$, $\mathcal{S}^*_R(\frac{1}{2})$ then for $f \in A$*

$$\arg \frac{z}{1+z} \leq \arg f(z) \leq \arg \frac{z}{1-z} \quad \textit{for} \quad z \in \Delta^+$$

$$\arg \frac{z}{1-z} \leq \arg f(z) \leq \arg \frac{z}{1+z} \quad \textit{for} \quad z \in \Delta^- \ .$$

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