Stud. Univ. Babeş-Bolyai Math. 57(2012), No. 2, 295–299

Families of positive operators with reverse order

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Abstract. The objective of this paper is to present several invariant subspace results for collections of positive operators on Banach lattices.For any family $\mathcal{C}$ of positive operators in $L(E)$,I will reverse order.

Mathematics Subject Classification (2010): Banach lattices, invariant.

Keywords: 47B60.

1. Introduction

The objective of this paper is to present several invariant subspace results for collections of positive operators on Banach lattices. For any vector in a Banach lattice define $x^+ = x \vee 0$, $x^- = (-x) \vee 0$, $|x| = x \vee (-x)$. Throughout this paper X will denote a real or a complex Banach space and E will denote a real Banach lattice.

A collection S of bounded operators on a Banach space is said to be a *multiplicative* (*additive*) *semigroup* if for each pair $S, T \in S$ the operator ST (resp. $S + T$) also belongs to S.

We assume that all collections or families of operators under consideration in this section are non-empty. $\mathcal{C}$ denotes a non-empty collection of positive operators on a Banach lattice E. For all $x \in X$, we let $\mathcal{C}x = \{Cx : C \in \mathcal{C}\}$, and therefore $\|\mathcal{C}x\| = \sup\{\|Cx\| : C \in \mathcal{C}\}$.

For a subset D of a Banach space, we let

$$||D|| = \sup_{x \in D} ||x|| .$$

Accordingly for a set $\mathcal{C} \subset \mathcal{L}(E)$,

$$||\mathcal{C}|| = \sup_{C \in \mathcal{C}} ||C|| .$$

For any $A \in \mathcal{L}(X)$ we define

$$A\mathcal{C} = \{AC : C \in \mathcal{C}\}$$

and

$$\mathcal{C}A = \{CA : C \in \mathcal{C}\} .$$

For each $n \in N$, we shall also use the notation

$$\mathcal{C}^n = \{C_1 C_2 ... C_n : C_1, ..., C_n \in \mathcal{C}\}$$

and hope that whenever this notation is used it will no cause any confusion with the standard Cartesian product notation. The commutant of $\mathcal{C}$ is the unital algebra of operators defined by

$$\mathcal{C}^{'} = \{A \in \mathcal{L}(X) : AC = CA \ for \ all \ C \in \mathcal{C}\}.$$

Definition 1.1. *A subspace $V \subset X$ is said to be $\mathcal{C} - invariant$ if V is C-invariant for each $C \in \mathcal{C}$.*

A collection $\mathcal{C}$ of operators is said to be $non - transitive$ if there exists a non-trivial closed $\mathcal{C}$-invariant subspace. Otherwise, the family $\mathcal{C}$ is called transitive.

Definition 1.2. *A family $\mathcal{C}$ of operators in $\mathcal{L}(X)$ is said to be:*
(1) *(locally) quasinilpotent at a point $x \in X$ if $\lim\limits_{n\to\infty} \sqrt[n]{\|\mathcal{C}^n x\|} = 0$ and*
(2) *finitely quasinilpotent at a point $x \in X$ if every finite subcollection of $\mathcal{C}$ is (locally) quasinilpotent at x. Finitely quasinilpotent algebras of operators were considered by V.S. Shulman.*

If $T : X \to X$ is a bounded operator on a Banach space, then $\mathcal{Q}_T$ denotes the subset of X consisting of all vectors at which T is locally quasinilpotent,

$$\mathcal{Q}_T = \left\{ x \in X : \lim_{n\to\infty} \|T^n x\|^{\frac{1}{n}} = 0 \right\}.$$

Definition 1.3. *Let $T : X \to X$ is a bounded operator on a Banach space and V is a subspace of X. Then V is non-trivial if $V \neq \{0\}$ $V \neq X$. We say that V is invariant under T if $T(V) \subseteq V$. Also, V is said to be hyperinvariant for T or $T-hyperinvariant$ whenever V is invariant under every bounded operator on X that commutes with T, i.e., if $S \in \mathcal{L}(X)$ and $ST = TS$ imply that $S(V) \subseteq V$.*

Definition 1.4. *Let $T : X \to X$ be a bounded operator on a Banach space and denoted by $\mathcal{Q}_T$, the set of all points where T is locally qusinilpotent, i.e.*

$$\mathcal{Q}_T = \left\{ x \in X : \lim_{n\to\infty} \|T^n x\|^{\frac{1}{n}} = 0 \right\}.$$

$\mathcal{Q}_T$ is a T-hyperinvariant vector subspace.

Let $x, y \in \mathcal{Q}_T$ and fix $\epsilon > 0$. Pick some n_0 such that $\|T^n x\| < \epsilon^n$ and $\|T^n y\| < \epsilon^n$ hold for all $n \geq n_0$. It follows that $\|T^n(x+y)\|^{\frac{1}{n}} \leq (\|T^n x\| + \|T^n y\|)^{\frac{1}{n}} < (2\epsilon^n)^{\frac{1}{n}} < 2\epsilon$ for all $n \geq n_0$.

Therefore, $\lim\limits_{n\to\infty} \|T^n(x+y)\|^{\frac{1}{n}} = 0$, and so $x + y \in \mathcal{Q}_T$. Also note that if λ is scalar, then

$$\lim_{n\to\infty} \|T^n(\lambda x)\|^{\frac{1}{n}} = \lim_{n\to\infty} \|\lambda T^n(x)\|^{\frac{1}{n}} = \lim_{n\to\infty} |\lambda|^{\frac{1}{n}} \lim_{n\to\infty} \|T^n x\|^{\frac{1}{n}} = 0$$

and so $\lambda x \in \mathcal{Q}_T$. Consequently $\mathcal{Q}_T$ is a vector subspace of X.

Finally, let us show that $\mathcal{Q}_T$ is a T-hyperinvariant subspace. To see this assume that an operator $S \in \mathcal{L}(X)$ satisfies $TS = ST$ and let $x_0 \in \mathcal{Q}_T$. Then we have,

$$\|T^n(Sx_0)\|^{\frac{1}{n}} = \|S(T^n x_0)\|^{\frac{1}{n}} \leq \|S\|^{\frac{1}{n}} \|T^n x_0\|^{\frac{1}{n}} \to 0.$$

This implies $Sx_0 \in \mathcal{Q}_T$, and so $\mathcal{Q}_T$ is a T-hyperinvariant subspace.

To generalize this to a collection of operators $\mathcal{C}$, we let

$$\mathcal{Q}_{\mathcal{C}}^f = \{x \in X : \mathcal{C} \text{ is finitely quasinilpotent at } x\}.$$

$\mathcal{C}$ denotes a non empty collection of positive operators on a Banach lattice E.

The presence of the order structure on E leads naturally to a modification of the set $\mathcal{Q}_c^f$.

$$\mathcal{Q}_{\mathcal{C}}^f = \left\{x \in E : |x| \in \mathcal{Q}_c^f\right\}.$$

For a positive operator $C : E \to E$ on a Banach lattice, we denote by $[C\rangle$ the collection of all positive operators $A : E \to E$ such that $[A, C] \geq 0$

$$[C\rangle = \{A \in \mathcal{L}(E)_+ : AC - CA \geq 0\}.$$

In accordance with this notation we also let

$$[\mathcal{C}\rangle = \{A \in \mathcal{L}(E)_+ : C \in \mathcal{C}, AC - CA \geq 0\}.$$

Similarly,

$$\langle\mathcal{C}] = \{A \in \mathcal{L}(E)_+ : C \in \mathcal{C}, AC - CA \leq 0\}.$$

We need to introduce two additional collections associated with an arbitrary collection $\mathcal{C}$ of positive operators on E.

The first of these collections is *multiplicative semigroup generated by* $\mathcal{C}$ in $\mathcal{L}(E)$. It is the smallest semigroup of operators that contains $\mathcal{C}$ and it will be denoted by $S_{\mathcal{C}}$. $S_{\mathcal{C}}$ consists of all finite products of operators in $\mathcal{C}$.

$$S_{\mathcal{C}} = \bigcup_{n=1}^{\infty} \mathcal{C}^n.$$

The second collection denoted by $\mathcal{D}_{\mathcal{C}}$, is also a large collection of positive operators that is defined,

$$\mathcal{D}_{\mathcal{C}} = \left\{ D \in \mathcal{L}(E)_+ : \exists\{T_1, ..., T_k\} \subseteq \langle\mathcal{C}] \text{ and } \{S_1, ..., S_k\} \subseteq S_{\mathcal{C}} \text{ such that } D \leq \sum_{i=1}^{k} S_i T_i \right\}.$$

Proposition 1.5. *For any family* $\mathcal{C}$ *of positive operators in* $L(E)$ *the set* $< \mathcal{C}]$ *is a norm closed additive and multiplicative semigroup in* $L(E)$ *and contains the zero and the identity operators.*

Proof. $\mathcal{C}$ is norm closed and the operators 0 and I belong to $< \mathcal{C}]$. Now take two arbitrary operators S, T in $< \mathcal{C}]$. Then for each operator $C \in \mathcal{C}$ we have $SC \leq CS$ and $TC \leq CT$. Adding up the two inequalities, we get $(S + T)\,C \leq C\,(S + T)$ then $S + T \in < \mathcal{C}]$. Consequently,

$$STC = S\,(TC) \leq SCT = (SC)\,T \leq CST.$$

Therefore, $ST \in < \mathcal{C}]$. □

Proposition 1.6. *If* $\mathcal{C}$ *is a family of positive operators, then the collection* $\mathcal{D}_{\mathcal{C}}$ *is an additive and multiplicative semigroup in* $\mathcal{L}(E)$.

Proof. Pick any two operators D_1 and D_2 in $D_{\mathcal{C}}$. Hence

$$D_j \leq \sum_{i=1}^{n_j} S_{j,i} T_{j,i}$$

for some $T_{j,i} \in \langle \mathcal{C}]$, where $j = 1, 2$. $D_1 + D_2$ belongs to $\mathcal{D}_{\mathcal{C}}$. Let us verify that $D_1 D_2 \in \mathcal{D}_{\mathcal{C}}$. Indeed,

$$D_1 D_2 \leq \left[\sum_{k=1}^{n_1} S_{1,k} T_{1,k}\right] \left[\sum_{i=1}^{n_2} S_{2,i} T_{2,i}\right] = \sum_{k=1}^{n_1} \sum_{i=1}^{n_2} S_{1,k} T_{1,k} S_{2,i} T_{2,i}.$$

Since $T_{j,i} \in \langle \mathcal{C}]$, it follows that $T_{j,i} \in \langle S_{\mathcal{C}}]$ and hence $T_{1,k} S_{2,i} \leq S_{2,i} T_{1,k}$. Therefore,

$$D_1 D_2 \leq \sum_{i=1}^{n_1} \sum_{k=1}^{n_2} S_{2,k} S_{2,i} T_{1,k} T_{2,i}.$$

Since $\langle \mathcal{C}]$ and $S_{\mathcal{C}}$ are semigroups, we have that $T_{1,k} T_{2,i} \in \langle \mathcal{C}]$, $S_{1,k} S_{2,i} \in S_{\mathcal{C}}$. □

Proposition 1.7. *Each ideal $[D_c x]$ is both $\mathcal{C} -$ invariant and $\langle \mathcal{C}]$-invariant.*

Proof. Take any $y \in [D_c x]$. Since D_c is an additive semigroup, it follows that $|y| \leq \lambda D x$ for some scalar λ and $D \in D_c$. By the definition of D_c there exist operators $T \in \langle \mathcal{C}]$ and $S_i \in S_c$ $(i = 1, 2, 3, .., n)$ such that $D \leq \sum_{i=1}^{n} S_i T_i$, and so

$$|y| \leq \lambda \sum_{i=1}^{n} S_i T_i x.$$

Fix $C \in \mathcal{C}$ and consider the vector Cy. From $CT_i \geq T_i C$ for each i, we see that

$$|Cy| \leq C\,|y| \leq \lambda \sum_{i=1}^{n} C S_i T_i x.$$

Since $CS_i \in S_{\mathcal{C}}$ for each i we see that

$$K = \sum_{i=1}^{n} (CS_i) T_i \in \mathcal{D}_{\mathcal{C}}.$$

Therefore,

$$|Cy| \leq \lambda \sum_{i=1}^{n} (CS_i) T_i x = \lambda K x$$

and $Cy \in [D_c x]$. $[D_c x]$ is $\mathcal{C}]$-invariant.

Let $T \in \langle \mathcal{C}]$. Since $\langle \mathcal{C}]$ is a multiplicative semigroup, $TT_i \in \langle \mathcal{C}]$ for each i, and hence the operator $L = \sum_{i=1}^{n} S_i (TT_i)$ belongs to $\mathcal{D}_{\mathcal{C}}$.

$$|Ty| \leq T\,|y| \leq \lambda \sum_{i=1}^{n} S_i T T_i x = \lambda L x.$$

Consequently, $Ty \in [\mathcal{D}_{\mathcal{C}} x]$. □

Proposition 1.8. *The ideal $\hat{\mathcal{Q}}_c^f$ is $\langle \mathcal{C}]$-invariant.*

Proof. Fix $x \in \hat{\mathcal{Q}}_c^f$ that is $\|\mathcal{G}^n |x|\|^{\frac{1}{n}} \to 0$ for each finite subset $\mathcal{G}$ of $\mathcal{C}$. We must prove that Tx belong to $\hat{\mathcal{Q}}_c^f$ for each $C \in \mathcal{C}$ and each $T \in \langle \mathcal{C}]$. Fix $C \in \mathcal{C}$, $T \in \langle \mathcal{C}]$ and let $\mathcal{F} = \{C_1, ..., C_k\}$ be a finite subset of $\mathcal{C}$.

$C_i T \geq T C_i$ for each $1 \leq i \leq k$. For each operator $F \in \mathcal{F}^n$ we have $FT \geq TF$, and therefore,

$$\|T\mathcal{F}^n |x|\|^{\frac{1}{n}} \leq \|\mathcal{F}^n T |x|\|^{\frac{1}{n}} \leq \|T\|^{\frac{1}{n}} \|\mathcal{F}^n |x|\|^{\frac{1}{n}} \to 0$$

Consequently, $\|\mathcal{F}^n |Tx|\|^{\frac{1}{n}} \to 0$, and so $Tx \in \hat{\mathcal{Q}}_c^f$. The ideal $\hat{\mathcal{Q}}_c^f$ is also $\langle \mathcal{C}]$-invariant. □

References

[1] Abramovich, Y.A., Aliprantis, C.D., *An invitation to operator theory*, Volume 50, American Math. Society.

[2] Abramovich, Y.A., Aliprantis, C.D., *Problems in Operator Theory*, Volume 51, American Math. Society.

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