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Distortion theorem and the radius of convexity for Janowski-Robertson functions

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Abstract. In this note, we consider another family of functions that includes the class of convex functions as a proper subfamily. For $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$, we say that $f(z) \in C_\alpha(A,B)$ if
i) $f(z) \in A$
ii) $f'(z) \neq 0 \in \mathbb{D}$, $e^{i\alpha}(1 + z\frac{f''(z)}{f'(z)}) = \cos\alpha p(z) + i\sin\alpha$, where $p(z)$ is analytic in $\mathbb{D}$ and satisfies the conditions $p(0) = 1$, $p(z) = \frac{1+A\phi(z)}{1+B\phi(z)}$, $-1 \leq B < A \leq 1$, $\phi(z)$ analytic in $\mathbb{D}$, and $\phi(0) = 0$, $|\phi(z)| < 1$ for every $z \in \mathbb{D}$. The class of $C_\alpha(A,B)$ is called Janowski-Robertson class. The aim of this paper is to give a distortion theorem and the radius of convexity for the class $C_\alpha(A,B)$.

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1. Introduction

Let Ω be the family of functions $\phi(z)$ analytic in the open unit disc

$$\mathbb{D} = \{z| \quad |z| < 1\}$$

and satisfy the conditions $\phi(0) = 0$, $|\phi(z)| < 1$ for every $z \in \mathbb{D}$.

Next, for arbitrary fixed numbers A, B, $-1 \leq B < A \leq 1$, denote by $P(A,B)$ the family of functions $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ analytic in $\mathbb{D}$, and such that $p(z)$ is in $P(A,B)$ if and only if

$$p(z) = \frac{1 + A\phi(z)}{1 + B\phi(z)} \tag{1.1}$$

for some $\phi(z) \in \Omega$, and every $z \in \mathbb{D}$.

Moreover, let $C_\alpha(A,B)$ denote the family of functions $f(z) = z + a_2 z^2 + \cdots$ analytic in $\mathbb{D}$ and such that $f(z)$ is in $C_\alpha(A,B)$ if and only if

$$e^{i\alpha}(1 + z\frac{f''(z)}{f'(z)}) = \cos\alpha p(z) + i\sin\alpha \tag{1.2}$$

for some functions $p(z) \in P(A,B)$, all $z \in \mathbb{D}$ and some real constant α ($|\alpha| < \frac{\pi}{2}$).

Finally, let $F(z) = z + \alpha_2 z^2 + \alpha_3 z^3 + \cdots$ and $G(z) = z + \beta_2 z^2 + \beta_3 z^3 + \cdots$ be analytic functions in $\mathbb{D}$, if there exists a function $\phi(z) \in \Omega$ such that $F(z) = G(\phi(z))$ for every $z \in \mathbb{D}$, then we say that $F(z)$ is subordinate to $G(z)$, and we write $F(z) \prec G(z)$. We also note that if $F(z) \prec G(z)$, then $F(\mathbb{D}) \subset G(\mathbb{D})$.

The radius of convexity for the family of analytic functions is defined in the following manner.

$$R(f) = \sup[r | Re(1 + z\frac{f''(z)}{f'(z)}) > 0, |z| < r]$$

2. Main Results

Theorem 2.1. *Let $f(z)$ be an element of $C_\alpha(A,B)$ then,*

$$\begin{cases} \frac{(1-Br)^{\frac{(A-B)}{2B}(1+\cos\alpha)\cos\alpha}}{(1+Br)^{\frac{(A-B)}{2B}(1-\cos\alpha)\cos\alpha}} \le |f'(z)| \le \frac{(1+Br)^{\frac{(A-B)}{2B}(1+\cos\alpha)\cos\alpha}}{(1-Br)^{\frac{(A-B)}{2B}(1-\cos\alpha)\cos\alpha}}; \ B \neq 0, \\ e^{-A\cos\alpha r} \le |f'(z)| \le e^{A\cos\alpha r}; \quad B = 0. \end{cases} \tag{2.1}$$

Proof. Since

$$f(z) \in C_\alpha(A,B) \Leftrightarrow e^{i\alpha}(1 + z\frac{f''(z)}{f'(z)}) = \cos\alpha p(z) + i\sin\alpha \tag{2.2}$$

and

$$B \neq 0 \Rightarrow \left| p(z) - \frac{1 - ABr^2}{1 - B^2r^2} \right| \le \frac{(A-B)r}{1 - B^2r^2} \tag{2.3}$$

Then we have,

$$\left| \frac{1}{\cos\alpha}[e^{i\alpha}(1 + z\frac{f''(z)}{f'(z)}) - i\sin\alpha] - \frac{1 - ABr^2}{1 - B^2r^2} \right| \le \frac{(A-B)r}{1 - B^2r^2} \tag{2.4}$$

After simple calculations from (2.4) we obtain,

$$\frac{-(A-B)\cos\alpha(1 + Br\cos\alpha)r}{(1-Br)(1+Br)} \le Re(z\frac{f''(z)}{f'(z)}) \le \frac{(A-B)\cos\alpha(1 - Br\cos\alpha)r}{(1-Br)(1+Br)} \tag{2.5}$$

On the other hand we have

$$Re(z\frac{f''(z)}{f'(z)}) = r\frac{\partial}{\partial r}\log|f'(z)| \tag{2.6}$$

Considering (2.5) and (2.6) together we get;

$$\frac{-(A-B)\cos\alpha(1 + Br\cos\alpha)}{(1-Br)(1+Br)} \le \frac{\partial}{\partial r}\log|f'(z)| \le \frac{(A-B)\cos\alpha(1 - Br\cos\alpha)}{(1-Br)(1+Br)} \tag{2.7}$$

then after integration, we obtain (2.1). On the other hand,

$$B = 0 \Rightarrow |p(z) - 1| \le Ar \Rightarrow \tag{2.8}$$

$$\left| \frac{1}{\cos\alpha}[e^{i\alpha}(1 + z\frac{f''(z)}{f'(z)}) - i\sin\alpha] - 1 \right| \le Ar \Rightarrow \tag{2.9}$$

$$\left| e^{i\alpha}(1 + z\frac{f''(z)}{f'(z)}) - i\sin\alpha - \cos\alpha \right| \leq A\cos\alpha.r \Rightarrow \tag{2.10}$$

$$\left| e^{i\alpha}(1 + z\frac{f''(z)}{f'(z)}) - e^{i\alpha} \right| \leq A\cos\alpha.r \tag{2.11}$$

$$\left| z\frac{f''(z)}{f'(z)} \right| \leq A\cos\alpha.r \tag{2.12}$$

$$-\left| z\frac{f''(z)}{f'(z)} \right| \geq -A\cos\alpha.r \tag{2.13}$$

Therefore,

$$-A\cos\alpha.r \leq Re(z\frac{f''(z)}{f'(z)}) = r\frac{\partial}{\partial r}\log|f'(z)| \leq \left| z\frac{f''(z)}{f'(z)} \right| \leq A\cos\alpha.r \tag{2.14}$$

$$-A\cos\alpha \leq \frac{\partial}{\partial r}\log|f'(z)| \leq A\cos\alpha \tag{2.15}$$

If we integrate the last inequality, then we obtain the result. □

Corollary 2.2. *If $f(z) \in C_\alpha(1,-1)$ then;*

$$\frac{(1+r)^{-\cos^2\alpha - \cos\alpha}}{(1-r)^{\cos^2 - \cos\alpha}} \leq |f'(z)| \leq \frac{(1+r)^{-\cos^2\alpha + \cos\alpha}}{(1-r)^{\cos^2 + \cos\alpha}} \tag{2.16}$$

This is the distortion theorem for Robertson functions.

Corollary 2.3. *If we give another special values to A and B we obtain another distortion inequalities.*

Remark 2.4. If we give special values to A and B, we obtain that new inequalities for the Janowski-Robertson functions. The special values of A and B can be ordered in the following manner.

i. $A = 1 - 2\alpha$, $B = -1$, $0 \leq \alpha < 1$
ii. $A = 1$, $B = 0$
iii. $A = \alpha$, $B = 0$, $0 < \alpha < 1$
iv. $A = \alpha$, $B = -\alpha$, $0 < \alpha < 1$
v. $A = 1$, $B = -1 + \frac{1}{M}$, $M > \frac{1}{2}$.

Corollary 2.5. *The radius of convexity of the class $C_\alpha(A,B)$ is;*

$$\begin{cases} r = \frac{2}{(A-B)\cos\alpha + \sqrt{(A+B)^2\cos^2\alpha + 4B^2\sin^2\alpha}}; & B \neq 0, \\ r = \frac{1}{A\cos\alpha}; & B = 0. \end{cases} \tag{2.17}$$

Proof. The inequality (2.4) can be written in the following form,

$$Re(1 + z\frac{f''(z)}{f'(z)}) \geq \frac{1 - (A-B)\cos\alpha r - B(A\cos^2\alpha + B\sin^2\alpha)r^2}{1 - B^2r^2}, B \neq 0 \tag{2.18}$$

$$Re(1 + z\frac{f''(z)}{f'(z)}) \geq 1 - A\cos\alpha.r, B = 0 \tag{2.19}$$

The inequality (2.18) and (2.19) show that this corollary is true. □

Corollary 2.6. *The radius of convexity of the class $C_\alpha(1,-1)$ is*

$$r = \frac{1}{\cos\alpha + |\sin\alpha|}$$

We also note that all these results are sharp because the extremal function is

$$f(z) = \begin{cases} \displaystyle\int_0^z (1+B\zeta)^{\frac{A-B}{B}} d\zeta; & B \neq 0, \\ \displaystyle\int_0^z e^{A\zeta} d\zeta; & B = 0. \end{cases} \tag{2.20}$$

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