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Convolution properties of Sălăgean-type harmonic univalent functions

Elif Yaşar and Sibel Yalçın

Abstract. Jahangiri et al. [5] defined "modified Salagean operator" for harmonic univalent functions in the unit disk. In that study, they also obtained necessary and sufficient coefficient conditions for the Salagean-type class of harmonic univalent functions. By using those coefficient conditions, we investigate some convolution properties for the Salagean-type class of harmonic univalent functions.

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1. Introduction

Let H denote the family of continuous complex valued harmonic functions which are harmonic in the open unit disk $U = \{z : |z| < 1\}$ and let A be the subclass of H consisting of functions which are analytic in U. A function harmonic in U may be written as $f = h + \overline{g}$, where h and g are members of A. In this case, f is sense-preserving if $|h'(z)| > |g'(z)|$ in U. See Clunie and Sheil-Small [2]. To this end, without loss of generality, we may write

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \qquad g(z) = \sum_{k=1}^{\infty} b_k z^k. \tag{1.1}$$

One shows easily that the sense-preserving property implies that $|b_1| < 1$.

Let SH denote the family of functions $f = h + \overline{g}$ which are harmonic, univalent, and sense-preserving in U for which $f(0) = f_z(0) - 1 = 0$.

For the harmonic function $f = h + \overline{g}$, we call h the analytic part and g the co-analytic part of f. Note that SH reduces to the class S of normalized analytic univalent functions in U if the co-analytic part of f is identically zero.

In 1984 Clunie and Sheil-Small [2] investigated the class SH as well as its geometric subclasses and obtained some coefficient bounds. Since then, there has been several related papers on SH and its subclasses such as Avcı and Zlotkiewicz [1], Silverman [9], Silverman and Silvia [10], Jahangiri [4] studied the harmonic univalent functions.

The differential operator D^n $(n \in \mathbb{N}_0)$ was introduced by Salagean [7]. For $f = h + \overline{g}$ given by (1.1), Jahangiri et al. [5] defined the modified Salagean operator of f as

$$D^n f(z) = D^n h(z) + (-1)^n \overline{D^n g(z)}, \tag{1.2}$$

where

$$D^n h(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k \quad \text{and} \quad D^n g(z) = \sum_{k=1}^{\infty} k^n b_k z^k.$$

For $0 \leq \alpha < 1$, Jahangiri et al. [5] defined the class $SH(n,\alpha)$ which consist of functions f of the form (1.1) such that

$$\operatorname{Re}\left(\frac{D^{n+1} f(z)}{D^n f(z)}\right) \geq \alpha, \quad 0 \leq \alpha < 1 \tag{1.3}$$

where $D^n f(z)$ defined by (1.2).

If the co-analytic part of $f = h + \overline{g}$ is identically zero, then the family $SH(n,\alpha)$ turns out to be the class $S(n,\alpha)$ introduced by Salagean [7] for the analytic case. The class $SH(n,\alpha)$ includes a variety of well-known subclasses of SH. Such as,

(i) $SH(0,0) = SH^*$, is the class of harmonic starlike functions ([1], [9], [10]),

(ii) $SH(0,\alpha) = SH^*(\alpha)$, is the class of harmonic starlike functions of order α ([3], [4]),

(iii) $SH(1,0) = KH$, is the class of harmonic convex functions ([1], [9], [10]),

(iv) $SH(1,\alpha) = KH(\alpha)$, is the class of harmonic convex functions of order α ([3], [4]) in U.

We let the subclass $\overline{SH}(n,\alpha)$ consist of harmonic functions $F = H + \bar{G}$ in SH so that H and G are of the form

$$H(z) = z - \sum_{k=2}^{\infty} a_k z^k, \; G(z) = (-1)^n \sum_{k=1}^{\infty} b_k z^k, \quad a_k, \; b_k \geq 0. \tag{1.4}$$

Let $f_j(z) \in SH$ $(j = 1, 2, \ldots, m)$ be given by

$$f_j(z) = z + \sum_{k=2}^{\infty} a_{k,j} \; z^k + \sum_{k=1}^{\infty} \overline{b_{k,j}} \; \overline{z}^k. \tag{1.5}$$

The convolution is defined by

$$(f_1 * \ldots * f_m)(z) = z + \sum_{k=2}^{\infty} \left(\prod_{j=1}^{m} a_{k,j}\right) z^k + \sum_{k=1}^{\infty} \left(\prod_{j=1}^{m} \overline{b_{k,j}}\right) \overline{z}^k. \tag{1.6}$$

Let $F_j(z) \in \overline{SH}(n,\alpha)$ $(j = 1, 2, \ldots, m)$ be given by

$$F_j(z) = z - \sum_{k=2}^{\infty} a_{k,j} z^k + (-1)^n \sum_{k=1}^{\infty} b_{k,j} \overline{z}^k, \quad a_{k,j}, \; b_{k,j} \geq 0. \tag{1.7}$$

The convolution is defined by

$$(F_1 * \ldots * F_m)(z) = z - \sum_{k=2}^{\infty} \left(\prod_{j=1}^{m} a_{k,j} \right) z^k + (-1)^n \sum_{k=1}^{\infty} \left(\prod_{j=1}^{m} b_{k,j} \right) \overline{z}^k. \tag{1.8}$$

Owa and Srivastava [6] studied convolution and generalized convolution properties of the classes $M_n(\alpha)$ and $N_n(\alpha)$, Al-Shaqsi and Darus [8] investigated such properties for the classes $SH^*(\alpha)$ and $KH(\alpha)$, by especially using Cauchy-Schwarz and Hölder inequalities. In this paper, we investigate convolution properties for the Salagean-type class of harmonic univalent functions by using above mentioned techniques.

2. Main Results

Lemma 2.1. [5] *Let $F = H + \overline{G}$ given by (1.4). Then $F \in \overline{SH}(n, \alpha)$ if and only if*

$$\sum_{k=2}^{\infty} k^n (k - \alpha) a_k + \sum_{k=1}^{\infty} k^n (k + \alpha) b_k \leq 1 - \alpha. \tag{2.1}$$

Theorem 2.2. *If $F_j(z) \in \overline{SH}(n, \alpha_j)$ $(j = 1, 2, \ldots, m)$ then $(F_1 * \ldots * F_m)(z) \in \overline{SH}(n, \beta)$, where*

$$\beta = 1 - \frac{\prod_{j=1}^{m} (1 - \alpha_j)}{2^n \prod_{j=1}^{m} (2 - \alpha_j) - \prod_{j=1}^{m} (1 - \alpha_j)}.$$

Proof. We use the principle of mathematical induction in our proof of Theorem 2.2. Let $F_1(z) \in \overline{SH}(n, \alpha_1)$ and $F_2(z) \in \overline{SH}(n, \alpha_2)$.
By using Lemma 2.1, we have

$$\sum_{k=2}^{\infty} k^n \left(\frac{k - \alpha_j}{1 - \alpha_j} \right) a_{k,j} + \sum_{k=1}^{\infty} k^n \left(\frac{k + \alpha_j}{1 - \alpha_j} \right) b_{k,j} \leq 1, \quad (j = 1, 2). \tag{2.2}$$

Then, we have

$$\left[\sum_{k=2}^{\infty} \left(\sqrt{k^n \left(\frac{k - \alpha_1}{1 - \alpha_1} \right) a_{k,1}} \right)^2 \times \sum_{k=2}^{\infty} \left(\sqrt{k^n \left(\frac{k - \alpha_2}{1 - \alpha_2} \right) a_{k,2}} \right)^2 \right]^{\frac{1}{2}}$$
$$+ \left[\sum_{k=1}^{\infty} \left(\sqrt{k^n \left(\frac{k + \alpha_1}{1 - \alpha_1} \right) b_{k,1}} \right)^2 \times \sum_{k=1}^{\infty} \left(\sqrt{k^n \left(\frac{k + \alpha_2}{1 - \alpha_2} \right) b_{k,2}} \right)^2 \right]^{\frac{1}{2}} \leq 1.$$

Thus, by applying the Cauchy-Schwarz inequality, we have

$$\sum_{k=2}^{\infty}\sqrt{k^{2n}\frac{(k-\alpha_1)(k-\alpha_2)}{(1-\alpha_1)(1-\alpha_2)}a_{k,1}a_{k,2}}$$

$$\leq \left[\sum_{k=2}^{\infty}\left(\sqrt{k^n\left(\frac{k-\alpha_1}{1-\alpha_1}\right)a_{k,1}}\right)^2\right]^{\frac{1}{2}}\times\left[\sum_{k=2}^{\infty}\left(\sqrt{k^n\left(\frac{k-\alpha_2}{1-\alpha_2}\right)a_{k,2}}\right)^2\right]^{\frac{1}{2}}$$

and

$$\sum_{k=1}^{\infty}\sqrt{k^{2n}\frac{(k+\alpha_1)(k+\alpha_2)}{(1-\alpha_1)(1-\alpha_2)}b_{k,1}b_{k,2}}$$

$$\leq \left[\sum_{k=1}^{\infty}\left(\sqrt{k^n\left(\frac{k+\alpha_1}{1-\alpha_1}\right)b_{k,1}}\right)^2\right]^{\frac{1}{2}}\times\left[\sum_{k=1}^{\infty}\left(\sqrt{k^n\left(\frac{k+\alpha_2}{1-\alpha_2}\right)b_{k,2}}\right)^2\right]^{\frac{1}{2}}.$$

Then, we get

$$\sum_{k=2}^{\infty}\sqrt{k^{2n}\frac{(k-\alpha_1)(k-\alpha_2)}{(1-\alpha_1)(1-\alpha_2)}a_{k,1}a_{k,2}}+\sum_{k=1}^{\infty}\sqrt{k^{2n}\frac{(k+\alpha_1)(k+\alpha_2)}{(1-\alpha_1)(1-\alpha_2)}b_{k,1}b_{k,2}}\leq 1.$$

Therefore, if

$$\sum_{k=2}^{\infty}k^n\left(\frac{k-\gamma}{1-\gamma}\right)a_{k,1}a_{k,2}\leq \sum_{k=2}^{\infty}\sqrt{k^{2n}\frac{(k-\alpha_1)(k-\alpha_2)}{(1-\alpha_1)(1-\alpha_2)}a_{k,1}a_{k,2}}\ ,$$

and

$$\sum_{k=1}^{\infty}k^n\left(\frac{k+\gamma}{1-\gamma}\right)b_{k,1}b_{k,2}\leq \sum_{k=1}^{\infty}\sqrt{k^{2n}\frac{(k+\alpha_1)(k+\alpha_2)}{(1-\alpha_1)(1-\alpha_2)}b_{k,1}b_{k,2}}$$

that is, if

$$\sqrt{a_{k,1}a_{k,2}}\leq \left(\frac{1-\gamma}{k-\gamma}\right)\sqrt{\frac{(k-\alpha_1)(k-\alpha_2)}{(1-\alpha_1)(1-\alpha_2)}}\quad (k=2,3,\ldots),$$

and

$$\sqrt{b_{k,1}b_{k,2}}\leq \left(\frac{1-\gamma}{k+\gamma}\right)\sqrt{\frac{(k+\alpha_1)(k+\alpha_2)}{(1-\alpha_1)(1-\alpha_2)}}\quad (k=1,2,\ldots)$$

then $(F_1 * F_2)(z)\in \overline{SH}(n,\gamma)$.

We also note that the inequality (2.2) yields

$$\sum_{k=2}^{\infty}\sqrt{k^n\left(\frac{k-\alpha_j}{1-\alpha_j}\right)a_{k,j}}\leq 1$$

and

$$\sum_{k=1}^{\infty}\sqrt{k^n\left(\frac{k+\alpha_j}{1-\alpha_j}\right)b_{k,j}}\leq 1$$

and so we get,

$$\sqrt{a_{k,j}} \leq \sqrt{\frac{1-\alpha_j}{k^n\left(k-\alpha_j\right)}} \quad (j=1,2;\ k=2,3,\ldots),$$

and

$$\sqrt{b_{k,j}} \leq \sqrt{\frac{1-\alpha_j}{k^n\left(k+\alpha_j\right)}} \quad (j=1,2;\ k=1,2,\ldots).$$

Consequently, if

$$\sqrt{\frac{\left(1-\alpha_1\right)\left(1-\alpha_2\right)}{k^{2n}\left(k-\alpha_1\right)\left(k-\alpha_2\right)}} \leq \frac{1-\gamma}{k-\gamma}\sqrt{\frac{\left(k-\alpha_1\right)\left(k-\alpha_2\right)}{\left(1-\alpha_1\right)\left(1-\alpha_2\right)}},\ (k=2,3,\ldots),$$

and

$$\sqrt{\frac{\left(1-\alpha_1\right)\left(1-\alpha_2\right)}{k^{2n}\left(k+\alpha_1\right)\left(k+\alpha_2\right)}} \leq \frac{1-\gamma}{k+\gamma}\sqrt{\frac{\left(k+\alpha_1\right)\left(k+\alpha_2\right)}{\left(1-\alpha_1\right)\left(1-\alpha_2\right)}},\ (k=1,2,\ldots)$$

that is, if

$$\frac{k-\gamma}{1-\gamma} \leq \frac{k^n\left(k-\alpha_1\right)\left(k-\alpha_2\right)}{\left(1-\alpha_1\right)\left(1-\alpha_2\right)}, \quad (k=2,3,\ldots)$$

and

$$\frac{k+\gamma}{1-\gamma} \leq \frac{k^n\left(k+\alpha_1\right)\left(k+\alpha_2\right)}{\left(1-\alpha_1\right)\left(1-\alpha_2\right)}, \quad (k=1,2,\ldots)$$

then we have $\left(F_1 * F_2\right)(z) \in \overline{SH}\left(n,\gamma\right).$
Then, we see that

$$\gamma \leq 1-\frac{(k-1)\left(1-\alpha_1\right)\left(1-\alpha_2\right)}{k^n\left(k-\alpha_1\right)\left(k-\alpha_2\right)-\left(1-\alpha_1\right)\left(1-\alpha_2\right)} = \phi(k) \quad (k=2,3,\ldots),$$

and

$$\gamma \leq 1-\frac{(k+1)\left(1-\alpha_1\right)\left(1-\alpha_2\right)}{k^n\left(k+\alpha_1\right)\left(k+\alpha_2\right)+\left(1-\alpha_1\right)\left(1-\alpha_2\right)} = \varphi(k) \quad (k=1,2,\ldots).$$

Since $\phi(k)$ for $k \geq 2$ and $\varphi(k)$ for $k \geq 1$ increasing,

$$\gamma \leq 1-\frac{\left(1-\alpha_1\right)\left(1-\alpha_2\right)}{2^n\left(2-\alpha_1\right)\left(2-\alpha_2\right)-\left(1-\alpha_1\right)\left(1-\alpha_2\right)},$$

and

$$\gamma \leq 1-\frac{2\left(1-\alpha_1\right)\left(1-\alpha_2\right)}{\left(1+\alpha_1\right)\left(1+\alpha_2\right)+\left(1-\alpha_1\right)\left(1-\alpha_2\right)},$$

and also,

$$\frac{\left(1-\alpha_1\right)\left(1-\alpha_2\right)}{2^n\left(2-\alpha_1\right)\left(2-\alpha_2\right)-\left(1-\alpha_1\right)\left(1-\alpha_2\right)} \leq \frac{2\left(1-\alpha_1\right)\left(1-\alpha_2\right)}{\left(1+\alpha_1\right)\left(1+\alpha_2\right)+\left(1-\alpha_1\right)\left(1-\alpha_2\right)},$$

then $\left(F_1 * F_2\right)(z) \in \overline{SH}\left(n,\gamma\right)$, where

$$\gamma = 1-\frac{\left(1-\alpha_1\right)\left(1-\alpha_2\right)}{2^n\left(2-\alpha_1\right)\left(2-\alpha_2\right)-\left(1-\alpha_1\right)\left(1-\alpha_2\right)}.$$

Next, we suppose that $(F_1 * \ldots * F_m)(z) \in \overline{SH}(n, \beta)$, where

$$\beta = 1 - \frac{\prod_{j=1}^{m}(1-\alpha_j)}{2^n \prod_{j=1}^{m}(2-\alpha_j) - \prod_{j=1}^{m}(1-\alpha_j)}.$$

We can show that, $(F_1 * \ldots * F_{m+1})(z) \in \overline{SH}(n, \delta)$, where

$$\delta = 1 - \frac{(1-\beta)(1-\alpha_{m+1})}{2^n(2-\beta)(2-\alpha_{m+1}) - (1-\beta)(1-\alpha_{m+1})}.$$

Since

$$(1-\beta)(1-\alpha_{m+1}) = \frac{\prod_{j=1}^{m+1}(1-\alpha_j)}{2^n \prod_{j=1}^{m}(2-\alpha_j) - \prod_{j=1}^{m}(1-\alpha_j)},$$

and

$$(2-\beta)(2-\alpha_{m+1}) = \frac{\prod_{j=1}^{m+1}(1-\alpha_j)}{2^n \prod_{j=1}^{m}(2-\alpha_j) - \prod_{j=1}^{m}(1-\alpha_j)},$$

we have

$$\delta = 1 - \frac{\prod_{j=1}^{m+1}(1-\alpha_j)}{2^n \prod_{j=1}^{m+1}(2-\alpha_j) - \prod_{j=1}^{m+1}(1-\alpha_j)}.$$

□

Corollary 2.3. *If* $F_j(z) \in \overline{SH}(n, \alpha)$ $(j = 1, 2, \ldots, m)$, *then* $(F_1 * \ldots * F_m)(z) \in \overline{SH}(n, \beta)$, *where*

$$\beta = 1 - \frac{(1-\alpha)^m}{2^n(2-\alpha)^m - (1-\alpha)^m}.$$

Corollary 2.4. *If* $F_j(z) \in SH^*(\alpha_j)$ $(j = 1, 2, \ldots, m)$, *then* $(F_1 * \ldots * F_m)(z) \in SH^*(\beta)$, *where*

$$\beta = 1 - \frac{\prod_{j=1}^{m}(1-\alpha_j)}{\prod_{j=1}^{m}(2-\alpha_j) - \prod_{j=1}^{m}(1-\alpha_j)}.$$

Corollary 2.5. *If* $F_j(z) \in KH(\alpha_j)$ $(j = 1, 2, \ldots, m)$, *then* $(F_1 * \ldots * F_m)(z) \in KH(\beta)$, *where*

$$\beta = 1 - \frac{\prod_{j=1}^{m}(1-\alpha_j)}{2\prod_{j=1}^{m}(2-\alpha_j) - \prod_{j=1}^{m}(1-\alpha_j)}.$$

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Elif Yaşar
Department of Mathematics, Faculty of Arts and Science,
Uludag University, 16059, Bursa, Turkey
e-mail: elifyasar@uludag.edu.tr

Sibel Yalçın
Department of Mathematics, Faculty of Arts and Science,
Uludag University, 16059, Bursa, Turkey
e-mail: syalcin@uludag.edu.tr