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A subclass of analytic functions

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Abstract. In the present paper, by means of Carlson-Shaffer operator and a multiplier transformation, we consider a new class of analytic functions $\mathcal{BL}(m, l, a, c, \mu, \alpha, \lambda)$. A sufficient condition for functions to be in this class and the angular estimates are provided.

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1. Introduction

Let $\mathcal{A}_n$ denote the class of functions f of the form

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k, \ z \in U \tag{1.1}$$

which are analytic in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$.

Let $\mathcal{H}(U)$ be the space of holomorphic functions in U.

For $a \in \mathbb{C}$ and $n \in \mathbb{N}$, we denote by

$$\mathcal{H}[a, n] = \left\{f \in \mathcal{H}(U), \quad f(z) = a + a_n z^n + a_{n+1} z^{n+1} + ..., z \in U\right\}$$

Let the function $\phi(a, c; z)$ be given by

$$\phi(a, c; z) := \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} z^{k+1} \quad (c \neq 0, -1, -2, ...; z \in U)$$

where $(x)_k$ is the *Pochhammer symbol* defined by

$$(x)_k := \begin{cases} 1, & k = 0 \\ x(x+1)(x+2)...(x+k-1), & k \in \mathbb{N}^* \end{cases}$$

Carlson and Shaffer[1] introduced a linear operator $L(a, c)$, corresponding to the function $\phi(a, c; z)$, defined by the following Hadamard product:

$$L(a, c) := \phi(a, c; z) * f(z) = z + \sum_{k=1}^{\infty} \frac{(a)_k}{(c)_k} a_{k+1} z^{k+1} \tag{1.2}$$

We note that

$$z\left(L(a,c)f(z)\right)' = aL(a+1,c)f(z) - (a-1)L(a,c)f(z)$$

For $a = m+1$ and $c = 1$ we obtain the *Ruscheweyh derivative of* f, (see [9]).

In [2], N.E. Cho and H. M. Srivastava introduced a linear operator of the form:

$$\mathcal{I}(m,l)f(z) = z + \sum_{k=2}^{\infty}\left(\frac{l+k}{l+1}\right)^m a_k z^k, \quad m \in \mathbb{Z},\ l \geq 0 \tag{1.3}$$

We note that

$$z\left(\mathcal{I}(m,l)f(z)\right)' = (l+1)\mathcal{I}(m+1,l)f(z) - l\mathcal{I}(m,l)f(z)$$

For $l = 0$ we obtain Sălăgean operator introduced in [10].

Let now $\mathcal{L}(m,l,a,c,\lambda)$ be the operator defined by:

$$\mathcal{L}(m,l,a,c,\lambda)f(z) = \lambda\mathcal{I}(m,l)f(z) + (1-\lambda)L(a,c)f(z)$$

For $\lambda = 0$ we get *Carlson-Shaffer* operator introduced in [1], for $\lambda = 1$ we get linear operator in [2] and for $a = m+1, c = 1, l = 0$ we get generalized Sălăgean and Ruscheweyh operator introduced by A. Alb Lupaş in [4]

By means of operator $\mathcal{L}(m,l,a,c,\lambda)$ we introduce the following subclass of analytic functions:

Definition 1.1. *We say that a function* $f \in \mathcal{A}_n$ *is in the class* $\mathcal{BL}(m,l,a,c,\mu,\alpha,\lambda)$, $n,m \in \mathbb{N}$, $l,\mu,\lambda \geq 0$, $\alpha \in [0,1)$ *if*

$$\left|\frac{\mathcal{L}(m+1,l,a+1,c,\lambda)f(z)}{z}\left(\frac{z}{\mathcal{L}(m,l,a,c,\lambda)f(z)}\right)^{\mu} - 1\right| < 1-\alpha,\ z \in U. \tag{1.4}$$

Remark 1.2. The class $\mathcal{BL}(m,l,a,c,\mu,\alpha,\lambda)$ includes various classes of analytic univalent functions, such as:

- $\mathcal{BL}(0,0,a,c,1,\alpha,1) \equiv \mathcal{S}^*(\alpha)$
- $\mathcal{BL}(1,0,a,c,1,\alpha,1) \equiv \mathcal{K}(\alpha)$
- $\mathcal{BL}(0,0,a,c,0,\alpha,1) \equiv \mathcal{R}(\alpha)$
- $\mathcal{BL}(0,0,a,c,2,\alpha,1) \equiv \mathcal{B}(\alpha)$ introduced by Frasin and Darus in [7]
- $\mathcal{BL}(0,0,a,c,\mu,\alpha,1) \equiv \mathcal{B}(\mu,\alpha)$ introduced by Frasin and Jahangiri in [6]
- $\mathcal{BL}(m,0,a,c,\mu,\alpha,1) \equiv \mathcal{BS}(m,\mu,\alpha)$ introduced by A. Alb Lupaş and A. Cătaş in [5]
- $\mathcal{BL}(m,0,m+1,1,\mu,\alpha,0) \equiv \mathcal{BR}(m,\mu,\alpha)$ introduced by A. Alb Lupaş and A. Cătaş in [4]
- $\mathcal{BL}(m,0,m+1,1,\mu,\alpha,\lambda) \equiv \mathcal{BL}(m,\mu,\alpha,\lambda)$ introduced by A. Alb Lupaş and A. Cătaş in [3].

To prove our main result we shall need the following lemmas:

Lemma 1.3. [6] *Let* p *be analytic in* U, *with* $p(0) = 1$, *and suppose that*

$$Re\left(1 + \frac{zp'(z)}{p(z)}\right) > \frac{3\alpha - 1}{2\alpha},\ z \in U \tag{1.5}$$

Then $Re\ p(z) > \alpha$ *for* $z \in U$ *and* $\alpha \in [-1/2, 1)$.

Lemma 1.4. [8] *Let $p(z)$ be an analytic function in U, $p(0) = 1$ and $p(z) \neq 0,\ z \in U$. If there exist a point $z_0 \in U$ such that*

$$|\arg(p(z))| < \frac{\pi}{2}\alpha, \quad |z| < |z_0|, \quad |\arg(p(z_0))| = \frac{\pi}{2}\alpha$$

with $0 < \alpha \leq 1$, then we have

$$\frac{z_0 p'(z_0)}{p(z_0} = ik\alpha$$

where

$$k \geq \frac{1}{2}\left(a + \frac{1}{a}\right) \geq 1 \quad \textit{when} \ \arg(p(z_0)) = \frac{\pi}{2}\alpha,$$

$$k \leq \frac{1}{2}\left(a + \frac{1}{a}\right) \leq -1 \quad \textit{when} \ \arg(p(z_0)) = -\frac{\pi}{2}\alpha,$$

$$p(z_0)^{1/\alpha} = \pm ai, \quad a > 0.$$

2. Main results

In the first theorem we provide sufficient condition for functions to be in the class $\mathcal{BL}(m, l, a, c, \mu, \alpha, \lambda)$

Theorem 2.1. *Let $f \in \mathcal{A}_n,\ n, m \in \mathbb{N}, l, \mu, \lambda \geq 0, \alpha \in [1/2, 1)$. If*

$$\frac{\lambda(l+1)\mathcal{I}(m+2,l)f(z) - \lambda l\mathcal{I}(m+1,l)f(z) + (1-\lambda)(a+1)L(a+2,c)f(z) - (1-\lambda)aL(a+1,c)f(z)}{\mathcal{L}(m+1,l,a+1,c,\lambda)}$$

$$-\mu\frac{\lambda(l+1)\mathcal{I}(m+1,l)f(z) - \lambda l\mathcal{I}(m,l)f(z) + (1-\lambda)aL(a+1,c)f(z) - (1-\lambda)(a-1)L(a,c)f(z)}{\mathcal{L}(m,l,a,c,\lambda)}$$

$$+\mu \prec 1 + \frac{3\alpha - 1}{2\alpha}z, \quad z \in U \tag{2.1}$$

then $f \in \mathcal{BL}(m, l, a, c, \mu, \alpha, \lambda)$

Proof. If we denote by

$$p(z) = \frac{\mathcal{L}(m+1, l, a+1, c, \lambda)f(z)}{z}\left(\frac{z}{\mathcal{L}(m, l, a, c, \lambda)f(z)}\right)^{\mu} \tag{2.2}$$

where

$$p(z) = 1 + p_1 z + p_2 z^2 + ...,\ p(z) \in \mathcal{H}[1, 1],$$

then, after a simple differentiation, we obtain

$$\frac{zp'(z)}{p(z)} = \frac{\lambda(l+1)\mathcal{I}(m+2,l)f(z) - \lambda l\mathcal{I}(m+1,l)f(z) + (1-\lambda)(a+1)L(a+2,c)f(z)}{\mathcal{L}(m+1,l,a+1,c,\lambda)}$$

$$-\frac{(1-\lambda)aL(a+1,c)f(z)}{\mathcal{L}(m+1,l,a+1,c,\lambda)} - \mu\frac{(1-\lambda)aL(a+1,c)f(z) - (1-\lambda)(a-1)L(a,c)f(z)}{\mathcal{L}(m,l,a,c,\lambda)}$$

$$-\mu\frac{\lambda(l+1)\mathcal{I}(m+1,l)f(z) - \lambda l\mathcal{I}(m,l)f(z)}{\mathcal{L}(m,l,a,c,\lambda)} - 1 + \mu$$

Using (2.1), we get

$$Re\left(1 + \frac{zp'(z)}{p(z)}\right) > \frac{3\alpha - 1}{2\alpha}.$$

Thus, from Lemma 1.3, we get

$$Re\ \left\{\frac{\mathcal{L}(m+1,l,a+1,c,\lambda)f(z)}{z}\left(\frac{z}{\mathcal{L}(m,l,a,c,\lambda)f(z)}\right)^{\mu}\right\} > \alpha.$$

Therefore, $f \in \mathcal{BL}(m,l,a,c,\mu,\alpha,\lambda)$, by Definition 1.1. □

If we take $a = l$ in Theorem 2.1, we obtain the following corollary:

Corollary 2.2. *Let $f \in \mathcal{A}_n,\ n,m \in \mathbb{N}, l,\mu,\lambda \geq 0, \alpha \in [1/2,1)$. If*

$$\frac{(l+1)\mathcal{L}(m+2,l,a+2,c,\lambda)}{\mathcal{L}(m+1,l,a+1,c,\lambda)} - \mu\frac{(l+1)\mathcal{L}(m+1,l,a+1,c,\lambda)}{\mathcal{L}(m,l,a,c,\lambda)} - l + \mu(l+1) \prec$$

$$\prec 1 + \frac{3\alpha-1}{2\alpha}z,\ \ z \in U$$

then $f \in \mathcal{BL}(m,l,a,c,\mu,\alpha,\lambda)$

Next, we prove the following theorem:

Theorem 2.3. *Let $f(z) \in \mathcal{A}$. If $f(z) \in \mathcal{BL}(m,l,l+1,c,\mu,\alpha,\lambda)$, then*

$$\left|\arg\frac{\mathcal{L}(m,l,l+1,c,\lambda)}{z}\right| < \frac{\pi}{2}\alpha$$

for $0 < \alpha \leq 1$ and $2/\pi \arctan(\alpha/(l+1)) - \alpha(\mu-1) = 1$

Proof. If we denote by

$$p(z) = \frac{\mathcal{L}(m,l,l+1,c,\lambda)}{z}$$

where

$$p(z) = 1 + p_1z + p_2z^2 + ...,\ \ p(z) \in \mathcal{H}[1,1],$$

then, after a simple differentiation, we obtain

$$\frac{\mathcal{L}(m+1,l,l+2,c,\lambda)f(z)}{z}\left(\frac{z}{\mathcal{L}(m,l,l+1,c,\lambda)f(z)}\right)^{\mu} = \left(\frac{1}{p(z)}\right)^{\mu-1}\left(1 + \frac{1}{l+1}\frac{zp'(z)}{p(z)}\right)$$

Suppose there exists a point $z_0 \in U$ such that

$$|\arg(p(z))| < \frac{\pi}{2}\alpha,\ \ |z| < |z_0|,\ \ |\arg(p(z_0))| = \frac{\pi}{2}\alpha$$

Then, from Lemma 1.4, we obtain:

- If $\arg(p(z_0)) = \pi/2\,\alpha$, then

$$\arg\left(\frac{\mathcal{L}(m+1,l,l+2,c,\lambda)f(z_0)}{z_0}\left(\frac{z_0}{\mathcal{L}(m,l,l+1,c,\lambda)f(z_0)}\right)^{\mu}\right)$$

$$= \arg\left(\frac{1}{p(z_0)}\right)^{\mu-1}\left(1 + \frac{1}{l+1}\frac{z_0p'(z_0)}{p(z_0)}\right) = -(\mu-1)\frac{\pi}{2}\alpha + \arg\left(1 + \frac{1}{l+1}ik\alpha\right)$$

$$\geq -(\mu-1)\frac{\pi}{2}\alpha + \arctan\frac{\alpha}{l+1} = \frac{\pi}{2}\left(\frac{2}{\pi}\arctan\frac{\alpha}{l+1} - \alpha(\mu-1)\right) = \frac{\pi}{2}$$

- If $\arg(p(z_0)) = -\pi/2\,\alpha$, then

$$\arg\left(\frac{\mathcal{L}(m+1,l,l+2,c,\lambda)f(z_0)}{z_0}\left(\frac{z_0}{\mathcal{L}(m,l,l+1,c,\lambda)f(z_0)}\right)^{\mu}\right) \leq -\frac{\pi}{2}$$

These contradict the assumption of the theorem.
Thus, the function $p(z)$ satisfy the inequality

$$|\arg(p(z))| < \frac{\pi}{2}\alpha, \quad z \in U. \qquad \square$$

If we get, in Theorem 2.2, $m = l = 0, \mu = 2$ and $\lambda = 1$, we obtain the following corollary, proved by B. A. Frasin and M. Darus in [7]:

Corollary 2.4. *Let $f(z) \in \mathcal{A}$. If $f(z) \in \mathcal{B}(\alpha)$, then*

$$\left|\arg\left(\frac{f(z)}{z}\right)\right| < \frac{\pi}{2}\alpha, \ z \in U$$

for some $\alpha(0 < \alpha < 1)$ and $(2/\pi)\tan^{-1}\alpha - \alpha = 1$.

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