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Relations between two kinds of derivatives on analytic functions II

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Abstract. We consider Ruscheweyh derivative $D^n f(z)$ and Sălăgean derivative $d^n f(z)$, for $n \in \{0,1,2,\ldots\}$, on a class

$$\mathcal{T} = \{f : f(z) = z - \sum_{n=2}^{\infty} a_n z^n \ (a_n \geqq 0,\ n = 2,3,\ldots) \text{ is analytic in } |z| < 1\}.$$

On this paper we study relations between two subclasses $\mathcal{TR}(n,m;\alpha)$ and $\mathcal{TS}(n,m;\beta)$ of $\mathcal{T}$, where $\mathcal{TR}(n,m;\alpha) = \left\{f \in \mathcal{T} : \mathrm{Re}\dfrac{D^{n+m}f(z)}{D^n f(z)} > \alpha, |z| < 1\right\}$ and $\mathcal{TS}(n,m;\beta) = \left\{f \in \mathcal{T} : \mathrm{Re}\dfrac{d^{n+m}f(z)}{d^n f(z)} > \beta, |z| < 1\right\}$ for $\alpha \in [0,1)$, $\beta \in [0,1)$, $n = 0,1,2,\ldots$ and $m = 1,2,\ldots$.

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1. Introduction

We consider a class of functions $f(z)$ defined by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

analytic in the unit open disk $|z| < 1$, and we denote by $\mathcal{A}$ the class of such functions. We also denote by $\mathcal{T}$ a subclass of the class $\mathcal{A}$ satisfying

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \qquad (a_n \geqq 0,\ n \in \mathcal{N}_2 = \mathcal{N} - \{1\}), \tag{1.2}$$

where $\mathcal{N}$ is the set of positive integers. For any $\beta \in [0,1) = \{x : 0 \leqq x < 1\}$ a function $f(z)$, which is in the class $\mathcal{A}$ and satisfies $\mathrm{Re}\left\{\dfrac{zf'(z)}{f(z)}\right\} > \beta$ in $|z| < 1$, is called starlike of order β and we denote by $\mathcal{S}^*(\beta)$ the class of such functions. We denote by $\mathcal{S}^*$ the class $\mathcal{S}^*(0)$.

We also consider two kinds of derivatives, namely Ruscheweyh derivative ([1]) D^n and Sălăgean derivative ([2]) d^n for $n \in \mathcal{N}_0 = \mathcal{N} \cup \{0\}$ by

$$D^0 f(z) = f(z), \quad D^1 f(z) = Df(z) = zf'(z), \quad D^n f(z) = \frac{z(z^{n-1}f(z))^{(n)}}{n!} \quad (n \in \mathcal{N}_2)$$

and

$$d^0 f(z) = f(z), \quad d^1 f(z) = df(z) = zf'(z), \quad d^n f(z) = d(d^{n-1}f(z)) \quad (n \in \mathcal{N}_2),$$

respectively.

For $n \in \mathcal{N}_0, m \in \mathcal{N}$ and $\beta \in [0,1)$, Sekine([4]) introduced the following class

$$\mathcal{S}(n,m;\beta) = \left\{ f \in \mathcal{A} : \mathrm{Re}\frac{d^{n+m}f(z)}{d^n f(z)} > \beta, |z| < 1 \right\} \tag{1.3}$$

as a subclass of the class $\mathcal{A}$. For $n \in \mathcal{N}_0$, $m \in \mathcal{N}$ and $\alpha \in [0,1)$, we([3]) introduced the following class

$$\mathcal{R}(n,m;\alpha) = \left\{ f \in \mathcal{A} : \mathrm{Re}\frac{D^{n+m}f(z)}{D^n f(z)} = \mathrm{Re}\frac{n!(z^{n+m-1}f(z))^{(n+m)}}{(n+m)!(z^{n-1}f(z))^{(n)}} > \alpha, |z| < 1 \right\} \tag{1.4}$$

as another subclass of the class $\mathcal{A}$. We express $\mathcal{R}(n,1;\frac{1}{2})$, $\mathcal{S}(n,1;0)$ and $\mathcal{S}(n,m;0)$ as $\mathcal{R}(n)$ ([1]), $\mathcal{S}(n)$ ([2]) and $\mathcal{S}(n,m)$, respectively. Next let $\mathcal{TR}(n,m;\alpha)$, $\mathcal{TS}(n,m;\beta)$, $\mathcal{TR}(n,m)$ and $\mathcal{TS}(n,m)$ denote the classes $\mathcal{T}\cap\mathcal{R}(n,m;\alpha)$, $\mathcal{T}\cap\mathcal{S}(n,m;\beta)$, $\mathcal{T}\cap\mathcal{R}(n,m)$ and $\mathcal{T}\cap\mathcal{S}(n,m)$, respectively.

In the papers ([6], [5]), we researched a relation among subclasses $\mathcal{TR}(n,m;\alpha)$ and $\mathcal{TS}(n,m;\beta)$, respectively. In this paper, we will discuss a relation among subclasses mixed with $\mathcal{TR}(n,m;\alpha)$ and $\mathcal{TS}(n,m;\beta)$.

2. Preliminaries

2.1. Fundamental results

In this subsection, we show some useful fundamental results to prove our main theorem.

Theorem 2.1. ([1]) *The relation $\mathcal{R}(n+1) \subset \mathcal{R}(n) \subset \mathcal{S}^*$ holds for all $n \in \mathcal{N}_0$.*

Theorem 2.2. ([2]) *The relation $\mathcal{S}(n+1) \subset \mathcal{S}(n) \subset \mathcal{S}^*$ holds for all $n \in \mathcal{N}_0$.*

Theorem 2.3. ([4]) *If $\displaystyle\sum_{k=2}^{\infty} \frac{k^n(k^m-\beta)}{1-\beta}|a_k| \leqq 1$ for $n \in \mathcal{N}_0, m \in \mathcal{N}, \beta \in [0,1)$ and $f \in \mathcal{A}$, then $f \in \mathcal{S}(n,m;\beta)$.*

Theorem 2.4. ([4]) *For $n \in \mathcal{N}_0, m \in \mathcal{N}, \beta \in [0,1)$ and $f \in \mathcal{T}$, we have that*

$$f \in \mathcal{TS}(n,m;\beta) \iff \sum_{k=2}^{\infty} \frac{k^n(k^m-\beta)}{1-\beta} a_k \leqq 1. \tag{2.1}$$

The following theorem is a result to indicate a sufficient condition for $f \in \mathcal{R}(n,m;\alpha)$.

Theorem 2.5. ([5]) *If for $n \in \mathcal{N}_0$, $m \in \mathcal{N}$, $\alpha \in [0,1)$ and $f \in \mathcal{A}$,*

$$\sum_{k=2}^{\infty} \frac{\begin{pmatrix} n+m+k-1 \\ k-1 \end{pmatrix} - \alpha \begin{pmatrix} n+k-1 \\ k-1 \end{pmatrix}}{1-\alpha} |a_k| \leqq 1,$$

where $\begin{pmatrix} a \\ 0 \end{pmatrix} = 1$ for $a \in \mathcal{N}$ and $\begin{pmatrix} a \\ b \end{pmatrix} = \dfrac{a(a-1)\times \ldots \times (a-b+1)}{b!}$ *for $a, b \in \mathcal{N}$ and $a \geqq b$, then $f \in \mathcal{R}(n,m;\alpha)$.*

The following theorem is a useful result to indicate a necessary and sufficient condition for $f \in \mathcal{TR}(n,m;\alpha)$.

Theorem 2.6. ([5]) *For $n \in \mathcal{N}_0$, $m \in \mathcal{N}$, $\alpha \in [0,1)$ and $f \in \mathcal{T}$, we have that*

$$f \in \mathcal{TR}(n,m;\alpha) \iff \sum_{k=2}^{\infty} \frac{\begin{pmatrix} n+m+k-1 \\ k-1 \end{pmatrix} - \alpha \begin{pmatrix} n+k-1 \\ k-1 \end{pmatrix}}{1-\alpha} a_k \leqq 1. \quad (2.2)$$

We obtain the following corollary of Theorem (2.5) replacing m by 1.

Corollary 2.7. ([5]) *If* $\displaystyle\sum_{k=2}^{\infty} \frac{\begin{pmatrix} n+k \\ k-1 \end{pmatrix} - \alpha \begin{pmatrix} n+k-1 \\ k-1 \end{pmatrix}}{1-\alpha} |a_k| \leqq 1$ *for $n \in \mathcal{N}_0$ and $f \in \mathcal{A}$, then $f \in \mathcal{R}(n,1;\alpha)$.*

We obtain the following corollary of Theorem (2.6) replacing m by 1.

Corollary 2.8. ([5]) For $n \in \mathcal{N}_0$ and $f \in \mathcal{T}$, we have that

$$f \in \mathcal{TR}(n,1;\alpha) \iff \sum_{k=2}^{\infty} \frac{\begin{pmatrix} n+k \\ k-1 \end{pmatrix} - \alpha \begin{pmatrix} n+k-1 \\ k-1 \end{pmatrix}}{1-\alpha} a_k \leqq 1.$$

2.2. Examples

Before proving our theorem, we present two examples. Their proof can be found in [3].

Example 2.9. The following relations hold true for $m \in \mathcal{N}_2$ and $0 \leqq \beta < 1$:
(a) $\mathcal{TR}(0,m;\alpha) \subsetneqq \mathcal{TS}(0,m;\beta)$ for $1 - \frac{1-\beta}{m!} \leqq \alpha < 1$,
(b) $\mathcal{TS}(0,m;\beta) \subsetneqq \mathcal{TR}(0,m;\alpha)$ for $0 \leqq \alpha < 1 - \frac{m}{2^m-1}(1-\beta)$,
(c) $\mathcal{TR}(0,m;\alpha) \not\subset \mathcal{TS}(0,m;\beta)$ for $1 - \frac{m}{2^m-1}(1-\beta) < \alpha < 1 - \frac{1-\beta}{m!}$
and
(d) $\mathcal{TS}(0,m;\beta) \not\subset \mathcal{TR}(0,m;\alpha)$ for $1 - \frac{m}{2^m-1}(1-\beta) < \alpha < 1 - \frac{1-\beta}{m!}$.

Example 2.10. The following relations hold true for $m \in \mathcal{N}_2$ and $0 \leqq \beta < 1$:
(a) $\mathcal{TR}(1,m;\alpha) \subsetneqq \mathcal{TS}(1,m;\beta)$ for $1 - \frac{1-\beta}{(m+1)!} \leqq \alpha < 1$,
(b) $\mathcal{TS}(1,m;\beta) \not\subset \mathcal{TR}(1,m;\alpha)$ and $\mathcal{TR}(1,m;\alpha) \not\subset \mathcal{TS}(1,m;\beta)$
for $1 - \frac{m}{2(2^m-1)}(1-\beta) < \alpha < 1 - \frac{(1-\beta)}{(m+1)!}$

and
(c) $\mathcal{TS}(1,m;\beta) \subsetneqq \mathcal{TR}(1,m;\alpha)$ for $0 \leqq \alpha \leqq 1 - \frac{m}{2(2^m-1)}(1-\beta)$.

3. Main result

Theorem 3.1. *The following relation holds true for* $n \in \mathcal{N}_0$, $m \in \mathcal{N}_2$ and $0 \leqq \beta < 1$:

$$\mathcal{TR}(n,m;\alpha) \subsetneqq \mathcal{TS}(n,m;\beta) \text{ for } 1 - \frac{1-\beta}{(m+n)!} \leqq \alpha < 1.$$

Proof. In order to prove that $\mathcal{TR}(n,m;\alpha) \subsetneqq \mathcal{TS}(n,m;\beta)$ for $1 - \frac{1-\beta}{(m+n)!} \leqq \alpha < 1$, $n \in \mathcal{N}_0$, $m \in \mathcal{N}_2$, $\beta \in [0,1)$, we have to prove that $G(\alpha,\beta;k,n,m) \leq 0$, where

$$G(\alpha,\beta;k,n,m) =$$
$$(1-\alpha)k^{n+m} - \frac{1-\beta}{(n+m)!}\prod_{l=0}^{n+m-1}(k+l) - \beta(1-\alpha)k^n + \frac{\alpha(1-\beta)}{n!}\prod_{l=0}^{n-1}(k+l),$$

for $k,m \in \mathcal{N}_2$, $n \in \mathcal{N}_0$, $1-\frac{1-\beta}{(m+n)!} \leqq \alpha < 1$ $\beta \in [0,1)$. We show that $G(\alpha,\beta;k,n,m)$ is a decreasing function of α for all β,k,n,m (with the conditions in the Theorem) and that

$$G(1 - \frac{1-\beta}{(n+m)!},\beta;k,n,m) \leq 0. \tag{3.1}$$

We have

$$\frac{\partial}{\partial\alpha}G(\alpha,\beta;k,n,m) = -k^{n+m} + \beta k^n + \frac{1-\beta}{n!}\prod_{l=0}^{n-1}(k+l) = H(n,m;\beta);$$

and we will prove that

$$\frac{\partial}{\partial\beta}H(n,m;\beta) = k^n - \frac{1}{n!}\prod_{l=0}^{n-1}(k+l) \geq 0, \quad n \in \mathcal{N},$$

or equivalently, that

$$n!k^n - \prod_{l=0}^{n-1}(k+l) \geq 0, \tag{3.2}$$

by the mathematical induction.

Case $n=1$. $\frac{\partial}{\partial\beta}H(1,m;\beta) = k - k = 0$;

Suppose that (3.2)holds true.

Case $(n+1)$. We have

$$(n+1)!k^{n+1} = (n+1)k\{n!k^n\} > (n+1)k\prod_{l=0}^{n-1}(k+l) > \prod_{l=0}^{n}(k+l)$$

because $(n+1)k > k+n$.

From $\frac{\partial}{\partial\beta}H(n,m;\beta)\ge 0$ we have that

$$H(n,m;\beta)\le H(n,m;1)=-k^{n+m}+k^n<0$$

for all $n\in\mathcal{N}_0$, $\beta\in[0,1)$. From $\frac{\partial}{\partial\alpha}G(\alpha,\beta;k,n,m)=H(n,m;\beta)<0$ we deduce that $G(\alpha,\beta;k,n,m)$ is a decreasing function of α; then

$$G(\alpha,\beta;k,n,m)\le G\left(1-\frac{1-\beta}{(n+m)!},\beta;k,n,m\right) \tag{3.3}$$

Now we have to show that (3.1) holds. We can write

$$G\left(1-\frac{1-\beta}{(n+m)!},\beta;k,n,m\right)=\frac{1-\beta}{(n+m)!}L(\beta;k;n,m), \tag{3.4}$$

where

$$L(\beta;k;n,m)=k^{n+m}-\prod_{l=0}^{n+m-1}(k+l)+\beta k^n+\frac{(n+m)!}{n!}\prod_{l=0}^{n-1}(k+l)-\frac{1-\beta}{n!}\prod_{l=0}^{n-1}(k+l).$$

But

$$\frac{\partial}{\partial\beta}L(\beta;k;n,m)=k^n+\frac{1}{n!}\prod_{l=0}^{n-1}(k+l)>0,$$

hence

$$L(\beta;k;n,m)\le L(1;k;n,m). \tag{3.5}$$

Now we prove that $L(1;k;n,m)\le 0$ by mathematical induction with respect to n. We have

$$L(1;k;1,m)=k^{1+m}-k(k+1)\times\ldots\times(k+m)+k+(1+m)!k=k\phi(k),$$

where the function

$$\phi(x)=x^m-(x+1)\times\ldots\times(x+m)+1+(1+m)!\text{ for }x\ge 2$$

is increasing, because

$$\phi'(x)=mx^{m-1}-(x+1)\times\ldots\times(x+m)\sum_{s=1}^{m}\frac{1}{x+s}<0.$$

Since

$$\phi(x)\le\phi(2)=2^m-3\times\ldots\times(m+2)+(m+1)!+1=2^m-\frac{(m+1)!m}{2}+1<0,$$

for $m\ge 2$, we obtain that $L(1;k;1,m)<0$ for $m,\,k\ge 2$.

Now we suppose that $L(1;k;n,m)<0$, that is

$$k^{n+m}+k^n<\prod_{l=0}^{n+m-1}(k+l)-\frac{(n+m)!}{n!}\prod_{l=0}^{n-1}(k+l) \tag{3.6}$$

We have (the case $(n+1)$)

$$k^{n+1+m}+k^{n+1}=k(k^{n+m}+k^n)<k\left[\prod_{l=0}^{n+m-1}(k+l)-\frac{(n+m)!}{n!}\prod_{l=0}^{n-1}(k+l)\right], \tag{3.7}$$

where we used (3.6). The following inequalities

$$k\left[\prod_{l=0}^{n+m-1}(k+l)-\frac{(n+m)!}{n!}\prod_{l=0}^{n-1}(k+l)\right]<\prod_{l=0}^{n+m}(k+l)-\frac{(n+m+1)!}{(n+1)!}\prod_{l=0}^{n}(k+l),$$

$$k\prod_{l=0}^{n-1}(k+l)\left[\prod_{l=n}^{n+m-1}(k+l)-\frac{(n+m)!}{n!}\right]<$$

$$(k+n)\prod_{l=0}^{n-1}(k+l)\left[\prod_{l=n}^{n+m}(k+l)-\frac{(n+m+1)!}{n!}\right],$$

$$k\left[\prod_{l=n}^{n+m-1}(k+l)-\frac{(n+m)!}{n!}\right]<(k+n)\left[\prod_{l=n}^{n+m}(k+l)-\frac{(n+m+1)!}{n!}\right],$$

$$k(k+n)\times\ldots\times(k+n+m-1)-k(n+1)\times\ldots\times(n+m)<$$

$$(k+n)(k+n+1)\times\ldots\times(k+n+m)-(k+n)(n+2)\times\ldots\times(n+m+1)$$

and

$$\begin{aligned}&(k+n)\times\ldots\times(k+n+m-1)[k-(k+n+m)]-\\&(n+2)\times\ldots\times(n+m)[k(n+1)-(k+n)(n+m+1)]<0\end{aligned}\tag{3.8}$$

are equivalent to (3.7). We denote

$$\begin{aligned}M(k,m,n)=&-(k+n)\times\ldots\times(k+n+m-1)(n+m)-\\&(n+2)\times\ldots\times(n+m)[k(n+1)-(k+n)(n+m+1)].\end{aligned}\tag{3.9}$$

Since $k\geq 2$ we deduce
$M(k,m,n)\leq -(2+n)\times\ldots\times(2+n+m-2)(k+n+m-1)(n+m)+(n+2)\times\ldots\times(n+m)[n^2+n+nm+km]$
$=(n+2)\times\ldots\times(n+m)[-(k+n+m-1)(n+m)+(n^2+n+nm+km)]$
$=(n+2)\times\ldots\times(n+m)[(2-k)n+m(1-n)-m^2]<0.$

From $M(k,m,n)<0$, notation (3.9) and the equivalence of the inequalities between (3.7) and (3.8), we obtain that (3.6) holds and this means that

$$L(1;k;n,m)\leq 0.\tag{3.10}$$

Combining (3.10), (3.5), (3.4) and (3.3) we complete the proof. □

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