

# A generalization of Goluzin's univalence criterion

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**Abstract.** In this paper we obtain a sufficient condition for univalence and quasiconformal extension of an analytic function, which generalizes the well known condition for univalency established by G. M. Goluzin.

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## 1. Introduction

Let  $\mathcal{U}_r = \{z \in \mathbb{C} : |z| < r\}$   $0 < r \leq 1$  be the disk of radius  $r$  centered at 0 and let  $\mathcal{U} = \mathcal{U}_1$  be the unit disk.

Denote by  $\mathcal{A}$  the class of analytic functions in  $\mathcal{U}$  which satisfy the usual normalization  $f(0) = f'(0) - 1 = 0$ .

During the time many criteria which guarantee the univalence of a function in  $\mathcal{A}$  have been obtained. Some of these univalence criteria (see [11], [12], [16], [17]) involve the expression  $z^2 f'(z)/f^2(z) - 1$ .

In this paper we obtain a generalization of a univalence criterion due to Goluzin (see [7]) in which the logarithmic derivative of  $z^2 f'(z)/f^2(z)$  is contained.

## 2. Loewner chains and quasiconformal extensions

Before proving our main result we need a brief summary of theory of Loewner chains.

A function  $L(z, t) : \mathcal{U} \times [0, \infty) \rightarrow \mathbb{C}$  is said to be a *Loewner chain* or a *subordination chain* if:

- (i)  $L(z, t)$  is analytic and univalent in  $\mathcal{U}$  for all  $t \geq 0$ .
- (ii)  $L(z, t) \prec L(z, s)$  for all  $0 \leq t \leq s < \infty$ , where the symbol " $\prec$ " stands for subordination.

The following result due to Pommerenke is often used to obtain univalence criteria.

**Theorem 2.1.** ([14], [15]) *Let  $L(z, t) = a_1(t)z + \dots$  be an analytic function in  $\mathcal{U}_r$  ( $0 < r \leq 1$ ) for all  $t \geq 0$ . Suppose that:*

- (i)  $L(z, t)$  is a locally absolutely continuous function of  $t \in [0, \infty)$ , locally uniform with respect to  $z \in \mathcal{U}_r$ .
- (ii)  $a_1(t)$  is a complex valued continuous function on  $[0, \infty)$  such that  $a_1(t) \neq 0$ ,  $\lim_{t \rightarrow \infty} |a_1(t)| = \infty$  and

$$\left\{ \frac{L(z, t)}{a_1(t)} \right\}_{t \geq 0}$$

is a normal family of functions in  $\mathcal{U}_r$ .

- (iii) There exists an analytic function  $p : \mathcal{U}_r \times [0, \infty) \rightarrow \mathbb{C}$  satisfying  $\Re p(z, t) > 0$  for all  $(z, t) \in \mathcal{U} \times [0, \infty)$  and

$$z \frac{\partial L(z, t)}{\partial z} = p(z, t) \frac{\partial L(z, t)}{\partial t}, \quad z \in \mathcal{U}_r, \text{ a.e. } t \geq 0. \tag{2.1}$$

Then, for each  $t \geq 0$ , the function  $L(z, t)$  has an analytic and univalent extension to the whole disk  $\mathcal{U}$ , i.e  $L(z, t)$  is a Loewner chain.

Let  $k$  be a constant in  $[0, 1)$ . Recall that a homeomorphism  $f$  of  $G \subset \mathbb{C}$  is said to be  $k$ -quasiconformal if  $\partial_z f$  and  $\partial_{\bar{z}} f$  are locally integrable on  $G$  and satisfy  $|\partial_{\bar{z}} f| \leq k |\partial_z f|$  almost everywhere in  $G$ .

The method of constructing quasiconformal extension criteria is based on the following result due to Becker (see [3], [4] and also [5]).

**Theorem 2.2.** *Suppose that  $L(z, t)$  is a subordination chain. Consider*

$$w(z, t) = \frac{p(z, t) - 1}{p(z, t) + 1}, \quad z \in \mathcal{U}, \quad t \geq 0$$

where  $p(z, t)$  is defined by (2.1). If

$$|w(z, t)| \leq k, \quad 0 \leq k < 1$$

for all  $z \in \mathcal{U}$  and  $t \geq 0$ , then  $L(z, t)$  admits a continuous extension to  $\bar{\mathcal{U}}$  for each  $t \geq 0$  and the function  $F(z, \bar{z})$  defined by

$$F(z, \bar{z}) = \begin{cases} L(z, 0) & , \text{ if } |z| < 1 \\ L\left(\frac{z}{|z|}, \log |z|\right) & , \text{ if } |z| \geq 1. \end{cases}$$

is a  $k$ -quasiconformal extension of  $L(z, 0)$  to  $\mathbb{C}$ .

Examples of quasiconformal extension criteria can be found in [1], [2], [13] and more recently in [8], [9], [10].

### 3. Univalence criterion

In this section making use of Theorem 2.1 we obtain a univalence criterion for analytic functions in  $\mathcal{U}$  which involves the logarithmic derivative of  $z^2 f'(z)/f^2(z)$ .

**Theorem 3.1.** *Let  $f \in \mathcal{A}$  and let  $m$  be a positive real number. If*

$$\left| (1 - |z|^{m+1})z \frac{d}{dz} \left( \log \frac{z^2 f'(z)}{f^2(z)} \right) - \frac{m-1}{2} |z|^{m+1} \right| \leq \frac{m+1}{2} |z|^{m+1} \tag{3.1}$$

for all  $z \in \mathcal{U}$ , then the function  $f$  is univalent in the unit disk.

*Proof.* Let  $a$  be a positive real number and let the function  $h(z, t)$  defined by

$$h(z, t) = 1 - (e^{mat} - e^{-at})z \left[ \frac{f'(e^{-at}z)}{f(e^{-at}z)} - \frac{e^{at}}{z} \right].$$

For all  $t \geq 0$  and  $z \in \mathcal{U}$  we have  $e^{-at}z \in \mathcal{U}$  and from the analyticity of  $f$  in  $\mathcal{U}$  it follows that  $h(z, t)$  is also analytic in  $\mathcal{U}$ . Since  $h(0, t) = 1$ , there exists a disk  $\mathcal{U}_{r_1}$ ,  $0 < r_1 < 1$  in which  $h(z, t) \neq 0$  for all  $t \geq 0$ . Then the function  $L(z, t)$  defined by

$$L(z, t) = f(e^{-at}z) + \frac{(e^{mat} - e^{-at})z f'(e^{-at}z)}{1 - (e^{mat} - e^{-at})z \left[ \frac{f'(e^{-at}z)}{f(e^{-at}z)} - \frac{e^{at}}{z} \right]} \tag{3.2}$$

is analytic in  $\mathcal{U}_{r_1}$ , for all  $t \geq 0$ . If  $L(z, t) = a_1(t)z + a_2(t)z^2 + \dots$  is Taylor expansion of  $L(z, t)$  in  $\mathcal{U}_{r_1}$ , then it can be checked that we have  $a_1(t) = e^{mat}$  and therefore  $a_1(t) \neq 0$  for all  $t \geq 0$  and  $\lim_{t \rightarrow \infty} |a_1(t)| = \infty$ .

From the analyticity of  $L(z, t)$  in  $\mathcal{U}_{r_1}$ , it follows that there exists a number  $r_2$ ,  $0 < r_2 < r_1$ , and a constant  $K = K(r_2)$  such that

$$|L(z, t)/a_1(t)| < K, \quad \forall z \in \mathcal{U}_{r_2}, \quad t \geq 0,$$

and thus  $\{L(z, t)/a_1(t)\}$  is a normal family in  $\mathcal{U}_{r_2}$ . From the analyticity of  $\partial L(z, t)/\partial t$ , for all fixed numbers  $T > 0$  and  $r_3$ ,  $0 < r_3 < r_2$ , there exists a constant  $K_1 > 0$  (that depends on  $T$  and  $r_3$ ) such that

$$\left| \frac{\partial L(z, t)}{\partial t} \right| < K_1, \quad \forall z \in \mathcal{U}_{r_3}, \quad t \in [0, T].$$

It follows that the function  $L(z, t)$  is locally absolutely continuous in  $[0, \infty)$ , locally uniform with respect to  $z \in \mathcal{U}_{r_3}$ . The function  $p(z, t)$  defined by

$$p(z, t) = z \frac{\partial L(z, t)}{\partial z} \bigg/ \frac{\partial L(z, t)}{\partial t}$$

is analytic in a disk  $\mathcal{U}_r$ ,  $0 < r < r_3$ , for all  $t \geq 0$ .

In order to prove that the function  $p(z, t)$  is analytic and has positive real part in  $\mathcal{U}$ , we will show that the function

$$w(z, t) = \frac{p(z, t) - 1}{p(z, t) + 1}, \quad z \in \mathcal{U}_r, \quad t \geq 0$$

is analytic in  $\mathcal{U}$  and

$$|w(z, t)| < 1 \text{ for all } z \in \mathcal{U} \text{ and } t \geq 0. \tag{3.3}$$

Elementary calculation gives

$$w(z, t) = \frac{(1+a)\mathcal{G}(z, t) + 1 - ma}{(1-a)\mathcal{G}(z, t) + 1 + ma}, \tag{3.4}$$

where  $\mathcal{G}(z, t)$  is given by

$$\mathcal{G}(z, t) = \left( e^{(m+1)at} - 1 \right) \left[ 2 + \frac{e^{-at} z f''(e^{-at} z)}{f'(e^{-at} z)} - 2 \frac{e^{-at} z f'(e^{-at} z)}{f(e^{-at} z)} \right]. \tag{3.5}$$

It is easy to prove that the condition (3.3) is equivalent to

$$\left| \mathcal{G}(z, t) - \frac{m-1}{2} \right| < \frac{m+1}{2} \quad \text{for all } z \in \mathcal{U} \text{ and } t \geq 0. \tag{3.6}$$

For  $t = 0$  and  $z = 0$  the inequality (3.6) becomes

$$\left| \mathcal{G}(z, 0) - \frac{m-1}{2} \right| = \left| \mathcal{G}(0, t) - \frac{m-1}{2} \right| = \left| \frac{m-1}{2} \right| < \frac{m+1}{2}. \tag{3.7}$$

Since  $m$  is a positive number, the last inequality holds true.

Let  $t$  be a fixed number,  $t > 0$  and let  $z \in \mathcal{U}$ ,  $z \neq 0$ . Since  $|e^{-at} z| \leq e^{-at} < 1$  for all  $z \in \bar{\mathcal{U}} = \{z \in \mathbb{C} : |z| \leq 1\}$ , from (3.1) we conclude that the function  $\mathcal{G}(z, t)$  is analytic in  $\bar{\mathcal{U}}$ . Using the maximum modulus principle it follows that for each  $t > 0$ , arbitrary fixed, there exists  $\theta = \theta(t) \in \mathbb{R}$  such that

$$|\mathcal{G}(z, t)| < \max_{|\xi|=1} |\mathcal{G}(\xi, t)| = |\mathcal{G}(e^{i\theta}, t)|. \tag{3.8}$$

Denote  $u = e^{-at} \cdot e^{i\theta}$ . Then  $|u| = e^{-at} < 1$ ,  $e^{(m+1)at} = 1/|u|^{m+1}$  and therefore

$$\begin{aligned} \mathcal{G}(e^{i\theta}, t) &= \left( \frac{1}{|u|^{m+1}} - 1 \right) \cdot u \cdot \left[ \frac{2}{u} + \frac{f''(u)}{f'(u)} - 2 \frac{f'(u)}{f(u)} \right] \\ &= \left( \frac{1}{|u|^{m+1}} - 1 \right) \cdot u \frac{d}{du} \left( \log \frac{u^2 f'(u)}{f^2(u)} \right) \end{aligned}$$

Since  $u \in \mathcal{U}$ , the inequality (3.1) implies

$$\left| \mathcal{G}(e^{i\theta}, t) - \frac{m-1}{2} \right| \leq \frac{m+1}{2} \tag{3.9}$$

From (3.7), (3.8) and (3.9) we conclude that the inequality (3.6) holds true for all  $z \in \mathcal{U}$  and  $t \geq 0$ . It follows that  $L(z, t)$  is a Loewner chain and hence the function  $L(z, 0) = f(z)$  is univalent in  $\mathcal{U}$ . □

**Remark 3.2.** For  $m = 1$  Theorem 3.1 specializes to a univalent criterion due to Goluzin [7].

The function  $f(z) = \frac{z}{1 + cz}$ ,  $z \in \mathcal{U}$  satisfies the condition (3.1) of the Theorem 3.1 for all positive real numbers  $m$  and all complex numbers  $c$ .

### 4. Quasiconformal extension

In this section we obtain a quasiconformal extension of the univalence condition given in Theorem 3.1.

**Theorem 4.1.** *Let  $f \in \mathcal{A}$ . Let also  $m \in \mathbb{R}_+$  and  $k \in [0, 1)$ . If the inequality*

$$\left| (1 - |z|^{m+1})z \frac{d}{dz} \left( \log \frac{z^2 f'(z)}{f^2(z)} \right) - \frac{m-1}{2} |z|^{m+1} \right| \leq k \frac{m+1}{2} |z|^{m+1} \tag{4.1}$$

*is true for all  $z \in \mathcal{U}$  then, the function  $f$  has an  $l$ -quasiconformal extension to  $\mathbb{C}$ , where*

$$l = \frac{(1-a)^2 + k|1-a^2|}{|1-a^2| + k(1-a)^2} \text{ and } a > 0.$$

*Proof.* In the proof of Theorem 3.1 has been shown that the function  $L(z, t)$  given by (3.2) is a subordination chain in  $\mathcal{U}$ . Applying Theorem 2.2 to the function  $w(z, t)$  given by (3.4), we obtain that the condition

$$\left| \frac{(1+a)\mathcal{G}(z, t) + 1 - ma}{(1-a)\mathcal{G}(z, t) + 1 + ma} \right| < l, \quad z \in \mathcal{U}, \quad t \geq 0 \text{ and } l \in [0, 1) \tag{4.2}$$

where  $\mathcal{G}(z, t)$  is defined by (3.5), implies  $l$ -quasiconformal extensibility of  $f$ .

Lengthy but elementary calculation shows that the last inequality (4.2) is equivalent to

$$\left| \mathcal{G}(z, t) - \frac{a(1+l^2)(m-1) + (1-l^2)(ma^2-1)}{2a(1+l^2) + (1-l^2)(1+a^2)} \right| \leq \frac{2al(1+m)}{2a(1+l^2) + (1-l^2)(1+a^2)}. \tag{4.3}$$

It is easy to check that, under the assumption (4.1) we have

$$\left| \mathcal{G}(z, t) - \frac{m-1}{2} \right| \leq k \frac{m+1}{2}. \tag{4.4}$$

Consider the two disks  $\Delta$  and  $\Delta'$  defined by (4.3) and (4.4) respectively, where  $\mathcal{G}(z, t)$  is replaced by a complex variable  $\zeta$ . Our theorem will be proved if we find the smallest  $l \in [0, 1)$  for which  $\Delta'$  is contained in  $\Delta$ . This will be so if and only if the distance apart of the centers plus the smallest radius is equal, at most, to the largest radius. So, we are required to prove that

$$\begin{aligned} & \left| \frac{a(1+l^2)(m-1) + (1-l^2)(ma^2-1)}{2a(1+l^2) + (1-l^2)(1+a^2)} - \frac{m-1}{2} \right| + k \frac{m+1}{2} \\ & \leq \frac{2al(1+m)}{2a(1+l^2) + (1-l^2)(1+a^2)} \end{aligned}$$

or equivalently

$$\frac{(1-l^2)|1-a^2|}{2[2a(1+l^2) + (1-l^2)(1+a^2)]} \leq \frac{2al}{2a(1+l^2) + (1-l^2)(1+a^2)} - \frac{k}{2} \tag{4.5}$$

with the condition

$$\frac{2al}{2a(1+l^2) + (1-l^2)(1+a^2)} - \frac{k}{2} \geq 0. \tag{4.6}$$

We will solve inequalities (4.5) and (4.6) for  $1 - a^2 > 0$ . In a similar way they can be solved for  $1 - a^2 < 0$ .

If in (4.5) the inequality sign is replaced by equal we obtain the following two solutions:

$$L_1 = \frac{(1-a)^2 + k(1-a^2)}{1-a^2 + k(1-a)^2}, \quad L_2 = -\frac{(1+a)^2 + k(1-a^2)}{1-a^2 + k(1-a)^2}.$$

Therefore, the solution of inequality (4.5) is  $l \leq L_2$  and  $L_1 \leq l$ . Since  $L_2 < 0$  it remains  $L_1 \leq l$ .

After similar calculations, from inequality (4.6), we have  $l \leq \mathcal{L}_2$  and  $\mathcal{L}_1 \leq l$ , where

$$\mathcal{L}_1 = \frac{-2a + \sqrt{4a^2 + (1-a^2)^2 k^2}}{k(1-a)^2}, \quad \mathcal{L}_2 = \frac{-2a - \sqrt{4a^2 + (1-a^2)^2 k^2}}{k(1-a)^2}.$$

Since  $\mathcal{L}_2 < 0$ , we get  $\mathcal{L}_1 \leq l$ .

It can be checked that  $\mathcal{L}_1 \leq L_1$ . It follows  $L_1 \leq l < 1$  and thus the proof is complete.  $\square$

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