Stud. Univ. Babeş-Bolyai Math. 57(2012), No. 2, 255–259

Growth and distortion theorem for the Janowski alpha-spirallike functions in the unit disc

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Abstract. Let A be the class of all analytic functions in the open unit disc $\mathbb{D} = \{z|\ |z| < 1\}$ of the form $f(z) = z + a_2z^2 + a_3z^3 + \cdots$. Let $g(z)$ be an element of A satisfying the condition

$$e^{i\alpha}z\frac{g'(z)}{g(z)} = \frac{1 + A\phi(z)}{1 + B\phi(z)}$$

where $|\alpha| < \frac{\pi}{2}$, $-1 \leq B < A \leq 1$ and $\phi(z)$ is analytic in $\mathbb{D}$ and satisfies the conditions $\phi(0) = 0$, $|\phi(z)| < 1$ for every $z \in \mathbb{D}$. Then $g(z)$ is called Janowski α-spirallike functions in the unit disc. The class of such functions is denoted by $S^*_\alpha(A, B)$. The aim of this paper is to give growth and distortion theorems for the class $S^*_\alpha(A, B)$.

Mathematics Subject Classification (2010): 30C45.

Keywords: Growth theorem, distortion theorem, radius of starlikeness.

1. Introduction

Let Ω be the family of functions $\phi(z)$ which are regular in the open unit disc $\mathbb{D}$ and satisfying the conditions $\phi(0) = 0$, $|\phi(z)| < 1$ for all $z \in \mathbb{D}$.

Next, for arbitrary fixed numbers A, B, $-1 \leq B < A \leq 1$, denote by $\mathcal{P}(A, B)$, the family of functions $p(z) = 1 + p_1z + p_2z^2 + \cdots$ regular in $\mathbb{D}$, such that $p(z)$ in $\mathcal{P}(A, B)$ if and only if

$$p(z) = \frac{1 + A\phi(z)}{1 - B\phi(z)} \tag{1.1}$$

for some function $\phi(z) \in \Omega$, and for all $z \in \mathbb{D}$. At the same time, this class can be represented by $Rep(z) > \frac{1-A}{1-B} > 0$.

Let $F(z) = z + \alpha_2z^2 + \alpha_3z^3 + \cdots$ and $G(z) = z + \beta_2z^2 + \beta_3z^3 + \cdots$ be analytic functions in $\mathbb{D}$. If there exists a function $\phi(z) \in \Omega$ such that $F(z) = G(\phi(z))$ for every $z \in \mathbb{D}$, then we say that $F(z)$ is subordinate to $G(z)$, and we write $F(z) \prec G(z)$. We also note that if $F(z) \prec G(z)$, then $F(\mathbb{D}) \subset G(\mathbb{D})$ ([1]).

Moreover, let $S^*_\alpha(A,B)$ denote the family of functions $f(z) = z + a_2z^2 + \cdots$ regular in $\mathbb{D}$, such that $f(z)$ is in $S^*_\alpha(A,B)$ if and only if there is a real number α for which,

$$e^{i\alpha} z\frac{f'(z)}{f(z)} = \cos\alpha p(z) + i\sin\alpha, |\alpha| < \frac{\pi}{2}, p(z) \in \mathcal{P}(A,B) \tag{1.2}$$

is true for every $z \in \mathbb{D}$. Then the class $S^*_\alpha(A,B)$ is called the Janowski α-spirallike functions.

The following lemma is due to I. S. Jack and plays very important role for our proof of Theorem 2.1 ([2]).

Lemma 1.1. *Let $\phi(z)$ be regular in the unit disc $\mathbb{D}$ with $\phi(0) = 0$. Then if $|\phi(z)|$ obtains its maximum value on the circle $|z| = r$ at the point z_1, one has $z_1\phi'(z_1) = k\phi(z_1)$, for some $k \geq 1$.*

2. Main results

Theorem 2.1.

$$f(z) \in S^*_\alpha(A,B) \Leftrightarrow (z\frac{f'(z)}{f(z)} - 1) \prec \begin{cases} \frac{e^{-i\alpha}(A-B)\cos\alpha . z}{1+Bz}; & B \neq 0, \\ e^{-i\alpha}(A\cos\alpha)z; & B = 0, \end{cases} \tag{2.1}$$

Proof. Let $f(z)$ be an element of $S^*_\alpha(A,B)$. We define the functions $\phi(z)$ by;

$$\frac{f(z)}{z} = \begin{cases} (1 + B\phi(z))^{\frac{(A-B)\cos\alpha e^{-i\alpha}}{B}}; & B \neq 0, \\ e^{A\cos\alpha e^{-i\alpha}\phi(z)}; & B = 0, \end{cases} \tag{2.2}$$

where $(1 + B\phi(z))^{\frac{(A-B)\cos\alpha e^{-i\alpha}}{B}}$ and $e^{A\cos\alpha e^{-i\alpha}\phi(z)}$ have the value 1 at $z = 0$. Then $\phi(z)$ is analytic and $\phi(0) = 0$. If we take the logarithmic derivative from (2.2) and after simple calculations, we get

$$(z\frac{f'(z)}{f(z)} - 1) = \begin{cases} \frac{(A-B)\cos\alpha e^{-i\alpha} z\phi'(z)}{1+B\phi(z)}; & B \neq 0, \\ A\cos\alpha . e^{-i\alpha}z\phi'(z); & B = 0, \end{cases} \tag{2.3}$$

We can easily conclude that this subordination is equivalent to $|\phi(z)| < 1$ for all $z \in \mathbb{D}$. On the contrary let's assume that there exists $z_1 \in \mathbb{D}$, such that $|\phi(z)|$ attains its maximum value on the circle $|z| = r$, that is $|\phi(z_1)| = 1$. Then when the conditions $z_1\phi'(z_1) = k\phi(z_1)$, $k \geq 1$ are satisfied for such $z_1 \in \mathbb{D}$ (Using I.S.Jack's Lemma), we obtain;

$$(z_1\frac{f'(z_1)}{f(z_1)} - 1) = \begin{cases} \frac{(A-B)\cos\alpha e^{-i\alpha} k\phi(z_1)}{1+B\phi(z_1)} = F_1(\phi(z_1)) \notin F_1(\mathbb{D}); & B \neq 0, \\ A\cos\alpha e^{-i\alpha}k\phi(z_1) = F_2(\phi(z_1)) \notin F_2(\mathbb{D}); & B = 0, \end{cases} \tag{2.4}$$

which contradicts (2.1) implying that the assumption is wrong , i.e., $|\phi(z)| < 1$ for all $z \in \mathbb{D}$. This shows that,

$$f(z) \in S^*_\alpha(A,B) \Rightarrow (z\frac{f'(z)}{f(z)} - 1) \prec \begin{cases} \frac{(A-B)\cos\alpha e^{-i\alpha} z}{1+Bz}; & B \neq 0, \\ A\cos\alpha . e^{-i\alpha}z; & B = 0, \end{cases} \tag{2.5}$$

Conversely,

$$(z\frac{f'(z)}{f(z)}-1)\prec\begin{cases}\frac{(A-B)\cos\alpha e^{-i\alpha}}{1+Bz}; & B\neq 0,\\ A\cos\alpha.e^{-i\alpha}z; & B=0,\end{cases}\Rightarrow$$

$$e^{i\alpha}z\frac{f'(z)}{f(z)}=\begin{cases}cos\alpha\frac{1+A\phi(z)}{1+B\phi(z)}+i\sin\alpha; & B\neq 0,\\ \cos\alpha(1+A\phi(z))+i\sin\alpha; & B=0,\end{cases}$$

This shows that $f(z)\in S^*_\alpha(A,B)$. □

Corollary 2.2. *Marx-Strohacker inequality for the class $S^*_\alpha(A,B)$ is;*

$$\begin{cases}\left|(\frac{f(z)}{z})^{\frac{B.e^{i\alpha}}{(A-B)\cos\alpha}}-1\right|<1; & B\neq 0,\\ \\ \left|\log(\frac{f(z)}{z})^{\frac{e^{i\alpha}}{A\cos\alpha}}\right|<1; & B=0,\end{cases}\tag{2.6}$$

Proof. The proof of this corollary is a simple consequence of Theorem 2.1. Indeed,

$$\frac{f(z)}{z}=(1+B\phi(z))^{\frac{A-B}{B}\cos\alpha e^{-i\alpha}}\Rightarrow\left|(\frac{f(z)}{z})^{\frac{B.e^{i\alpha}}{(A-B)\cos\alpha}}-1\right|<1$$

$$\frac{f(z)}{z}=e^{A\cos\alpha e^{-i\alpha}\phi(z)}\Rightarrow\left|\log(\frac{f(z)}{z})^{\frac{e^{i\alpha}}{A\cos\alpha}}\right|<1$$

□

Theorem 2.3. *The radius of starlikeness of the class $S^*_\alpha(A,B)$ is,*

$$r=\begin{cases}\frac{2}{(A-B)\cos\alpha+\sqrt{((A-B)^2\cos^2\alpha+4[AB\cos^2\alpha+B^2\sin^2\alpha])}}; & B\neq 0,\\ \frac{1}{A\cos\alpha}; & B=0,\end{cases}\tag{2.7}$$

This radius is sharp because the extremal function is;

$$f(z)=\begin{cases}z(1+Bz)^{\frac{A-B}{B}\cos\alpha e^{-i\alpha}}; & B\neq 0,\\ ze^{A\cos\alpha e^{-i\alpha}z}; & B=0,\end{cases}\tag{2.8}$$

with $\zeta=\frac{r(r-e^{i\alpha})}{1-re^{i\alpha}}$ and we obtain,

$$\zeta\frac{f'(\zeta)}{f(\zeta)}=\begin{cases}\frac{1-(A-B)\cos\alpha r-(AB\cos^2\alpha+B^2\sin^2\alpha)r^2}{1-B^2r^2}; & B\neq 0,\\ 1-Ar\cos\alpha; & B=0,\end{cases}\tag{2.9}$$

Proof. Using (1.2) we get;

$$p(z)=\frac{1}{\cos\alpha}(e^{i\alpha}z\frac{f'(z)}{f(z)}-i\sin\alpha)\tag{2.10}$$

On the other hand, since $p(z)\in\mathcal{P}(A,B)$, then we have,

$$\left|p(z)-\frac{1-ABr^2}{1-B^2r^2}\right|\leq\frac{(A-B)r}{1-B^2r^2}\tag{2.11}$$

The inequality (2.11) was obtained by W. Janowski [5]. Using (2.10) in (2.11) and after straightforward calculations we get:

$$\begin{cases} \frac{1-(A-B)\cos\alpha r-(AB\cos^2\alpha+B^2\sin^2\alpha)r^2}{1-B^2r^2} \le Rez\frac{f'(z)}{f(z)} \\ \quad \le \frac{1+(A-B)\cos\alpha r-(AB\cos^2\alpha+B^2\sin^2\alpha)r^2}{1-B^2r^2}; \qquad B\neq 0, \\ 1-A\cos\alpha r \le Rez\frac{f'(z)}{f(z)} \le 1+A\cos\alpha r; \qquad B=0, \end{cases} \tag{2.12}$$

The inequalities (2.12) shows that this theorem is true. □

Corollary 2.4. *If we take $A = 1$, $B = -1$ we obtain,*

$$r = \frac{1}{\cos\alpha + |\sin\alpha|} \tag{2.13}$$

This is the radius of starlikeness of class of α-spirallike functions. This result was obtained independently and using different methods by both Robertson [4] and Libera [3]. We also note that if we give another special values to A and B, we obtain the radius of starlikeness of the subclass of α-spirallike functions.

Corollary 2.5. *Let $f(z)$ be an element of $S^*_\alpha(A,B)$, then*

$$r(1-Br)^{\frac{(A-B)\cos\alpha(\cos\alpha+1)}{2B}}(1+Br)^{\frac{(A-B)\cos\alpha(\cos\alpha-1)}{2B}} \le |f(z)| \le$$
$$r(1-Br)^{\frac{(A-B)\cos\alpha(\cos\alpha-1)}{2B}}(1+Br)^{\frac{(A-B)\cos\alpha(\cos\alpha+1)}{2B}}; B\neq 0,$$
$$re^{-(A\cos\alpha)r} \le |f(z)| \le re^{(A\cos\alpha)r}; B=0$$

Proof. Using (2.12),

$$Re(z\frac{f'(z)}{f(z)}) = r\frac{\partial}{\partial r}\log|f(z)|$$

and after the straightforward calculations we get the result. Also we note that these inequalities are sharp. Because the extremal function was given in Theorem 2.3. □

Corollary 2.6. *If $f(z) \in S^*_\alpha(A,B)$, then*

$$[(1-Ar)\cos\alpha-(1-Br)\sin\alpha]F(A,B,\cos\alpha,-r) \le |f'(z)| \le$$
$$[(1+Ar)\cos\alpha+(1+Br)\sin\alpha]F(A,B,\cos\alpha,r)$$

where

$$F(A,B,\cos\alpha,r) = (1+Br)^{\frac{(A-B)\cos\alpha(\cos\alpha+1)}{2B}-1}(1-Br)^{\frac{(A-B)\cos\alpha(\cos\alpha-1)}{2B}}$$

This inequality is sharp.

Proof. The proof of this corollary is based on the following observations

i. $p(z)\in P(A,B) \Rightarrow \frac{1-Ar}{1-Br} \le |p(z)| \le \frac{1+Ar}{1+Br}$

ii. $p(z) = \frac{1}{\cos\alpha}(e^{i\alpha}z\frac{f'(z)}{f(z)} - i\sin\alpha)$, $f(z)\in S^*_\alpha(A,B), p(z)\in P(A,B)$

iii. Corollary 2.5. using (i) and (ii) and after simple calculations we get:

$$\frac{(1-Ar)\cos\alpha-(1-Br)\sin\alpha}{1-Br} \le \left|z\frac{f'(z)}{f(z)}\right| \le \frac{(1+Ar)\cos\alpha+(1+Br)\sin\alpha}{1+Br} \tag{2.14}$$

Considering (2.14) and Corollary 2.5 we get the result. □

References

[1] Goodman, A.W. , *Univalent functions*, Volume I. Mariner Publishing, Tampa Florida, 1984.

[2] Jack, I.S. , *Functions starlike and convex of order* α, J. London. Math. Soc., **3**(1971), no. 2.

[3] Libera, R.J., *Univalent* α*-spiral functions*, Canad. J. Math., **19**(1967), 449-456.

[4] Robertson, M.S., *Radii of star-likeness and close-to-convexity*, Proc. Amer. Math. Soc., **14**(1965), 847-852.

[5] Janowski, W., *Some extremal problems for certain families of analytic functions*, J. Ann. Polon. Math., **28**(1973), 297-326.

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