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Some properties on generalized close-to-star functions

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Abstract. Let $f(z) = a_1 z + a_1 z^2 + \cdots, a_1 \neq 0$, be regular in $|z| < 1$ and have there no zeros except at the origin. Reade ([3]) and the Sakaguchi ([2]) showed that a necessary and sufficient condition for $f(z)$ to be a member of the class $C(k)$ is that $f(z)$ has a representation of the form

$$f(z) = s(z)(p(z))^k$$

where $s(z)$ is a regular function starlike with respect to the origin for $|z| < 1$, k is a positive constant, and $p(z)$ is a regular function with positive real part in $|z| < 1$. The class of close-to-star functions introduced by Reade ([4]) is equivalent to $C(1)$. In this paper we define the class $C(k, A, B)$ $(-1 \leq B < A \leq 1, k$ is positive constant) which contains the functions of the form

$$f(z) = s(z)(p(z))^k$$

where $s(z)$ is a regular Janowski starlike function, and $p(z)$ is a regular function with positive real part in $|z| < 1$. The aim of this paper is to give some properties and distortion theorems for this class.

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1. Introduction

Let Ω be the family of functions $\phi(z)$ regular in $\mathbb{D} = \{z \quad | \quad |z| < 1\}$ and satisfying the conditions $\phi(0) = 0$, $|\phi(z)| < 1$ for all $z \in \mathbb{D}$.

The set $\mathcal{P}$ is the set of all functions of the form

$$f(z) = 1 + p_1 z + p_2 z^2 + \cdots = 1 + \sum_{n=1}^{\infty} p_n z^n$$

that are regular in $\mathbb{D}$, and such that for $z \in \mathbb{D}$,

$$Re(f(z)) > 0.$$

Any function in $\mathcal{P}$ is called a function with positive real part in $\mathbb{D}$ ([1]).

Next, for arbitrary fixed numbers A, B, given by $-1 \leq B < A \leq 1$, we denote by $\mathcal{P}(A, B)$ the family of functions $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ regular in $\mathbb{D}$ and such that $p(z)$ is in $\mathcal{P}(A, B)$ if and only if

$$p(z) = \frac{1 + A\phi(z)}{1 + B\phi(z)}$$

for some function $\phi(z) \in \Omega$ and every $z \in \mathbb{D}$ ([5]).

Let $\mathcal{S}^*(A, B)$ denote the family of functions $s(z) = z + c_2 z^2 + c_3 z^3 + \cdots$ regular in $\mathbb{D}$, and such that $s(z)$ is in $\mathcal{S}^*(A, B)$ if and only if

$$z\frac{s'(z)}{s(z)} = p(z)$$

for some $p(z)$ is in $\mathcal{P}(A, B)$ and all $z \in \mathbb{D}$ ([5]).

Moreover, $F(z) = z + \alpha_2 z^2 + \alpha_3 z^3 + \cdots$ and $G(z) = z + \beta_2 z^2 + \beta_3 z^3 + \cdots$ are analytic functions in $\mathbb{D}$, if there exist a function $\phi(z) \in \Omega$ such that $F(z) = G(\phi(z))$ for every $z \in \mathbb{D}$, then we say that $F(z)$ is subordinate to $G(z)$ and we write $F(z) \prec G(z)$ ([1]).

Finally, let $f(z) = z + a_2 z^2 + \cdots$ be analytic function in $\mathbb{D}$, if there exists a function $s(z) \in \mathcal{S}^*(A, B)$, such that

$$\frac{f(z)}{s(z)} = (p(z))^k$$

where $p(z) \in \mathcal{P}$, k is a positive constant, then the function is called the generalized close-to-star. The class of these functions is denoted by $C(k, A, B)$.

Lemma 1.1. [1] *Let $p(z)$ be an element of $\mathcal{P}$, then*

$$\frac{-2r}{1-r^2} \leq \left| z\frac{p'(z)}{p(z)} \right| \leq \frac{2r}{1-r^2} \tag{1.1}$$

$$\frac{-2r}{1-r^2} \leq Re\left(z\frac{p'(z)}{p(z)} \right) \leq \frac{2r}{1-r^2}. \tag{1.2}$$

Lemma 1.2. [5] *Let $s(z)$ be an element of $\mathcal{S}^*(A, B), (-1 \leq B < A \leq 1)$, then*

$$\frac{1-Ar}{1-Br} \leq \left| z\frac{s'(z)}{s(z)} \right| \leq \frac{1+Ar}{1+Br} \tag{1.3}$$

$$\frac{1-Ar}{1-Br} \leq Re\left(z\frac{s'(z)}{s(z)} \right) \leq \frac{1+Ar}{1+Br}. \tag{1.4}$$

2. Main Results

Theorem 2.1. *Let $f(z)$ be an element of $C(k, A, B)$, $(-1 \leq B < A \leq 1$, k is positive constant$)$, then*

$$\left| \frac{-2kr}{1-r^2} + \frac{1-Ar}{1-Br} \right| \leq \left| z\frac{f'(z)}{f(z)} \right| \leq \frac{2kr}{1-r^2} + \frac{1+Ar}{1+Br}. \tag{2.1}$$

Proof. If $f(z)$ be an element of $C(k,A,B)$, then we write

$$f(z)=s(z)(p(z))^k.$$

If we take the logarithmic derivative of the last equality, then we have

$$z\frac{f'(z)}{f(z)}=z\frac{s'(z)}{s(z)}+kz\frac{p'(z)}{p(z)}, \tag{2.2}$$

by applying triangle inequality for the equality (2.2), we obtain

$$\left|z\frac{s'(z)}{s(z)}\right|-k\left|z\frac{p'(z)}{p(z)}\right|\le\left|z\frac{f'(z)}{f(z)}\right|\le\left|z\frac{s'(z)}{s(z)}\right|+k\left|z\frac{p'(z)}{p(z)}\right|, \tag{2.3}$$

using lemma 1.1 in the inequality (2.3), then we obtain (2.1). □

Corollary 2.2. *For $A=1,\ B=-1$, then*

$$\left|\frac{1-2(k+1)r+r^2}{1-r^2}\right|\le\left|z\frac{f'(z)}{f(z)}\right|\le\frac{1+2(k+1)r+r^2}{1-r^2}. \tag{2.4}$$

This result was obtained by Sakaguchi ([3]).

Corollary 2.3. *If $f(z)\in C(k,A,B)$ $(-1\le B<A\le 1, k$ is positive constant), then*

$$\frac{1-(2k+A)r+(2kB-1)r^2+Ar^3}{(1-r^2)(1-Br)}\le Re\left(z\frac{f'(z)}{f(z)}\right)$$
$$\le\frac{1+(2k+A)r+(2kB-1)r^2-Ar^3}{(1-r^2)(1+Br)}.$$

This inequality is simple consequence of inequality (2.1).

Corollary 2.4. [6] *The radius of starlikeness of the class $C(k,A,B)$ is the smallest positive root of the equations*

$$\psi(r)=1-(2k+A)r+(2kB-1)r^2+Ar^3=0.$$

Proof. If $f(z)\in C(k,A,B)$, then we have

$$Re\left(z\frac{f'(z)}{f(z)}\right)\ge\frac{1-(2k+A)r+(2kB-1)r^2+Ar^3}{(1-r^2)(1-Br)}=\frac{\psi(r)}{(1-r^2)(1-Br)}. \tag{2.5}$$

The denominator of the expression on the right-hand side of the inequality (2.5) is positive for $0\le r<1$,

$$\psi(0)=1,$$
$$\psi(1)=1-(2k+A)+(2kB-1)+A=-2k+2kB=-2k(1-B)\le 0.$$

Thus the smallest positive root r_0 of the equation $\psi(r)=0$ lies between 0 and 1.

Therefore the inequality $Re\left(z\frac{f'(z)}{f(z)}\right)>0$ is valid $r=|z|=r_0$. Hence the radius of starlikeness for $C(k,A,B)$ is not less than r_0. Thus the corollary is proved. □

Theorem 2.5. *Let $f(z)$ be an element of $C(k,A,B)$ $(-1 \leq B < A \leq 1$, k is positive constant), then*

$$\frac{r(1-r)^k}{(1+r)^k(1-Br)^{\frac{B-A}{B}}} \leq |f(z)| \leq \frac{r(1+r)^k}{(1-r)^k(1+Br)^{\frac{B-A}{B}}}, \quad B \neq 0;$$
$$e^{-Ar} r \left(\frac{1-r}{1+r}\right)^k \leq |f(z)| \leq e^{Ar} r \left(\frac{1+r}{1-r}\right)^k, \quad B = 0.$$

Proof. Using corollary 2.3 and the equality

$$Re\left(z\frac{f'(z)}{f(z)}\right) = r\frac{\partial}{\partial r}\log|f(z)|$$

then we have

$$\frac{1-(2k+A)r+(2kB-1)r^2+Ar^3}{r(1-r^2)(1-Br)} \leq \frac{\partial}{\partial r}\log|f(z)| \leq \frac{1+(2k+A)r+(2kB-1)r^2-Ar^3}{r(1-r^2)(1+Br)}, \quad B \neq 0; \tag{2.6}$$

$$\frac{1-(2k+A)r-r^2+Ar^3}{r(1-r^2)} \leq \frac{\partial}{\partial r}\log|f(z)| \leq \frac{1+(2k+A)r-r^2-Ar^3}{r(1-r^2)}, \quad B = 0. \tag{2.7}$$

Integrating both sides of the inequalities (2.6) and (2.7) we get the results. □

Corollary 2.6. *Let $f(z)$ be an element of $C(k,A,B)$ $(-1 \leq B < A \leq 1$, k is positive constant), then*

$$\left(\frac{1-r}{1+r}\right)^k \frac{1}{(1-Br)^{\frac{B-A}{B}}} \left|-\frac{2kr}{1-r^2} + \frac{1-Ar}{1-Br}\right| \leq |f'(z)| \leq \left(\frac{1+r}{1-r}\right)^k \frac{1}{(1+Br)^{\frac{B-A}{B}}} \left[\frac{2kr}{1-r^2} + \frac{1+Ar}{1+Br}\right], B \neq 0;$$

$$\left(\frac{1-r}{1+r}\right)^k e^{-Ar} \left|-\frac{2kr}{1-r^2} + 1 - Ar\right| \leq |f'(z)| \leq \left(\frac{1+r}{1-r}\right)^k e^{Ar} \left[\frac{2kr}{1-r^2} + 1 + Ar\right], B = 0.$$

This corollary is simple consequence of theorem 2.1 *and theorem* 2.5.

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