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Dominants and best dominants in fuzzy differential subordinations

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Abstract. The theory of differential subordination was introduced by S.S. Miller and P.T. Mocanu in [1] and [2] then developed in many other papers. Using the notion of differential subordination, in [5] the authors define the notion of fuzzy subordination and in [6] they define the notion of fuzzy differential subordination. In this paper, we determine conditions for a function to be a dominant of the fuzzy differential subordination and we also give the best dominant.

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1. Introduction and preliminaries

Let U denote the unit disc of the complex plane:

$$U = \{z \in \mathbb{C} : \ |z| < 1\},\ \overline{U} = \{z \in \mathbb{C} : \ |z| \leq 1\},\ \partial U = \{z \in \mathbb{C} : \ |z| = 1\}$$

and $\mathcal{H}(U)$ denote the class of analytic functions in U.

For $a \in \mathbb{C}$ and $n \in \mathbb{N}$, we denote by

$$\mathcal{H}[a,n] = \{f \in \mathcal{H}(U);\ f(z) = a + a_n z^n + a_{n+1}z^{n+1} + \dots,\ z \in U\}$$

and

$$A_n = \{f \in \mathcal{H}(U) :\ f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \dots,\ z \in U\},$$

with $A_1 = A$.

Let

$$S = \{f \in A;\ f \text{ univalent in } U\}$$

be the class of holomorphic and univalent functions in the open unit disc U, with conditions $f(0) = 0$, $f'(0) = 1$, that is the holomorphic and univalent functions with the following power series development

$$f(z) = z + a_2 z^2 + \dots, \quad z \in U.$$

Denote by

$$S^* = \left\{ f \in A;\ \operatorname{Re} \frac{zf'(z)}{f(z)} > 0,\ z \in U \right\},$$

the class of normalized starlike functions in U,

$$K = \left\{ f \in A;\ \operatorname{Re} \frac{zf''(z)}{f'(z)} + 1 > 0,\ z \in U \right\}$$

the class of normalized convex functions in U and by

$$C = \left\{ f \in A :\ \exists\ \varphi \in K;\ \operatorname{Re} \frac{f'(z)}{\varphi'(z)} > 0,\ z \in U \right\}$$

the class of normalized close-to-convex functions in U.

An equivalent formulation for close-to-convexity would involve the existence of a starlike function h (not necessarily normalized) such that

$$\operatorname{Re} \frac{zf'(z)}{h(z)} > 0, \quad z \in U.$$

Kaplan [1] and Sakaguchi [9] showed that $f \in S$ if

$$\operatorname{Re} \left[\frac{zf''(z)}{f'(z)} + 1 \right] > -\frac{1}{2}.$$

In order to prove our original results, we use the following definitions and lemmas:

Definition 1.1. [3, p. 21, Definition 2.26] *We denote by Q the set of functions q that are analytic and injective on $\overline{U} \setminus E(q)$, where*

$$E(q) = \left\{ \zeta \in \partial U :\ \lim_{z \to \zeta} q(z) = \infty \right\},$$

and are such that $q'(\zeta) \neq 0$, for $\zeta \in \partial U \setminus E(q)$. The set $E(q)$ is called exception set.

Definition 1.2. [5] *Let X be a non-empty set. An application $F : X \to [0,1]$ is called fuzzy subset.*

An alternate definition, more precise, would be the following:

A pair (A, F_A), where $F_A : X \to [0,1]$ and

$$A = \{x \in X :\ 0 < F_A(x) \le 1\} = \operatorname{supp}(A, F_A),$$

is called fuzzy subset.

The function F_A is called membership function of the fuzzy subset (A, F_A).

Definition 1.3. [6] *Let two fuzzy subsets of X, (M, F_M) and (N, F_N). We say that the fuzzy subsets M and N are equal if and only if $F_M(x) = F_N(x)$, $x \in X$ and we denote this by $(M, X_M) = (N, F_N)$. The fuzzy subset (M, F_M) is contained in the fuzzy subset (N, F_N) if and only if $F_M(x) \le F_N(x)$, $x \in X$ and we denote the inclusion relation by $(M, F_M) \subseteq (N, F_N)$.*

Definition 1.4. [5] *Let $D \subset \mathbb{C}$, $z_0 \in D$ be a fixed point, and let the functions $f, g \in \mathcal{H}(D)$. The function f is said to be fuzzy subordinate to g and write $f \prec_F g$ or $f(z) \prec_F g(z)$ if the following conditions are satisfied:*

(i) $f(z_0) = g(z_0)$

(ii) $F_{f(D)}f(z) \leq F_{g(D)}g(z)$, $z \in U$,

where

$$f(D) = \operatorname{supp}(D, F_{f(D)}) = \{z \in \mathbb{C} \mid 0 < F_{f(D)}(z) \leq 1\},$$

$$g(D) = \operatorname{supp}(D, F_{g(D)}) = \{z \in \mathbb{C} \mid 0 < F_{g(D)}(z) \leq 1\}.$$

Definition 1.5. [6] *Let Ω be a set in $\mathbb{C}$, $q \in Q$ and n be a positive integer. The class of admissible functions $\Psi_n[\Omega, q]$ consists of those functions $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ that satisfy the admissibility condition:*

$$F_\Omega \psi(r, s, t; z) = 0, \quad z \in U, \tag{1.1}$$

whenever $r = q(\zeta)$, $s = m\zeta q'(\zeta)$,

$$\operatorname{Re} \frac{t}{s} + 1 \geq m \operatorname{Re} \left[\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right], \quad z \in U,$$

$\zeta \in \partial U \setminus E(q)$ and $m \geq n$. We write $\Psi_1[\Omega, q]$ as $\Psi[\Omega, q]$. In the special case when Ω is a simply connected domain, $\Omega \neq \mathbb{C}$, and h is conformal mapping of U into Ω we denote this class by $\Psi_n[h(U), q]$ or $\Psi_n[h, q]$.

If $\mathbb{C}^2 \times U \to \mathbb{C}$, then the admissibility condition (1.1) reduces to

$$F_\Omega \psi(q(\zeta), m\zeta q'(\zeta); z) = 0 \tag{1.2}$$

when $z \in U$, $\zeta \in \partial U \setminus E(q)$ and $m \geq n$.

Definition 1.6. [6] *Let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ and let h be univalent in U with $h(0) = \psi(a, 0, 0; 0)$. If p is analytic in U with $p(0) = a$ and satisfies the (second-order) fuzzy differential subordination*

$$F_{\psi(\mathbb{C}^3 \times U)} \psi(p(z), zp'(z), z^2 p''(z)) \leq F_{h(U)} h(z), \quad z \in U, \tag{1.3}$$

then p is called a fuzzy solution of the fuzzy differential subordination. The univalent function q is called a fuzzy dominant of the fuzzy solutions of the fuzzy differential subordination, or more simple a fuzzy dominant, if $p(0) = q(0)$ and $F_{p(U)}f(z) \leq F_{q(U)}q(z)$, $z \in U$, for all p satisfying (1.3). A fuzzy dominant $\widetilde{q}$ that satisfies $\widetilde{q}(0) = q(0)$ and $F_{\widetilde{q}(U)}\widetilde{q}(z) \leq F_{q(U)}q(z)$, $z \in U$, for all fuzzy dominants q of (1.3) is said to be the fuzzy best dominant of (1.3). Note that the fuzzy best dominant is unique up to a rotation of U. If we require the more restrictive condition $p \in \mathcal{H}[a, n]$, then p will be called an (a, n)-fuzzy solution, q an (a, n)-fuzzy dominant, and $\widetilde{q}$ the best (a, n)-fuzzy dominant.

Definition 1.7. [8] *A function $L(z, t)$, $z \in U$, $t \geq 0$, is a fuzzy subordination chain if $L(\cdot, t)$ is analytic and univalent in U for all $t \geq 0$, $L(z, t)$ is continuously differentiable on $[0, \infty)$ for all $z \in U$, and $F_{L[U \times (0,\infty)]}L(z, t_1) \leq F_{L[U \times (0,\infty)]}L(z, t_2)$, when $t_1 \leq t_2$.*

Lemma 1.8. [6, Th. 2.4] *Let h and q be univalent in U, with $q(0)=a$, and let $h_\rho(z)=h(\rho z)$ and $q_\rho(z)=q(\rho z)$. Let $\psi:\mathbb{C}^3\times U\to\mathbb{C}$ satisfy one of the following conditions:*

(i) $\psi\in\Psi_n[h,q_\rho]$ for some $\rho\in(0,1)$, or

(ii) there exists $\rho_0\in(0,1)$ such that $\psi\in\Psi_n[h_\rho,q_\rho]$, for all $\rho\in(\rho_0,1)$.

If $p\in\mathcal{H}[a,1]$ and $\psi(p(z),zp'(z),z^2p''(z))$ is analytic in U, $\psi(a,0,0;0)=h(0)$ and

$$F_{\psi(\mathbb{C}^3\times U)}\psi(p(z),zp'(z),z^2p''(z))\le F_{h(U)}h(z),\quad z\in U,$$

then

$$F_{p(U)}p(z)\le F_{q(U)}q(z),\quad z\in U.$$

Lemma 1.9. [6, Th. 2.6] *Let h be univalent in U, and let $\psi:\mathbb{C}^3\times U\to\mathbb{C}$. Suppose that the differential equation*

$$\psi(q(z),zq'(z),n(n-1)zq'(z)+n^2z^{2n}q''(z);z)=h(z),$$

has a solution q, with $q(0)=a$, and one of the following conditions is satisfied:

(i) $q\in Q$ and $\psi\in\Psi_n[h,q]$;

(ii) q is univalent in U and $\psi\in\Psi_n[h,q_\rho]$ for some $\rho\in(0,1)$, or

(iii) q is univalent in U and there exists $\rho_0\in(0,1)$ such that $\psi\in\Psi_n[h_\rho,q_\rho]$ for all $\rho\in(\rho_0,1)$.

If $p\in\mathcal{H}[a,n]$, $\psi(p(z),zp'(z),z^2p''(z))$ is analytic in U, and p satisfies

$$F_{\psi(\mathbb{C}^3\times U)}\psi(p(z),zp'(z),z^2p''(z))\le F_{h(U)}h(z),$$

then

$$F_{p(U)}p(z)\le F_{q(U)}q(z),\quad z\in U.$$

Lemma 1.10. [7] *If $L_\gamma:A\to A$ is the integral operator defined by $L_\gamma[f]=F$, given by*

$$L_\gamma[f](z)=F(z)=\frac{\gamma+1}{z^\gamma}\int_0^z f(t)t^{\gamma-1}dt,$$

and $\operatorname{Re}\gamma>0$ *then*

(i) $L_\gamma[S^*]\subset S^*$;

(ii) $L_\gamma[K]\subset K$;

(iii) $L_\gamma[\mathbb{C}]\subset\mathbb{C}$.

Lemma 1.11. [3, Lemma 2.2.d, p. 24] *Let $q\in Q$ with $q(0)=a$, and let*

$$p(z)=a+a_nz^n+a_{n+1}z^{n+1}+\dots$$

be analytic in U with $p(z)\not\equiv a$ and $n\ge1$. If p is not subordinate to q, there exist points $z_0=r_0e^{i\theta_0}\in U$ and $\zeta_0\in\partial U\setminus E(q)$ and an $m\ge n\ge1$ for which $p(U_{r_0})\subset q(U)$,

(i) $p(z_0)=q(\zeta_0)$;

(ii) $z_0p'(z_0)=m\zeta_0q'(\zeta_0)$, and

(iii) $\operatorname{Re}\dfrac{z_0p''(z_0)}{p'(z_0)}+1\ge m\operatorname{Re}\left[\dfrac{\zeta_0q''(\zeta_0)}{q'(\zeta_0)}+1\right].$

Lemma 1.12. [8, p. 159] *The function*

$$L(z,t) = a_1(t)z + a_2(t)z^2 + \dots$$

with $a_1(t) \neq 0$ *for* $t \geq 0$ *and* $\lim_{t\to\infty} |a_1(t)| = \infty$ *is a subordination chain if and only if*

$$\operatorname{Re}\left[\frac{z\partial L(z,t)/\partial z}{\partial L(z,t)/\partial t}\right] > 0, \quad z \in U,\ t \geq 0.$$

2. Main results

Theorem 2.1. *Let* h *be analytic in* U*, let* ϕ *be analytic in a domain* D *containing* $h(U)$ *and suppose*

(a) $\operatorname{Re}\phi[h(z)] > 0,\ z \in U$ *and*

(b) $h(z)$ *is convex.*

If p *is analytic in* U*, with* $p(0) = h(0)$*,* $p(U) \subset D$ *and* $\psi : \mathbb{C}^2 \times U \to \mathbb{C}$*,*

$$\psi(p(z), zp'(z)) = p(z) + zp'(z) \cdot \phi[p(z)]$$

is analytic in U*, then*

$$F_{\psi(\mathbb{C}^2\times U)}\psi(p(z), zp'(z)) \leq F_{h(U)}h(z), \tag{2.1}$$

implies

$$F_{p(U)}p(z) \leq F_{h(U)}h(z), \quad z \in U,$$

where

$$\begin{aligned}\psi(\mathbb{C}^2 \times U) &= \operatorname{supp}\,(\mathbb{C}^2 \times U, F_{\psi(\mathbb{C}^2\times U)}\psi(p(z), zp'(z)))\\ &= \{z \in \mathbb{C};\ 0 < F_{\psi(\mathbb{C}^2\times U)}\psi(p(z), zp'(z)) \leq 1\},\\ h(U) &= \operatorname{supp}\,\{U, F_{h(U)}h(z)) = \{z \in \mathbb{C}:\ 0 < F_{h(U)}h(z) \leq 1\}.\end{aligned}$$

Proof. Without loss of generality we can assume that p and h satisfy the conditions of the theorem on the closed disc $\overline{U}$. If not, then we can replace $p(z)$ by $p_\rho(z) = p(\rho z)$, and $h(z)$ by $h_\rho(z) = h(\rho z)$, where $0 < \rho < 1$. These new functions satisfy the conditions of the theorem on $\overline{U}$. We would then prove that

$$F_{p_\rho(U)}p_\rho(z) \leq F_{p(U)}p(z),\ \text{for all } 0 < \rho < 1.$$

By letting $\rho \to 1$, we obtain

$$F_{p(U)}p(z) \leq F_{h(U)}h(z), \quad z \in U.$$

In order to prove the theorem, we apply Lemma 1.8, and we show that $\psi \in \Psi_1[h_\rho, h_\rho]$, for all $\rho \in (0, 1)$.

Suppose (a) and (b) are satisfied, but p is not fuzzy subordinate to h.

According to Lemma 1.11, there are points $z_0 \in U$ and $\zeta_0 \in \partial U$, and $m \geq 1$, with $p(z_0) = h(\zeta_0)$, $z_0p'(z_0) = m\zeta_0 q'(\zeta_0)$ such that

$$\begin{aligned}\psi_0 &= \psi(p(z_0), zp'(z_0))\\ &= p(z_0) + z_0p'(z_0) \cdot \phi[p(z_0)] = h(\zeta_0) + m\zeta_0h'(\zeta_0) \cdot \phi[h(\zeta_0)].\end{aligned} \tag{2.2}$$

From (2.2), we have

$$\psi_0 = h(\zeta_0) + m\zeta_0h'(\zeta_0)\phi[h(\zeta_0)], \quad \zeta_0 \in \partial U,\ |\zeta_0| = 1,\ m \geq 1 \tag{2.3}$$

which gives

$$\frac{\rho_0 - h(\zeta_0)}{\zeta_0 h'(\zeta_0)} = m\phi[h(\zeta_0)]. \tag{2.4}$$

Using the conditions from the hypothesis of the theorem, we have:

$$\operatorname{Re}\frac{\psi_0 - h(\zeta_0)}{\zeta_0 h'(\zeta_0)} = \operatorname{Re} m\phi[h(\zeta_0)] > 0, \tag{2.5}$$

which implies

$$\left|\arg\frac{\psi_0 - h(\zeta_0)}{\zeta_0 h(\zeta_0)}\right| < \frac{\pi}{2}$$

which is equivalent to

$$|\arg[\psi_0 - h(\zeta_0)] - \arg[\zeta_0 h'(\zeta_0)]| < \frac{\pi}{2}. \tag{2.6}$$

Since $\zeta_0 h'_\rho(\zeta_0)$ is the outer normal at the border of the convex domain $h_\rho(U)$ at $h_\rho(\zeta_0)$, from (2.6) we get that $\psi_0 \notin h_\rho(U)$ which means

$$F_{h(U)}\psi(p(z_0), z_0 p'(z_0), z_0) = F_{h(U)}\psi(h(\zeta_0), m\zeta_0 h'(\zeta_0), z_0) = 0. \tag{2.7}$$

Using Definition 1.5, from (2.7) we have $\psi \in \Psi[h(U), h]$.

Using condition (i) from Lemma 1.8, we have

$$F_{p(U)}p(z) \le F_{q(U)}q(z), \quad z \in U.$$

By carefully selecting the function ϕ we obtain the following corollaries.

If we let $\phi(w) = \beta w + r$ is Theorem 2.1 we obtain the following corollary:

Corollary 2.2. *Let β and γ be complex numbers with $\beta \neq 0$ and let β and h be analytic in U with $h(0) = p(0)$. If*

$$Q(z) = \beta h(z) + \gamma$$

satisfies

(a) $\operatorname{Re} Q(z) = \operatorname{Re}[\beta h(z) + \gamma] > 0$,

and

(b) p is convex

then

$$F_{\psi(\mathbb{C}^2\times U)}[p(z) + zp'(z)(\beta p(z) + \gamma)] \le F_{h(U)}h(z)$$

implies that

$$F_{p(U)}p(z) \le F_{h(U)}h(z), \quad z \in U.$$

If we let $\phi(w) = \dfrac{1}{\beta w + \gamma}$ in Theorem 2.1, we obtain the following corollary:

Corollary 2.3. *Let β and γ be complex numbers with $\beta \neq 0$, and let p and h be analytic in U with $h(0) = p(0)$.*

If $Q(z) = \beta h(z) + \gamma$ satisfies

a) $\operatorname{Re} Q(z) > 0$, $z \in U$

and

b) Q is convex
then

$$F_{\psi(\mathbb{C}^2\times U)}\left[p(z)+\frac{zp'(z)}{\beta p(z)+\gamma}\right]\le F_{h(U)}h(z),$$

implies

$$F_{p(U)}p(z)\le F_{h(U)}h(z),\quad z\in U.$$

If we let $\phi(w)=\dfrac{1}{(\beta w+\gamma)^2}$ in Theorem 2.1, we obtain the following corollary:

Corollary 2.4. *Let β and γ be complex numbers with $\beta\neq 0$ and let p and h be analytic in U with $h(0)=p(0)$.*

If $Q(z)=\beta h(z)+\gamma$ satisfies
(i) $\operatorname{Re}Q^2(z)>0,\ z\in U$ and
(ii) Q is convex, then

$$F_{\psi(\mathbb{C}^2\times U)}\left[p(z)+\frac{zp'(z)}{(\beta p(z)+\gamma)^2}\right]\le F_{h(U)}h(z),$$

implies

$$F_{p(U)}p(z)\le F_{q(U)}q(z),\quad z\in U.$$

Theorem 2.5. *Let h be convex in U and let $P:U\to\mathbb{C}$, with $\operatorname{Re}P(z)>0$. If p is analytic in U and $\psi:\mathbb{C}^2\times U\to\mathbb{C}$,*

$$\psi(p(z),zp'(z))=p(z)+P(z)zp'(z), \tag{2.8}$$

is analytic in U, then

$$F_{\psi(\mathbb{C}^2\times U)}[p(z)+P(z)zp'(z)]\le F_{h(U)}h(z), \tag{2.9}$$

implies

$$F_{p(U)}p(z)\le F_{h(U)}h(z),\quad z\in U.$$

Proof. We next show that ψ is a proper admissible function. It seems like we can use Lemma 1.8 with $q=h$, and show that $\psi\in\Psi[h,h]$.

Unfortunately, we do not know the specific boundary behavior of h and thus cannot use this result. Instead we require the use of the limiting form of the theorem as given in part (ii) of Lemma 1.8. We only need to show that $\psi\in\Psi[h_\rho,h_\rho]$ for $0<\rho<1$, where $h_\rho(z)=h(\rho z)$. In this case the admissibility condition (1.1) reduces to showing

$$\psi_0=\psi[h_\rho(\zeta),m\zeta_0h'_\rho(\zeta_0);z]=h_\rho(\zeta_0)+mP(z)\zeta_0h'_\rho(\zeta)\notin h_\rho(U), \tag{2.10}$$

where $|\zeta_0|=1$, $z\in U$, and $m\ge 1$.

From (2.10), we have

$$\lambda=\frac{\psi_0-h_\rho(\zeta_0)}{\zeta_0h_\rho(\zeta_0)}=mP(z),\quad z\in U. \tag{2.11}$$

From $\operatorname{Re}P(z)>0$, $m\ge n$, we obtain

$$\operatorname{Re}\lambda=\operatorname{Re}mP(z)>0,$$

which gives

$$\left|\arg \frac{\psi_0 - h_\rho(\zeta_0)}{\zeta h'(\zeta_0)}\right| = |\arg mP(z)| < \frac{\pi}{2}. \tag{2.12}$$

Since $h_\rho(U)$ is convex, $h_\rho(\zeta_0) \in h_\rho(\partial U)$, and $\zeta_0 h'_\rho(\zeta_0)$ is the outer normal to $h_\rho(\partial U)$ at $h_\rho(\zeta_0)$, from (2.12) we conclude that $\psi_0 \notin h_\rho(U)$, which gives

$$F_{h_\rho(U)}[h_\rho(\zeta_0) + P(z)m\zeta_0 h'_\rho(\zeta_0); z] = 0, \tag{2.13}$$

and using Definition 1.5, we have

$$\psi \in \Psi_n[h_\rho(U), h_\rho].$$

Using Lemma 1.8, we deduce

$$F_{p(U)}p(z) \leq F_{q(U)}q(z), \quad z \in U.$$

Theorem 2.6. (Hallenbeck and Ruscheweyh) *Let h be a convex function with $h(0) = a$, and let $\gamma \in \mathbb{C}^*$ be a complex number with* $\operatorname{Re}\gamma \geq 0$. *If $p \in \mathcal{H}[a, n]$ with $p(0) = a$ and*

$$\psi : \mathbb{C}^2 \times U \to \mathbb{C}, \quad \psi(p(z), zp'(z)) = p(z) + \frac{1}{\gamma} zp'(z)$$

is analytic in U, then

$$F_{\psi(\mathbb{C}^2 \times U)}\left[p(z) + \frac{1}{\gamma} zp'(z)\right] \leq F_{h(U)}h(z), \tag{2.14}$$

implies

$$F_{p(U)}p(z) \leq F_{q(U)}q(z) \leq F_{h(U)}h(z), \quad z \in U,$$

where

$$q(z) = \frac{\gamma}{nz^{\gamma/n}} \int_0^z h(t)t^{\frac{\gamma}{n}-1}dt. \tag{2.15}$$

The function q is convex and is the fuzzy best (a, n)-dominant.

Proof. We can apply Theorem 2.5. From (2.14) we obtain

$$F_{p(U)}p(z) \leq F_{h(U)}h(z), \quad z \in U. \tag{2.16}$$

The integral given by (2.15), with the exception of a different normations $q(0) = a$ has the form

$$q(z) = \frac{\gamma}{nz^{\gamma/n}} \int_0^z h(t)t^{\frac{\gamma}{n}-1}dt = \frac{\gamma}{nz^{\gamma/n}} \int_0^z (a + a_n t^n + \ldots)t^{\frac{\gamma}{n}-1}dt$$

$$= a + \frac{a_n}{\frac{\gamma}{n} + n} z^n + \ldots, \quad z \in U,$$

which gives $q \in \mathcal{H}[a, n]$.

Since h is convex and $\operatorname{Re}\dfrac{\gamma}{n} \geq 0$, we deduce from part (ii) of Lemma 1.10 that q is convex and univalent.

A simple calculation shows that q also satisfies the differential equation

$$q(z) + \frac{nz}{\gamma} zq'(z) = h(z) = \psi[q(z), zq'(z)], \quad z \in U. \tag{2.17}$$

Since q is the univalent solution of the differential equation (2.17) associated with (2.14), we can prove that it is the best dominant by applying Lemma 1.9. Without loss of generality, we can assume that h and q are analytic and univalent on $\overline{U}$, and $q'(\zeta) \neq 0$ for $|\zeta| = 1$. If not, then we could replace h with $h_\rho(z) = h(\rho z)$, and q with $q_\rho(z) = q(\rho z)$.

These new functions would then have the desired properties and we would prove the theorem using part (iii) of Lemma 1.9.

With our assumption, we will use part (i) of Lemma 1.9 and so we only need to show that $\psi \in \Psi_n[h, q]$. This is equivalent to showing that

$$\psi_0 = \psi(q(\zeta), m\zeta q'(\zeta)) = q(\zeta) + \frac{m\zeta q'(\zeta)}{\gamma} \notin h(U) \tag{2.18}$$

when $|\zeta| = 1$, $z \in U$ and $m \geq n$.

From (2.17) we obtain

$$\psi_0 = q(\zeta) + \frac{m}{n}[h(\zeta) - q(\zeta)].$$

Since $h(U)$ is a convex domain, and

$$H_{q(U)}q(z) \leq F_{h(U)}h(z), \quad z \in U,$$

and $\frac{m}{n} \geq 1$, we conclude that $\psi_0 \notin h(U)$, which implies

$$F_{h(U)}\psi(q(\zeta), m\zeta q'(\zeta); z) = 0.$$

Using Definition 1.5, from condition (1.1) we get

$$\psi \in \Psi_n[h(U), q].$$

Using Lemma 1.9, from condition (i) we obtain

$$F_{p(U)}p(z) \leq F_{q(U)}q(z), \quad z \in U.$$

Therefore, q is the fuzzy best (a, n)-dominant.

Theorem 2.7. *Let q be a convex function in U and let the function*

$$h(z) = q(z) + n\alpha z q'(z), \tag{2.19}$$

where $\alpha > 0$ and $n \in \mathbb{N}^*$.

If the function $p \in H[q(0), n]$, and $\psi : \mathbb{C}^2 \times U \to \mathbb{C}$,

$$\psi(p(z), zp'(z)) = p(z) + \alpha z p'(z)$$

is analytic in U, then

$$F_{\psi(\mathbb{C}^2 \times U)}[p(z) + \alpha n z p'(z)] \leq F_{h(U)}h(z), \tag{2.20}$$

implies

$$F_{p(U)}p(z) \leq F_{q(U)}q(z), \quad z \in U$$

and q is fuzzy best $(q(0), n)$-dominant.

Proof. Step I. We prove that function h is univalent.

Differentiating (2.19), we have

$$h'(z) = q'(z) + n\alpha[q'(z) + zq''(z)] \tag{2.21}$$

which gives

$$\frac{h'(z)}{q'(z)} = 1 + n\alpha\left[1 + \frac{zq''(z)}{q'(z)}\right], \quad z \in U. \tag{2.22}$$

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