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Differential sandwich theorems involving certain convolution operator

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Abstract. In the present paper a certain convolution operator of analytic functions is defined. Moreover, subordination- and superordination- preserving properties for a class of analytic operators defined on the space of normalized analytic functions in the open unit disk is obtained. We also apply this to obtain sandwich results and generalizations of some known results.

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1. Introduction

Let $H = H(\Delta)$ denote the class of analytic functions in the open unit disk

$$\Delta = \{z : |z| < 1\}$$

and

$$A := \{f \in H : f(0) = f'(0) - 1 = 0\}.$$

For a positive integer number n and $a \in \mathbb{C}$, let

$$\mathcal{H}[a, n] = \{f \in H(\Delta) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots\}.$$

Let f and F be members of the analytic function class H. The function f is said to be subordinate to F or F is said to be superordinate of f, if there exists a function w analytic in Δ, with $w(0) = 0$, and $|w(z)| < 1$ $(z \in \Delta)$ such that $f(z) = F(w(z))$ and we write $f \prec F$ or $f(z) \prec F(z)$ $(z \in \Delta)$. If the function F is univalent in $(z \in \Delta)$, then we have

$$f \prec F \iff f(0) = F(0) \ and \ f(\Delta) \subset F(\Delta).$$

Let $\varphi : \mathbb{C}^2 \times \Delta \longrightarrow \mathbb{C}$ and h be analytic in Δ. If p is analytic in Δ and satisfies the(first-order) differential subordination

$$\varphi(p(z), zp'(z); z) \prec h(z) \tag{1.1}$$

then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordination, or dominant if $p \prec q$ for all p satisfying (1.1). A dominant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all dominants of (1.1) is said to be the best dominant.

Let $\varphi : \mathcal{C}^2 \times \Delta \longrightarrow \mathcal{C}$ and h be analytic in Δ. If p and $\varphi(p(z), zp'(z); z)$ are univalent in Δ and satisfies the(first-order) differential superordination

$$h(z) \prec \varphi(p(z), zp'(z); z) \tag{1.2}$$

then p is called a solution of the differential superordination. An analytic function q is called a subordinant of the solution of the differential superordination, or more simply a subordinant if $q \prec p$ for all q satisfying (1.2). A univalent subordinant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all subordinant of (1.2) is said to be the best subordinant.

Ali et al [1] have obtained sufficient conditions for certain normalized analytic functions $f(z)$ to satisfy $q_1 \prec \frac{zf'(z)}{f(z)} \prec q_2(z)$, where q_1 and q_2 are given univalent functions in Δ with $q_1(0) = 1$ and $q_2(0) = 1$.
For two functions $f_j(z)(j = 1, 2)$, given by

$$f_j(z) = z + \sum_{k=2}^{\infty} a_{k,j} z^k \qquad (j = 1, 2)$$

we define the Hadamard product (or convolution) of $f_1(z)$ and $f_2(z)$ by

$$(f_1 * f_2)(z) := z + \sum_{k=2}^{\infty} a_{k,1} a_{k,2} z^k = (f_2 * f_1)(z) \qquad (z \in \Delta)$$

In terms of the Pochhammer symbol(or the shifted factorial), define $(\kappa)_n$ by

$$(\kappa)_0 = 1, \ \textit{and} \ (\kappa)_n = \kappa(\kappa+1)(\kappa+2)\dots(\kappa+n-1) \qquad (n \in \mathbb{N} := \{1, 2, \dots\})$$

also, define a function $\phi_a^\lambda(b, c; z)$ by

$$\phi_a^\lambda(b, c; z) := 1 + \sum_{n=1}^{\infty} (\frac{a}{a+n})^\lambda \frac{(b)_n}{(a)_n (c)_n} z^n, \qquad (z \in \Delta) \tag{1.3}$$

where

$$b \in \mathbb{C}, c \in \mathbb{R} \backslash Z_0^-, a \in \mathbb{C} \backslash Z_0^- (Z_0^- = \{0, -1, -2, \dots\}); \lambda \geq 0$$

Corresponding to the function $\phi_a^\lambda(b, c; z)$, given by (1.3), we introduce the following convolution operator

$$L_a^\lambda(b, c; \beta) f(z) := \phi_a^\lambda(b, c; z) * (\frac{f(z)}{z})^\beta \qquad (f \in A, \beta \in \mathbb{C} \backslash 0, z \in \Delta) \tag{1.4}$$

It is easy to see that

$$z(\phi_a^\lambda(b, c; z))' = a\phi_a^\lambda(b, c; z) - a\phi_a^{\lambda+1}(b, c; z) \tag{1.5}$$

and

$$z(L_a^{\lambda+1}(b, c; \beta) f(z))' = aL_a^\lambda(b, c; \beta) f(z) - aL_a^{\lambda+1}(b, c; \beta) f(z) \tag{1.6}$$

The operator $L_a^\lambda(b, c; \beta) f(z)$ includes, as its special cases, Komatu integral operator(see [4], [6], [11]), some fractional calculus operators(see [4], [13], [14]) and Carlson-Shaffer operator(see [2]).

Making use of the principle of subordinantion between analytic functions Miller et all [9] obtained some interesting subordination theorems involving certain operators. Also Miller and Mocanu [8] considered subordination-preserving properties of certain integral operator investigations as the dual concept of differential subordination. In the present investigation, we obtain the subordination and superordination-preserving properties of the convolution operator L_a^λ defined by (1.4) with the Sandwich-type theorems.

2. Definitions and preliminaries

The following definitions and Lemmas will be required in our present investigation.

Definition 2.1. *If $0 \leq \alpha < 1$ and $\lambda \geq 0, a \in \mathbb{C}\backslash Z_0^- (Z_0^- = \{0, -1, -2, \ldots\})$, let $\mathcal{L}_a^\lambda(\alpha)$ denote the class of functions $f \in A$ wich satisfies the inequality*

$$Re[L_a^\lambda(b, c; \beta)f(z)] > \alpha$$

For $a = 1$, we set $\mathcal{L}_1^\lambda(\alpha) = \mathcal{L}^\lambda(\alpha)$.

Definition 2.2. [7] *We denote by Q the set of function q that are analytic and injective on $\overline{\Delta}\backslash E(q)$ where*

$$E(q) = \{\xi \in \Delta : \lim_{z \to \xi} q(z) = \infty\}$$

and $h'(\xi) \neq 0$ for $\xi \in \partial\Delta\backslash E(q)$.

Lemma 2.3. [7] *Let $h(z)$ be analytic and convex univalent in Δ and $h(0) = a$. Also let $p(z)$ be analytic in Δ with $p(0) = a$. If $p(z) + \frac{zp'(z)}{\gamma} \prec h(z)$, where $\gamma \neq 0$ and $Re\gamma \geq 0$, then $p(z) \prec q(z) \prec h(z)$, where*

$$q(z) = \frac{\gamma}{z^\gamma} \int_0^z h(t)t^{\gamma-1}dt$$

Furthermore $q(z)$ is a convex function and is the best dominant.

Lemma 2.4. [8] *Let $h(z)$ be convex in Δ, $h(0) = a, \gamma \neq 0$ and $Re\gamma \geq 0$. Also $p \in \mathcal{H}[a, n] \cap Q$. If $p(z) + \frac{zp'(z)}{\gamma}$ is univalent in Δ, $h(z) \prec p(z) + \frac{zp'(z)}{\gamma}$ and*

$$q(z) = \frac{\gamma}{z^\gamma} \int_0^z h(t)t^{\gamma-1}dt$$

then $q(z) \prec p(z)$, and $q(z)$ is a convex function and is the best subordinant.

Lemma 2.5. [12] *Let $q(z)$ be a convex univalent function in Δ and $\psi, \gamma \in \mathbb{C}$ with $Re(1 + \frac{zq''(z)}{q'(z)}) > max\{0, -Re\frac{\psi}{\gamma}\}$, $h(0) = a, \gamma \neq 0$ and $Re\gamma \geq 0$. If $p(z)$ is analytic in Δ and $\psi p(z) + \gamma zp'(z) \prec \psi q(z) + \gamma zq'(z)$ then $p(z) \prec q(z)$, and $q(z)$ is the best dominant.*

Lemma 2.6. [10] *Let $q(z)$ be a convex univalent function in Δ and $\eta \in \mathbb{C}$, assume that $Re\eta > 0$. If $p(z) \in \mathcal{H}[a, n] \cap Q$and $p(z) + \eta zp'(z) \prec q(z) + \eta zq'(z)$ which implies that $q(z) \prec p(z)$, and $q(z)$ is the best subordinant.*

3. Differential subordination defined by convolution operator

In this section some differential subordinations are set using the convolution operator and concrete example of convex functions.

Theorem 3.1. *If* $0 \leq \alpha < 1$ *and* $\lambda \geq 0, a \in \mathbb{C} \backslash Z_0^- (Z_0^- = \{0, -1, -2, \dots\})$, *then we have*

$$\mathcal{L}_a^\lambda(\alpha) \subset \mathcal{L}_a^{\lambda+1}(\delta)$$

where

$$\delta(\alpha, a) = a\beta(a) + a(2\alpha - 1)\beta(a + 1)$$

and

$$\beta(x) = \int_0^1 \frac{t^{x-1}}{1+t} dt$$

the result is sharp.

Proof. First not that $f \in \mathcal{L}_a^\lambda(\alpha)$ and

$$z(L_a^{\lambda+1}(b, c; \beta)f(z))' = aL_a^\lambda(b, c; \beta)f(z) - aL_a^{\lambda+1}(b, c; \beta)f(z) \tag{3.1}$$

we define $p(z) = L_a^{\lambda+1}(b, c; \beta)f(z)$ from the relation(1.1) we have

$$L_a^\lambda(b, c; \beta)f(z) = p(z) + \frac{zp'(z)}{a}$$

now from Lemma 2.3, for $\gamma = a$ it follows that

$$p(z) = L_a^{\lambda+1}(b, c; \beta)f(z) \prec q(z) = \frac{a}{z^a} \int_0^z \frac{1 + (2\alpha - 1)t}{1 + t} t^{a-1} dt$$

therefore we have

$$\mathcal{L}_a^\lambda(\alpha) \subset \mathcal{L}_a^{\lambda+1}(\delta)$$

where

$$\delta = MinReq(z)_{|z| \leq 1} = q(1) = a\beta(a) + a(2\alpha - 1)\beta(a + 1)$$

Furthermore $q(z)$ is a convex function and is the best dominant. □

For the class $\mathcal{L}^\lambda$ we obtain the next corollary.

Corollary 3.2. *If* $0 \leq \alpha < 1$ *and* $\lambda \geq 0$, *then we have*

$$\mathcal{L}^\lambda(\alpha) \subset \mathcal{L}^{\lambda+1}(\delta)$$

where

$$\delta = \delta(\alpha) = 2\alpha - 1 + 2(1 - \alpha) \ln 2$$

and the result is sharp.

Theorem 3.3. *Let* $h \in H(\Delta)$, *with* $h(0) = 1, h'(0) \neq 0$, *which verifies the inequality*

$$Re[1 + \frac{zh''(z)}{h'(z)}] > -\frac{1}{2}(z \in \Delta).$$

If $f \in A$ *and satisfies the differential subordination*

$$L_a^\lambda(b, c; \beta)f(z) \prec h(z) \tag{3.2}$$

then

$$L_a^{\lambda+1}(b,c;\beta)f(z) \prec q(z)(z \in \Delta) \tag{3.3}$$

where

$$q(z) = \frac{a}{z^a}\int_0^z h(t)t^{a-1}dt$$

The function $q(z)$ is convex and is the best dominant.

Proof. Let

$$p(z) = L_a^{\lambda+1}(b,c;\beta)f(z) \tag{3.4}$$

Differentiating (3.4) with respect to z, we have $p'(z) = (L_a^{\lambda+1}(b,c;\beta)f(z))'$. From the relation (3.1) we have

$$\frac{zp'(z)}{a} + p(z) = L_a^{\lambda}(b,c;\beta)f(z)$$

now, in view of (3.4), we obtain the following subordination

$$\frac{zp'(z)}{a} + p(z) \prec h(z)$$

then from Lemma 2.3 for $\gamma = a$ we conclude that

$$p(z) = L_a^{\lambda+1}(b,c;\beta)f(z) \prec q(z)$$

where

$$q(z) = \frac{a}{z^a}\int_0^z h(t)t^{a-1}dt$$

and $q(z)$ is the best dominant. □

Taking $\lambda = 0$ in the Theorem 3.3 we arrive the following corollary.

Corollary 3.4. *Let $h \in H(\Delta)$, with $h(0) = 1, h'(0) \neq 0$, and $Re(1 + \frac{zh''(z)}{h'(z)}) > -\frac{1}{2}$ $(z \in \Delta)$. If $f \in A$ and satisfies $(\frac{f(z)}{z})^\beta \prec h(z)$, then $K_a(b,c;\beta) \prec q(z)$ where*

$$q(z) = \frac{a}{z^a}\int_0^z h(t)t^{a-1}dt.$$

The function $q(z)$ is the best dominant.

Putting $\gamma \in \mathbb{C}$. By setting $a = \gamma + \beta, \lambda = 0$, and $b = c = 1$ in the Theorem 3.3, we get the following corollary.

Corollary 3.5. *Let $h \in H(\Delta)$, with $h(0) = 1, h'(0) \neq 0$, which satisfies the inequality*

$$Re(1 + \frac{zh''(z)}{h'(z)}) > -\frac{1}{2}\ (z \in \Delta).$$

If $f \in A$ and satisfies the differential subordination $(\frac{f(z)}{z})^\beta \prec h(z)$, then

$$\frac{\gamma+\beta}{z^{\gamma+\beta}}\int_0^z u^{\gamma-1}(f(u))^\beta du \prec \frac{1}{z}\int_0^z h(u)du.$$

The function $\frac{1}{z}\int_0^z h(u)du$ is the best dominant.

Corollary 3.6. *Let $0 < R \leq 1$ and let $h(z)$ be convex in Δ, defined by*

$$h(z) = 1 + Rz + \frac{Rz}{2 + Rz},$$

with $h(0) = 1$. If $f \in A$ satisfies the following differential subordination

$$L_a^{\lambda}(b, c; \beta)f(z) \prec h(z)$$

then

$$L_a^{\lambda+1}(b, c; \beta)f(z) \prec q(z)(z \in \Delta)$$

where

$$q(z) = \frac{a}{z^a}\int_0^z 1 + Rt + \frac{Rt}{2 + Rt}t^{a-1}dt,$$

$$q(z) = z^{a-1} + Ra(\frac{z^a}{a+1} + \frac{M(z)}{z})$$

where

$$M(z) = \int_0^z \frac{t^a}{2 + Rt}dt$$

The function $q(z)$ is convex and is the best dominant.

If $a = 1$, the Corollary 3.6 becomes:

Corollary 3.7. *Let $0 < r \leq 1$ and let $h(z)$ be convex in Δ, defined by*

$$h(z) = 1 + Rz + \frac{Rz}{2 + Rz},$$

with $h(0) = 1$. If $f \in A$ and suppose that

$$L^{\lambda}(b, c; \beta)f(z) \prec h(z)$$

then

$$L^{\lambda+1}(b, c; \beta)f(z) \prec q(z)(z \in \Delta)$$

where

$$q(z) = \frac{1}{z}\int_0^z 1 + Rt + \frac{Rt}{2 + Rt}dt,$$

$$q(z) = 2 + \frac{Rz}{2} - \frac{2}{Rz}\log(2 + Rz)$$

The function $q(z)$ is convex and is the best dominant.

By taking $R = 1$ in the Corollary 3.7 we have the following corollary.

Corollary 3.8. *Let $h(z)$ be convex in Δ, defined by $h(z) = 1 + z + \frac{z}{2+z}$, with $h(0) = 1$. If $f \in A$, satisfies the differential subordination*

$$L^{\lambda}(b, c; \beta)f(z) \prec h(z)$$

then

$$L^{\lambda+1}(b, c; \beta)f(z) \prec q(z)(z \in \Delta)$$

where

$$q(z) = 2 + \frac{z}{2} - \frac{2}{z}\log(2 + z)$$

The function $q(z)$ is convex and is the best dominant.

Corollary 3.9. *Let $h(z)$ be convex in Δ, defined by $h(z) = 1 + z + \frac{z}{2+z}$, with $h(0) = 1$, and suppose that $\gamma \in \mathbb{C}, a = \gamma + \beta, \lambda = 0, b = c = 1$. If $f \in A$ and satisfies the differential subordination $(\frac{f(z)}{z})^{\beta} \prec h(z)$, then*

$$\frac{\gamma+\beta}{z^{\gamma+\beta}} \int_0^z u^{\gamma-1}(f(u))^{\beta} du \prec q(z) = 2 + \frac{z}{2} - \frac{2}{z} \log(2+z)$$

The function $q(z)$ is convex and is the best dominant.

Corollary 3.10. *Let $h(z) = \frac{1+(2\alpha-1)z}{1+z}$ be convex function in Δ, with $h(0) = 1$. If $f \in \mathcal{L}^{\lambda}(\alpha)$ and $L^{\lambda}(b,c;\beta)f(z) \prec h(z)$ then*

$$L^{\lambda+1}(b,c;\beta)f(z) \prec q(z)(z \in \Delta)$$

where

$$q(z) = 2\alpha - 1 + 2(1-\alpha)\frac{\log(1+z)}{z}$$

The function $q(z)$ is convex and is the best dominant.

Theorem 3.11. *Let $q(z)$ be a convex function $q(0) = 1$, and let h be a function such that*

$$h(z) = q(z) + \frac{zq'(z)}{q(z)} \ (z \in \Delta).$$

If $f \in H(\Delta)$ and verifies the differential subordination

$$L_a^{\lambda}(b,c;\beta)f(z) \prec h(z) \tag{3.5}$$

then

$$L_a^{\lambda+1}(b,c;\beta)f(z) \prec q(z)(z \in \Delta)$$

and this result is sharp.

Proof. We have

$$z(L_a^{\lambda+1}(b,c;\beta)f(z))' = aL_a^{\lambda}(b,c;\beta)f(z) - aL_a^{\lambda+1}(b,c;\beta)f(z) \tag{3.6}$$

Let $p(z) = L_a^{\lambda+1}(b,c;\beta)f(z)$, then from (3.5) and (3.6), we have

$$p(z) + \frac{zp'(z)}{a} \prec q(z) + \frac{zq'(z)}{a}$$

An application of Lemma 2.6, we conclude that $p(z) \prec q(z)$ or $L_a^{\lambda+1}(b,c;\beta)f(z) \prec q(z)(z \in \Delta)$ and this result is sharp. □

Theorem 3.12. *Let $h \in H(\Delta)$, with $h(0) = 1, h'(0) \neq 0$, which satisfies the inequality*

$$Re(1 + \frac{zh''(z)}{h'(z)}) > -\frac{1}{2} \ (z \in \Delta).$$

If $f \in A$ and verifies the differential subordination

$$(L_a^{\lambda+1}(b,c;\beta)f(z))' \prec h(z)$$

then

$$\frac{L_a^{\lambda+1}(b,c;\beta)f(z)}{z} \prec q(z)$$

where

$$q(z) = \frac{1}{z}\int_0^z h(t)t^{a-1}dt$$

the function $q(z)$ is the best dominant.

Proof. Let us define the function f by

$$f(z) = \frac{L_a^{\lambda+1}(b, c; \beta)f(z)}{z} \tag{3.7}$$

Differentiating logarithmically with respect to z, we have

$$\frac{zp'(z)}{p(z)} = \frac{z(L_a^{\lambda+1}(b, c; \beta)f(z))'}{L_a^{\lambda+1}(b, c; \beta)f(z)} - 1$$

and

$$p(z) + zp'(z) = (L_a^{\lambda+1}(b, c; \beta)f(z))'$$

Now, from (3.7) we obtain

$$p(z) + zp'(z) \prec h(z)$$

Then, by Lemma 2.3 , for $\gamma = 1$ we have $p(z) \prec q(z)$ or

$$\frac{L_a^{\lambda+1}(b, c; \beta)f(z)}{z} \prec \frac{1}{z}\int_0^z h(t)dt$$

and the function $q(z)$ is the best dominant. Therefore, we complete the proof of theorem 3.12. □

Suppose that$\lambda = 0$ and in the Theorem 3.12 we have the following result.

Corollary 3.13. *Let $h \in H(\Delta)$, with $h(0) = 1, h'(0) \neq 0$, which satisfies the inequality*

$$Re(1 + \frac{zh''(z)}{h'(z)}) > -\frac{1}{2} \ (z \in \Delta).$$

If $f \in A$ and $(K_a(b, c; \beta)f(z))' \prec h(z)$ then

$$\frac{K_a(b, c; \beta)f(z)}{z} \prec \frac{1}{z}\int_0^z h(t)dt,$$

and the function $\frac{1}{z}\int_0^z h(t)dt$ is the best dominant.

By taking $\gamma \in \mathbb{C}, a = \gamma + \beta, \lambda = 0$, and $b = c = 1$ in Theorem 3.12 we get the following result.

Corollary 3.14. *Let $h \in H(\Delta), h(0) = 1, h'(0) \neq 0$. If*

$$Re(1 + \frac{zh''(z)}{h'(z)}) > -\frac{1}{2} \ (z \in \Delta)$$

and if $f \in A$

$$\frac{-(\gamma + \beta)}{z^{\gamma+\beta+1}}\int_0^z u^{\gamma-1}(f(u))^\beta du + \frac{\gamma + \beta}{z^{\beta+1}} \prec h(z)$$

then

$$\frac{\gamma + \beta}{z^{\gamma+\beta-1}}\int_0^z u^{\gamma-1}(f(u))^\beta du \prec \frac{1}{z}\int_0^z h(u)du$$

The function $\frac{1}{z}\int_0^z h(u)du$ is the best dominant.

Corollary 3.15. *Let $0 < R \leq 1$ and let $h(z)$ be convex in Δ, defined by*

$$h(z) = 1 + Rz + \frac{Rz}{2 + Rz},$$

with $h(0) = 1$. If $f \in A$ satisfies the following differential subordination

$$(L^{\lambda+1}(b, c; \beta)f(z))' \prec h(z)$$

then

$$\frac{L^{\lambda+1}(b, c; \beta)f(z)}{z} \prec q(z)$$

where

$$q(z) = \frac{1}{z}\int_0^z 1 + Rt + \frac{Rt}{2 + Rt}dt,$$

$$q(z) = 1 + \frac{Rz}{2} + \frac{RM(z)}{z}$$

where

$$M(z) = \frac{z}{R} - \frac{2}{R^2}(\ln(2 + Rz)) - \frac{2}{R}\ln 2, (z \in \Delta)$$

The function $q(z)$ is convex and is the best dominant.

Suppose that $\gamma \in \mathbb{C}, a = \gamma + \beta, \lambda = 0$, and $b = c = 1$ in the Corollary 3.15 we have the following corollary.

Corollary 3.16. *Let $h(z)$ be convex in Δ, defined by $h(z) = 1 + z + \frac{z}{2+z}$, with $h(0) = 1$. If $f \in A$, satisfies the differential subordination*

$$\frac{-(\gamma + \beta)}{z^{\gamma+\beta+1}}\int_0^z u^{\gamma-1}(f(u))^{\beta}du + \frac{\gamma + \beta}{z^{\beta+1}} \prec h(z)$$

then

$$\frac{\gamma + \beta}{z^{\gamma+\beta-1}}\int_0^z u^{\gamma-1}(f(u))^{\beta}du \prec \frac{1}{z}\int_0^z h(u)du$$

where

$$q(z) = 2 + \frac{z}{2} - \frac{2}{z}\log(2 + z)$$

The function $q(z)$ is convex and is the best dominant.

Corollary 3.17. *Let $h(z) = \frac{1+(2\alpha-1)z}{1+z}$ be convex function in Δ, with $h(0) = 1$. If $f \in \mathcal{L}^{\lambda}(\alpha)$ and*

$$(L^{\lambda+1}(b, c; \beta)f(z))' \prec h(z)$$

then

$$\frac{L^{\lambda+1}(b, c; \beta)f(z)}{z} \prec q(z)$$

where

$$q(z) = 2\alpha - 1 + 2(1 - \alpha)\frac{\log(1 + z)}{z}$$

The function $q(z)$ is convex and is the best dominant.

Theorem 3.18. *Let $q(z)$ be a convex function, $q(0) = 1$, and*

$$h(z) = q(z) + \frac{zq'(z)}{q(z)} \ (z \in \Delta).$$

If $f \in H(\Delta)$ and satisfies the differential subordination

$$(L_a^{\lambda+1}(b, c; \beta)f(z))' \prec h(z) \tag{3.8}$$

then

$$\frac{L_a^{\lambda+1}(b, c; \beta)f(z)}{z} \prec q(z)(z \in \Delta)$$

and this result is sharp.

Proof. Let

$$p(z) = \frac{L_a^{\lambda+1}(b, c; \beta)f(z)}{z} \tag{3.9}$$

Logarithmic differentiation of (3.9) and through a little simplification we obtain

$$p(z) + zp'(z) = (L_a^{\lambda+1}(b, c; \beta)f(z))'$$

now by using Lemma 2.6, we conclude that

$$\frac{L_a^{\lambda+1}(b, c; \beta)f(z)}{z} \prec q(z)$$

and this result is sharp. □

4. Differential superordination defined by convolution operator

The results of this section are obtained with differential superordination method.

Theorem 4.1. *Let $h \in H(\Delta)$ be convex function in Δ, with $h(0) = 1$, and $f \in A$. Assume that $L_a^{\lambda}(b, c; \beta)f(z)$ is univalent with $L_a^{\lambda+1}(b, c; \beta)f(z) \in \mathcal{H}[1, n] \cap Q$. If $h(z) \prec L_a^{\lambda}(b, c; \beta)f(z)$ then*

$$q(z) \prec L_a^{\lambda+1}(b, c; \beta)f(z) \tag{4.1}$$

where

$$q(z) = \frac{a}{z^a} \int_0^z h(t)t^{a-1}dt$$

The function $q(z)$ is the best subordinant.

Proof. If we let $p(z) = L_a^{\lambda+1}(b, c; \beta)f(z)$ then from the relation (1.6) we have $p(z) + \frac{zp'(z)}{a} = L_a^{\lambda}(b, c; \beta)f(z)$. Now according to Lemma 2.4 we get the desired result (4.1). □

Corollary 4.2. *Suppose that $\gamma \in \mathbb{C}, a = \gamma + \beta, \lambda = 0$, and $b = c = 1$. Let $h \in H(\Delta)$ be convex function in Δ, with $h(0) = 1$, and $f \in A$. Assume that $(\frac{f(z)}{z})^{\beta}$ is univalent with $\frac{\gamma+\beta}{z^{\gamma+\beta}} \int_0^z u^{\gamma-1}(f(u))^{\beta}du \in \mathcal{H}[1, n] \cap Q$. If $h(z) \prec (\frac{f(z)}{z})^{\beta}$ then*

$$\frac{1}{z}\int_0^z h(u)du \prec \frac{\gamma+\beta}{z^{\gamma+\beta}} \int_0^z u^{\gamma-1}(f(u))^{\beta}du$$

and $\frac{1}{z}\int_0^z h(u)du$ is the best subordinant.

Corollary 4.3. *Let $h(z)$ be convex in Δ, defined by $h(z) = 1 + z + \frac{z}{2+z}$, with $h(0) = 1$. Suppose that $\gamma \in \mathbb{C}, a = \gamma + \beta, \lambda = 0, b = c = 1$, and $f \in A$ and $(\frac{f(z)}{z})^\beta$ is univalent with $\frac{\gamma+\beta}{z^{\gamma+\beta}} \int_0^z u^{\gamma-1}(f(u))^\beta du \in \mathcal{H}[1,n] \cap Q$.If $h(z) \prec (\frac{f(z)}{z})^\beta$ then $q(z) \prec \frac{\gamma+\beta}{z^{\gamma+\beta}} \int_0^z u^{\gamma-1}(f(u))^\beta du$ where $q(z) = 2 + \frac{z}{2} - \frac{2}{z}\log(2+z)$. The function $q(z)$ is the best subordinant.*

Corollary 4.4. *Let $h(z) = \frac{1+(2\alpha-1)z}{1+z}$ be convex function in Δ, with $h(0) = 1$. Assume that $f \in \mathcal{L}^{\lambda+1}(\alpha)$ and $L^\lambda(b,c;\beta)f(z)$ is univalent with $L^{\lambda+1}(b,c;\beta)f(z) \in \mathcal{H}[1,n] \cap Q$. If $h(z) \prec L^\lambda(b,c;\beta)f(z)$ then*

$$q(z) \prec L_a^{\lambda+1}(b,c;\beta)f(z)$$

where

$$q(z) = 2\alpha - 1 + 2(1-\alpha)\frac{\log(1+z)}{z}$$

The function $q(z)$ is the best subordinant.

Theorem 4.5. *Let $h \in H(\Delta)$ be convex function in Δ, with $h(0) = 1$, and $f \in A$. Assume that $(L_a^{\lambda+1}(b,c;\beta)f(z))'$ is univalent with $\frac{L_a^{\lambda+1}(b,c;\beta)f(z)}{z} \in \mathcal{H}[1,n] \cap Q$. If $h(z) \prec (L_a^{\lambda+1}(b,c;\beta)f(z))'$ then*

$$q(z) \prec \frac{L_a^{\lambda+1}(b,c;\beta)f(z)}{z}$$

where

$$q(z) = \frac{1}{z}\int_0^z h(t)dt$$

The function $q(z)$ is the best subordinant.

5. Sandwich results

Combining results of differential subordinations and superordinations, we arrive at the following "Sandwich results".

Theorem 5.1. *Let $q_1(z)$ be convex univalent in the open unit disk, and $q_2(z)$ univalent in the open unite disk Δ and $f \in A$. Also let $L_a^\lambda(b,c;\beta)f(z)$ be univalent with $L_a^{\lambda+1}(b,c;\beta)f(z) \in \mathcal{H}[1,n] \cap Q$. The following subordinate relationship $q_1(z) \prec L_a^\lambda(b,c;\beta)f(z) \prec q_1(z)$ implies $q_1(z) \prec L_a^{\lambda+1}(b,c;\beta)f(z) \prec q_2(z)$. Moreover the functions $q_1(z), q_2(z)$, are, respectively the best subordinant and the best dominant.*

Theorem 5.2. *Suppose that $q_1(z)$ is convex univalent, and let $q_2(z)$ be univalent Δ and $f \in A$. If $(L_a^{\lambda+1}(b,c;\beta)f(z))'$ is univalent with $\frac{L_a^{\lambda+1}(b,c;\beta)f(z)}{z} \in \mathcal{H}[1,n] \cap Q$. If $q_1(z) \prec (L_a^{\lambda+1}(b,c;\beta)f(z))' \prec q_2(z)$ then $q_1(z) \prec \frac{L_a^{\lambda+1}(b,c;\beta)f(z)}{z} \prec q_2(z)$ and $q_1(z), q_2(z)$, are, respectively the best subordinant and the best dominant.*

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