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Univalence criterion for a certain general integral operator

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Abstract. In this paper we consider a general integral operator, the class of analytic functions defined in the open unit disk and two classes of univalent functions. By imposing supplementary conditions for these functions we determine sufficient univalence conditions for the considered general operator. Some particular results are also presented.

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1. Introduction

Let $\mathcal{A}$ be the class of analytic functions f defined in the open unit disk of the complex plane $U = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions

$$f(0) = f'(0) - 1 = 0.$$

We consider S the class of all functions in $\mathcal{A}$ which are univalent in U and denote by P the class of the functions h which are analytic in U, $h(0) = 1$ and Re $h(z) > 0$ for all $z \in U$.

We define the class $S(\alpha)$ with $0 < \alpha \leq 2$ consisting of all functions $f \in \mathcal{A}$ that satisfy the conditions $f(z) \neq 0$ and $\left| \left(\frac{z}{f(z)} \right)'' \right| \leq \alpha,\ z \in U$. Singh [4] proved that if $f \in S(\alpha)$ then the following relation is true:

$$\left| \frac{z^2 f'(z)}{f^2(z)} - 1 \right| \leq \alpha |z|^2,\ z \in U.$$

In this paper we introduce a general integral operator

$$H_{n,p}(z) = \left\{ \beta \int_0^z t^{\beta-1} \prod_{i=1}^{n} \left(\frac{f_i(t)}{t} \right)^{\frac{1}{\gamma}} \prod_{j=1}^{p} (h_j(t))^{\delta} \mathrm{d}t \right\}^{\frac{1}{\beta}}, \tag{1.1}$$

with $f_i \in S(\alpha_i)$ for all $i = 1, 2, ..., n$ and $h_j \in P$ for $j = 1, 2, ..., p$ and we obtain sufficient conditions for its univalency.

For proving our main results we need the following theorems:

Theorem 1.1. [3]. *Let α be a complex number, Re $\alpha > 0$ and $f \in \mathcal{A}$. If*

$$\frac{1-|z|^{2Re\ \alpha}}{Re\ \alpha}\left|\frac{zf''(z)}{f'(z)}\right| \le 1 \tag{1.2}$$

for all $z \in U$, then for any complex number β, Re $\beta \ge$ Re α, the function

$$F_\beta(z) = \left\{\beta \int_0^z u^{\beta-1} f'(u)du\right\}^{\frac{1}{\beta}} \tag{1.3}$$

is in the class S.

Lemma 1.2. [2]. *(General Schwarz Lemma). Let the function f be regular in the disk $U_R = \{z \in \mathbb{C} : |z| < R\}$, with $|f(z)| < M$ for fixed M. If f has one zero with multiplicity order bigger than m for $z = 0$, then*

$$|f(z)| \le \frac{M}{R^m} \cdot |z|^m \quad (z \in U_R).$$

The equality can hold only if

$$f(z) = e^{i\theta} \cdot \frac{M}{R^m} \cdot z^m,$$

where θ is constant.

2. Main results

Theorem 2.1. *Let $f_i \in S(\alpha_i)$, $0 < \alpha_i \le 2$, $f_i(z) = z + a_3^i z^3 + a_4^i z^4 + ...$, $M_i \ge 1$, for all $i = 1, 2, ...n$, $h_j \in P$, $N_j > 0$, for all $j = 1, 2, ...p$ and $\delta, \gamma \in \mathbb{C}$ with*

$$\text{Re } \gamma \ge \frac{1}{|\gamma|}\left(n + \sum_{i=1}^{n}(\alpha_i + 1)M_i\right) + |\delta| \sum_{j=1}^{p} N_j. \tag{2.1}$$

If

$$|f_i(z)| \le M_i \text{ for all } i = 1, 2, ..., n, \ (z \in U) \tag{2.2}$$

and

$$\left|\frac{zh_j'(z)}{h_j(z)}\right| \le N_j \text{ for all } j = 1, 2, ..., p, \ (z \in U) \tag{2.3}$$

then for every complex number β, Re $\beta \ge$ Re γ *the integral operator $H_{n,p}(z)$ defined by (1.1) is in the class S.*

Proof. Let us define the function

$$g(z) = \int_0^z \prod_{i=1}^{n}\left(\frac{f_i(t)}{t}\right)^{\frac{1}{\gamma}} \prod_{j=1}^{p}(h_j(t))^\delta \mathrm{d}t.$$

We have

$$g'(z) = \prod_{i=1}^{n} \left(\frac{f_i(z)}{z}\right)^{\frac{1}{\gamma}} \prod_{j=1}^{p} (h_j(z))^{\delta}$$

and, hence

$$\frac{zg''(z)}{g'(z)} = \frac{1}{\gamma}\sum_{i=1}^{n}\left(\frac{zf_i'(z)}{f_i(z)} - 1\right) + \delta\sum_{j=1}^{p}\frac{zh_j'(z)}{h_j(z)}. \tag{2.4}$$

From (2.4) we obtain

$$\frac{1-|z|^{2\mathrm{Re}\gamma}}{\mathrm{Re}\gamma}\left|\frac{zg''(z)}{g'(z)}\right| \le \frac{1-|z|^{2\mathrm{Re}\gamma}}{\mathrm{Re}\gamma}\left[\frac{1}{|\gamma|}\sum_{i=1}^{n}\left(\left|\frac{zf_i'(z)}{f_i(z)}\right| + 1\right) + |\delta|\sum_{j=1}^{p}\left|\frac{zh_j'(z)}{h_j(z)}\right|\right]$$

$$\le \frac{1-|z|^{2\mathrm{Re}\gamma}}{\mathrm{Re}\gamma}\left[\frac{1}{|\gamma|}\sum_{i=1}^{n}\left(\left|\frac{z^2f_i'(z)}{f_i^2(z)}\right|\cdot\left|\frac{f_i(z)}{z}\right| + 1\right) + |\delta|\sum_{j=1}^{p}\left|\frac{zh_j'(z)}{h_j(z)}\right|\right] \tag{2.5}$$

From (2.2) applying general Schwarz Lemma we have $\left|\frac{f_i(z)}{z}\right| \le M_i$ for all $i = 1, 2, ..., n$. Using the last relation, (2.3) and (2.5) we obtain

$$\frac{1-|z|^{2\mathrm{Re}\gamma}}{\mathrm{Re}\gamma}\left|\frac{zg''(z)}{g'(z)}\right| \le \frac{1-|z|^{2\mathrm{Re}\gamma}}{\mathrm{Re}\gamma}\left\{\frac{1}{|\gamma|}\sum_{i=1}^{n}\left[\left(\left|\frac{z^2f_i'(z)}{f_i^2(z)} - 1\right| + 1\right)M_i + 1\right] + \right.$$

$$\left. + |\delta|\sum_{j=1}^{p}N_j\right\} \tag{2.6}$$

Because $f_i \in S(\alpha_i)$ for all $i = 1, 2, ..., n$ we have

$$\left|\frac{z^2f_i'(z)}{f_i^2(z)} - 1\right| \le \alpha_i|z|^2 \text{ for all } i = 1, 2, ..., n, \ (z \in U). \tag{2.7}$$

From (2.6) and (2.7) it results

$$\frac{1-|z|^{2\mathrm{Re}\gamma}}{\mathrm{Re}\gamma}\left|\frac{zg''(z)}{g'(z)}\right| \le \frac{1-|z|^{2\mathrm{Re}\gamma}}{\mathrm{Re}\gamma}\left\{\frac{1}{|\gamma|}\sum_{i=1}^{n}\left[\left(\alpha_i|z|^2 + 1\right)M_i + 1\right] + \right.$$

$$\left. + |\delta|\sum_{j=1}^{p}N_j\right\} \le \frac{1}{\mathrm{Re}\gamma}\left[\frac{1}{|\gamma|}\left(n + \sum_{i=1}^{n}(\alpha_i + 1)M_i\right) + |\delta|\sum_{j=1}^{p}N_j\right] \tag{2.8}$$

From (2.1) and (2.8) we have

$$\frac{1-|z|^{2\mathrm{Re}\gamma}}{\mathrm{Re}\gamma}\left|\frac{zg''(z)}{g'(z)}\right| \le 1, \ (z \in U)$$

and applying Theorem 1.1 we obtain that the integral operator $H_{n,p}(z)$ defined in (1.1) is in the class S. □

Letting $\alpha_i = \alpha$ for all $i = 1, 2, ..., n$ in Theorem 2.1 we have

Corollary 2.2. *Let* $f_i \in S(\alpha)$, $0 < \alpha \leq 2$, $f_i(z) = z + a_3^i z^3 + a_4^i z^4 + ...$, $M_i \geq 1$, *for all* $i = 1, 2, ...n$, $h_j \in P$, $N_j > 0$, *for all* $j = 1, 2, ...p$ *and* $\delta, \gamma \in \mathbb{C}$ *with*

$$\mathrm{Re}\ \gamma \geq \frac{n + (\alpha + 1)\sum\limits_{i=1}^{n} M_i}{|\gamma|} + |\delta| \sum_{j=1}^{p} N_j.$$

If

$$|f_i(z)| \leq M_i \text{ for all } i = 1, 2, ..., n, \ (z \in U)$$

and

$$\left| \frac{z h_j'(z)}{h_j(z)} \right| \leq N_j \text{ for all } j = 1, 2, ..., p, \ (z \in U)$$

then for every complex number β, $\mathrm{Re}\ \beta \geq \mathrm{Re}\ \gamma$ *the integral operator* $H_{n,p}(z)$ *defined by* (1.1) *is in the class* S.

Letting $M_i = M$ for all $i = 1, 2, ..., n$ and $N_j = N$ for all $j = 1, 2, ..., p$ in Corollary 2.2 we have

Corollary 2.3. *Let* $f_i \in S(\alpha)$, $0 < \alpha \leq 2$, $f_i(z) = z + a_3^i z^3 + a_4^i z^4 + ...$ *for all* $i = 1, 2, ...n$, $h_j \in P$ *for all* $j = 1, 2, ...p$, $M \geq 1$, $N > 0$, *and* $\delta, \gamma \in \mathbb{C}$ *with*

$$\mathrm{Re}\ \gamma \geq \frac{n[1 + (\alpha + 1)M]}{|\gamma|} + p|\delta|N.$$

If

$$|f_i(z)| \leq M \text{ for all } i = 1, 2, ..., n, \ (z \in U)$$

and

$$\left| \frac{z h_j'(z)}{h_j(z)} \right| \leq N \text{ for all } j = 1, 2, ..., p, \ (z \in U)$$

then for every complex number β, $\mathrm{Re}\ \beta \geq \mathrm{Re}\ \gamma$ *the integral operator* $H_{n,p}(z)$ *defined by* (1.1) *is in the class* S.

Letting $n = 1$ and $p = 1$ in Corollary 2.3 we have

Corollary 2.4. *Let* $f \in S(\alpha)$, $0 < \alpha \leq 2$, $f(z) = z + a_3 z^3 + a_4 z^4 + ...$, $h \in P$, $M \geq 1$, $N > 0$, *and* $\delta, \gamma \in \mathbb{C}$ *with*

$$\mathrm{Re}\ \gamma \geq \frac{1 + (\alpha + 1)M}{|\gamma|} + |\delta|N.$$

If

$$|f(z)| \leq M \ (z \in U)$$

and

$$\left| \frac{z h'(z)}{h(z)} \right| \leq N \ (z \in U)$$

then for every complex number β, $\mathrm{Re}\ \beta \geq \mathrm{Re}\ \gamma$ *the integral operator*

$$H(z) = \left\{ \beta \int_0^z t^{\beta - 1} \left(\frac{f(t)}{t} \right)^{\frac{1}{\gamma}} (h(t))^{\delta} \, \mathrm{d}t \right\}^{\frac{1}{\beta}}$$

is in the class S.

References

[1] Breaz, D., *An integral univalent operator of the class $S(p)$ and T_2*, Novi Sad J. Math., **37**(2007), no. 1, 9-15.

[2] Nehari, Z., *Conformal Mapping*, Dover, New York, NY, USA, 1975.

[3] Pascu, N.N., *An improvement of Becker's univalence criterion*, Proceedings of the Commemorative Session Simion Stoilow, University of Brasov, 1987, 43-48.

[4] Singh, V., *On class of univalent functions*, Internat. J. Math. Sci., **23**(2000), no. 12, 855-857.

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