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On a complex multidimensional approximation theorem and its applications to a complex operator-valued moment problems

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Abstract. The present note gives a proof to an approximation theorem of some holomorphic function, positively defined on a compact set in the unit polydisc in C^{2m}, with complex polynomials of a special type. This theorem is only enunciated in [3], without proof. We also underline in Theorem 5.1, section 5, some useful applications of the approximation theorem in solving a complex, operator-valued moment problem on the closed unit polydisc in C^m. The presented Theorem 5.1 gives another proof of Theorem 3.4. and Theorem 1.2.17. in [27], [28].

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1. Introduction

The present note, in Section 3 and Section 4, presents the proof of a theorem of uniformly approximation on compacta of a holomorphic, multivariable, complex function, positively defined on some compact set in the multidimensional unit polydisc $\mathrm{D}^m \times \mathrm{D}^m \subset \mathrm{C}^{2m}$, with polynomials in $2m$ complex variables, of a special type. This theorem was only enunciated in [3], without proof. The proof of this theorem in the present note is based on establishing an equivalence of the theorem with an intrinsic characterization theorem of subnormal tuples of commuting operators. The complex multidimensional approximation theorem is a useful tool in the approach of the complex, operator-valued moment problem on the closed unit polydisc $\overline{\mathrm{D}}^m \subset \mathrm{C}^m$. Based on this theorem, in Theorem 5.1, Section 5 of this note, it is given another proof of Theorem 3.4 in [26] and Theorem 1.2.17 in [27]. The complex operator-valued moment problem on the unit polydisc or on semialgebraic, nonvoid, compact sets in R^{2m} was simplified by testing the positivity of the moment functional associated with the moment sequence not on the whole space of the positively defined polynomials on the semialgebraic compact sets, but on some "smaller test subsets" e.g. [13], [21], [26], [27]. The proofs of Theorem 3.4 in [26] and Theorem 1.2.17 in [27] used Cassier's method for solving multidimensional real moment problems on semialgebraic nonvoid compact sets [5]. The operator-valued complex moment problem in theorem 2 is solved by applying the mentioned approximation Theorem 4.1 to obtain the same "test subset" as in Theorem 3.4, Theorem 1.2.27 [27], [28].

2. Preliminaries

Let $m \in \mathrm{N}^*$ be arbitrary, $I = (i_1, ..., i_m) \in \mathrm{N}^m$, $z = (z_1, ..., z_m) \in \mathrm{C}^m$, $t = (t_1, ..., t_m) \in \mathrm{R}^m$ denote the complex, respectively the real Euclidean space, $z^I = z_1^{i_1} ... z_m^{i_m}$, $|I| = i_1 + ... + i_m$ the length of $I = (i_1, ..., i_m) \in \mathrm{Z}_+^m$ and

$$\overline{D}^m = \{z = (z_1, ..., z_m) \in \mathrm{C}^m,\ |z_i| \leq 1,\ 1 \leq i \leq m\}$$

the closed multidimensional unit polydisc. Let also H be a complex, separable Hilbert space and $B(H)$ denote the algebra of bounded, linear operators on H. For the commuting operators $S_1, ..., S_m$ in $B(H)$, the operator tuple $(S_1, ..., S_m)$ is called subnormal if there exists a Hilbert space $K \supset H$ and m commuting normals $(N_1, ..., N_m)$ such that $N_i H \subset H$ and $N_i|_H = S_i$ for all $1 \leq i \leq m$. The operator tuple $N = (N_1, ..., N_m)$ is referred to as a commuting normal extension of S. We denote with $S^I = S^{i_1} \circ ... \circ S^{i_m}$, with $C_K^P = C_{k_1}^{p_1} \times ... \times C_{k_m}^{p_m}$ for $K = (k_1, ..., k_m), P = (p_1, ..., p_m)$ multiindices in Z_+^m with $p_i \leq k_i$, $\forall 1 \leq i \leq m$. For characterizing a subnormal tuple of commuting operators $S = (S_1, ..., S_m)$ in $B(H)$, we used in this note Ito's and Lubin's criterions of subnormality (Theorem 1 in [11], respectively Theorem 3.2 in [12]). This criterions are:

"A commuting tuple of operators $S = (S_1, ...S_m)$ in $B(H)$ is subnormal if and only if for any finite number of multiindices $I, J \in \mathrm{N}^m$ and any finite number of elements $x_I,\ x_J \in H$ we have the positivity condition:

$$\sum_{I,J} < S^I x_J, S^J x_I >_H \geq 0 (th.1[11],"$$

"$T_1, ..., T_m$ have commuting normal extension if and only if there exists a positive operator valued measure ρ defined on some m-dimensional rectangle R such that $T^{\bullet J} T^J = \int_R t^{2J} \mathrm{d}\rho t$ for all J (Theorem 3.2 [12])."

We also denote with $P(\mathrm{C}^m)$ the algebra of all complex polynomial functions in $z_1, ..., z_m, \overline{z_1}, ..., \overline{z_m}$.

3. A subnormality characterization of a commuting tuple of operators

Proposition 3.1. *Let $S_1, ..., S_m$ be m commuting operators in $B(H)$. Statements (i) and (ii) below are equivalent:*

(i) $S = (S_1, ...S_m)$ is a subnormal tuple where each S_i is a contraction.

The following inequality occurs:

(ii) $\sum_{0 \le p_i \le k_i; 1 \le i \le m} (-1)^{|P|} C_K^P S^{*P} \circ S^P \ge 0,$ for $\forall\ K = (k_1, ..., k_m) \in \mathrm{Z}_+^m$ *with* $P = (p_1, ..., p_m)$.

Proof. We shall prove that $(i) \Rightarrow (ii)$. Let $S = (S_1, ..., S_m)$ be a commuting tuple of subnormal contractions on an arbitrary, complex, separable Hilbert space H. In this case, there exists a Hilbert space $K \supset H$, commuting normals $N_1, ..., N_m$ such that $N_i H \subseteq H$ and $N_i|_H = S_i,\ \forall 1 \le i \le m$. According to Halmos's paper [9], if B is a normal operator on $K \supseteq H$, $A = B|_H$ and P is the projection of K on H, we have $A \circ P = B \circ P$ and $A^* \circ P = P \circ B^*$. Using these equalities for $B = N_i$ and $A = S_i$, for an arbitrary $u \in H$, statement (ii) becomes:

$$\sum_{0 \le p_i \le k_i; 1 \le i \le m} (-1)^{p_1+p_2+...+p_m} C_{k_1}^{p_1} ... C_{k_m}^{p_m} S_1^{*p_1} \circ ... \circ S_m^{*p_m} \circ S_1^{p_1} \circ ... \circ S_m^{p_m} u$$

$$= \sum_{0 \le p_i \le k_i; 1 \le i \le m} (-1)^{p_1+p_2+...+p_m} C_{k_1}^{p_1} ... C_{k_m}^{p_m} S_1^{*p_1} \circ ... \circ S_m^{*p_m} \circ S_1^{p_1} \circ ... \circ S_m^{p_m} P u$$

$$= P(\sum_{0 \le p_i \le k_i; 1 \le i \le m} (-1)^{p_1} C_{k_1}^{p_1} N_1^{*p_1} N_1^{P_1} ... (-1)^{p_m} C_{k_m}^{p_m} N_m^{*p_m} N_m^{p_m}) u$$

$$= P[\prod_{1 \le i \le m} (1 - z_i w_i)^{k_i} (N_i N_i^*)] u.$$

Because all S_i are contractions and $(N_1, ..., N_m)$ is the normal minimal extension of $(S_1, ..., S_m)$, we have $||N_i|| = ||S_i|| \le 1$; this imply

$$\prod_{0 \le i \le m} (1 - z_i w_i)^{k_i} (N_i, N_i^*) \ge 0$$

e.g. $(< (1 - N_i^* \circ N_i)^2) u, u > = < (1 - N_i^* N_i) u, (1 - N_i^* N_i) u > \ge 0.)$ Applying again Halmos's equalities in [4] and returning to S_i operators, we obtain

$$\sum_{0 \le p_i \le k_i; 1 \le i \le m} (-1)^{|P|} C_K^P S^{*P} \circ S^P \ge 0, \text{ for } K = (k_1, ..., km), P = (p_1, ..., p_m) \in \mathrm{Z}_+^m$$

the required inequality.

Conversely. $(ii) \Rightarrow (i)$. If

$$\sum_{0 \le p_i \le k_i; 1 \le i \le m} (-1)^{|P|} C_K^P S^{*P} \circ S^P \ge 0,$$

is true for any $K = (k_1, ..., k_m),\ P = (p_1, ..., p_m) \in \mathrm{Z}_+^m,\ p_i \le k_i,\ \forall 1 \le i \le m$ taking account that the $\{S_i\}_{i=1}^m$ is a family of commuting one, by multiplying the above inequality with $S^{*T} \circ S^T,\ T \in \mathrm{N}^m$, we obtain

$$\sum_{0 \le p_i \le k_i; 1 \le i \le m} (-1)^{p_1+p_2+...+p_m} C_{k_1}^{p_1} ... C_{k_m}^{p_m} < S^{T+P} u, S^{T+P} u > \ge 0$$

for any vector $u \in H$ and any multiindicees $K,\ T,\ P \in \mathrm{N}^m$. For every $u \in H$, we define the linear functional $\varphi_u : \mathrm{N}^m \to \mathrm{R}$, by $\varphi_u(T) = < S^{T+P} u, S^{T+P} u >$. Let Δ_i the operators $\Delta_i : L(\mathrm{N}^m, \mathrm{R}) \to L(\mathrm{N}^m, \mathrm{R})$,

$$\Delta_i \varphi_u(T) = \varphi_u(T_1, ..., T_i + 1, ..., T_m) - \varphi_u(T_1, ..., T_m).$$

For $\Delta_i - s$ operators, condition (ii) in hypothesis is $\Delta_1^{k_1} \circ ... \circ \Delta_1^{k_m} \varphi_u(T) \geq 0$; that is the functional φ_u is completely monotonic for every $u \in H$. From [10], in these conditions, for a completely monotonic functional there exists positive, scalar, Borel measures μ_u such that $\varphi_u(P) = \int_{[0,1]^m} x^P \mathrm{d}\mu_u(x)$. With the help of the scalar measures μ_u, we define the semispectral measure: $< \rho(A)u, u >_H = \mu_u(A)$ for any Borel set $A \in Bor([0,1]^m)$. In these conditions,

$$\varphi_u(P) = \int_{[0,1]^m} x^P \mathrm{d} < \rho(x)u, u > = < S^P u, S^P u >_H = < S^{*P} S^P u, u > .$$

By the uniqueness of the representation theorem of a normal operator with respect to a spectral measure, we obtain $[S^*S]^P = \int_{[0,1]^m} x^P \mathrm{d}\rho(x)$. We prove in the sequel that $(S_1, ..., S_m)$ verify Ito's necessary and sufficient condition for subnormality by using the above identity; that is we prove that $\sum_{I,J} < S^{I+J} x_I, S^{I+J} x_J >_H \geq 0$ for any finite family $\{x_I\} \in H$. With the obtained representation, we have:

$$\sum_{I,J} < S^{I+J} x_I, S^{I+J} x_J >_H = \sum_{I,J} \int_{[0,1]^m} z^{I+J} \mathrm{d} < \rho(z) x_I, x_J >_H$$

$$= \sum_{I,J} \int_{[0,1]^m} \mathrm{d} < \rho^{\frac{1}{2}}(z) \sum_{I,J} z^I x_I, \rho^{\frac{1}{2}} \sum_{I,J} z^J x_J >_H \geq 0.$$

From this calculus, via Theorem 3.2 [12], it results that $S = (S_1, ..., S_m)$ is a subnormal tuple.

4. The main approximation theorem

In this section we prove the following:

Theorem 4.1. *Let D^m, $m \in \mathrm{N}^*$ denote the open unit polydisc in the complex space C^m and $A^2(D^m \times D^m)$ denote the space of continuous functions on the closed polydisc $\overline{D}^m \times \overline{D}^m$, analytic on $D^m \times D^m$, equipped with the topology of uniform convergence on compacta. Let $M = \{f \in A^2(D^m \times D^m),\ f(z, \overline{z}) \geq 0\ , \forall\ z \in D^m\}$. If C denotes the convex hull of the set $\{f \in A^2(D^m \times D^m),\ f(z, w) = p(z) \prod_{i=1}^m (1 - z_i w_i)^{k_i} \overline{p(\overline{w})}\}$ where k_i are non-negative integers and p is an m-variable complex polynomial, then $\overline{\mathrm{C}} = M$, where $\overline{\mathrm{C}}$ denotes the closure of C in the topology of uniform convergence on compacta.*

This theorem appears in [3], only enounced, without proof. The proof of this approximation theorem in the present note is based on establishing an equivalence of Theorem 4.1 in this section with Proposition 3.1 in section 3 (a criterion of subnormality of a m-tuple $(A_1, ..., A_m)$ of m commuting bounded operators, acting on a prescribed Hilbert space). In the sequel, we give some applications of this theorem to an operator-valued complex moment problem.

Proof. First of all we shall prove that Proposition 3.1 $\Rightarrow$ Theorem 4.1. Let $f \in C$, that is $f(z, w) = p(z) \prod_{i=1}^m (1 - z_i w_i)^{k_i} \overline{p(\overline{w})}$ and $|z_i| \leq 1$, $|w_i| \leq 1$, $1 \leq i \leq m$; in these conditions, $f \in A^2(D^m \times D^m)$ and $f(z, \overline{z}) \geq 0$ that is $f \in M$, consequently $\overline{C} \subseteq M$.

Conversely, we prove that, also $M \subseteq \overline{C}$; otherwise, if $M - \overline{C} \neq \Phi$, let $h_0 \in M - \overline{C}$. Because $\overline{C}$ is a close set, from Hahn Banach theorem, there exists a linear functional $\Lambda : A^2(D^m \times D^m) \to$ C such that

$$\mathrm{Re}\Lambda\,(h_0) < 0 \leq \mathrm{Re}\Lambda\,(f),\ \forall f \in \overline{C}. \tag{4.1}$$

We extend Λ to the space of continuous functionals on $C(\overline{D}^m \times \overline{D}^m)$. By Riesz representation theorem, there exists $\nu_\Lambda \in Bor(\overline{D}^m \times \overline{D}^m)$ such that $\Lambda(f) = \int_{\overline{D}^m \times \overline{D}^m} f \mathrm{d}\nu_\Lambda$. With this identification, we have

$$\mathrm{Re}\,[\int_{\overline{D^m} \times \overline{D^m}} h_0 \mathrm{d}\nu_\Lambda] < 0 \leq \mathrm{Re} \int_{\overline{D^m} \times \overline{D^m}} f \mathrm{d}\nu_\Lambda,\ \forall f \in \overline{C}. \tag{4.2}$$

Let $\varphi : \overline{D}^m \times \overline{D}^m \to \overline{D}^m \times \overline{D}^m$, $\varphi(z,w) = (\overline{w}, \overline{z})$, the measure $\mu = \frac{1}{2}(\nu_\Lambda + \overline{\nu_\Lambda \circ \varphi})$ and $h \in M$ an arbitrary element. With the help of h, we define

$$g(z,w) = h(z,w) - \overline{h \circ \varphi}(z,w) = h(z,w) - \overline{h(\overline{w}, \overline{z})}.$$

From this construction, g is holomorphic on $D^m \times D^m$, continuous on $\overline{D}^m \times \overline{D}^m$ and $g(z, \overline{z}) = 0$ for all $(z, \overline{z}) \in D^m \times D^m$. If we consider the set $B = \{(z, \overline{z}), z \in D^m\}$, because $g|_B = 0$ and $g \in A^2(D^m \times D^m)$, $2m \geq 2$, it results that $g(z,w) = 0$ for all $(z,w) \in D^m \times D^m$. From this remark, $h = \overline{h \circ \varphi}$ on $\overline{D}^m \times \overline{D}^m$. Inequality (4.2) becomes

$$\begin{aligned}
\mathrm{Re}\,[\int_{\overline{D}^m \times \overline{D}^m} h \mathrm{d}\nu_\Lambda] &= \frac{1}{2}[\int_{\overline{D}^m \times \overline{D}^m} h \mathrm{d}\nu_\Lambda + \overline{h} \mathrm{d}\overline{\nu}_\Lambda] \\
&= \frac{1}{2}[\int_{\overline{D}^m \times \overline{D}^m} h \mathrm{d}\nu_\Lambda + \int_{\overline{D}^m \times \overline{D}^m} \overline{h \circ \varphi} \mathrm{d}\overline{\nu_\Lambda \circ \varphi} \\
&= \frac{1}{2}[\int_{\overline{D}^m \times \overline{D}^m} h \mathrm{d}\nu_\Lambda + \int_{\overline{D}^m \times \overline{D}^m} h \mathrm{d}\overline{\nu_\Lambda \circ \varphi}] \\
&= \int_{\overline{D}^m \times \overline{D}^m} h \frac{1}{2} \mathrm{d}[\nu_\Lambda + \overline{\nu \circ \varphi}] = \int_{\overline{D}^m \times \overline{D}^m} h \mathrm{d}\mu.
\end{aligned} \tag{4.3}$$

Using the new constructed measure μ, inequality (4.2) becomes:

$$\int_{\overline{D}^m \times \overline{D}^m} h_0 \mathrm{d}\mu < 0 \leq \int_{\overline{D}^m \times \overline{D}^m} f \mathrm{d}\mu \quad \forall f \in \overline{C}. \tag{4.4}$$

We shall construct, with the help of Proposition 3.1, a scalar measure σ defined on $\overline{D}^m$, generated by the spectral measure associated with a commuting tuple of subnormals defined on $\overline{D}^m$, and prove, that on some dense set in $A^2(D^m \times D^m)$, the two measures are equal.

Let P the C- vector space of polynomials with complex coefficients in $z-$variable and $N = \{p \in P$ with $\int_{\overline{D}^m \times \overline{D}^m} p(z)\overline{p(\overline{w})} \mathrm{d}\mu(z,w) = 0\}$. The set N is a C-vector subspace in P; on the coset P/N, with the help of the definition of the subspace N, we define the hermitian product $< p + N, q + N >= \int_{\overline{D}^m \times \overline{D}^m} p(z)\overline{q(\overline{w})} \mathrm{d}\mu(z,w)$ and consider $H^2(\mu)$ the Hilbert space obtained as the separate completion of P/N with respect to this hermitian product. Let the operators $S_i : P \to P$, $[S_i p](z) = z_i p(z)$ for all $1 \leq i \leq m$. Because $||p||^2 - ||S_i||^2 = (\int_{\overline{D}^m \times \overline{D}^m} (1 - z_i w_i) p(z)\overline{p(\overline{w})} \mathrm{d}\mu(z,w) \geq 0$ on the subspace P, ($||\ ||$ represents the norm generated by the introduced hermitian product), the operators S_i are contractions; we extend S_i to $H^2(\mu)$ with preserving the norm and the commutation relations; the extensions will be denoted also $S_i,\ 1 \le i \le m$. For the tuple $(S_1, ..., S_m)$ we shall verify condition (ii) in Proposition 3.1, for testing if it is a subnormal one. That is:

$$< \sum_{0 \le p_i \le k_i; 1 \le i \le m} (-1)^{|P|} C_K^P S_1^{*p_1} \circ ... \circ S_m^{*p_m} \circ S_1^{p_1} \circ ... \circ S_m^{p_m} p, p >_{H^2(\mu)}$$

$$= \int_{\overline{D}^m \times \overline{D}^m} \sum_{0 \le p_i \le k_i; 1 \le i \le m} (-1)^{|P|} C_K^P z_1^{p_1} ... z_m^{p_m} w_1^{p_1} ... w_m^{p_m} p(z) \overline{p(\overline{w})} \mathrm{d}\mu(z, w)$$

$$= \int_{\overline{D}^m \times \overline{D}^m} p(z) \prod_{i=1}^{m} (1 - z_i w_i)^{k_i} \overline{p(\overline{w})} \mathrm{d}\mu(z, w) \ge 0,$$

with $P = (p_1, ..., p_m),\ K = (k_1, ..., k_m)$, inequality that is exactly condition (ii) in Proposition 3.1. Because this proposition is true, there exists the joint spectral measure associated to the subnormal tuple $(S_1, ..., S_m)$ and, consequently, the scalar measures generated by it. Let $\mathrm{d}\sigma(A) =< E_S(A)1, 1 >_{H^2(\mu)}$ defined for all Borel sets A in $Bor(D^m)$ where E_S is the joint spectral measure associated with S. We have then,

$$[S_1^{p_1} \circ ... \circ S_m^{p_m} 1, S_1^{n_1} \circ ... \circ S_m^{n_m} 1] = [N_1^{p_1} \circ ... \circ N_m^{p_m} 1, N_1^{n_1} \circ ... \circ N_m^{n_m} 1] = \int_{\overline{D}^m} z^P \overline{z}^N \mathrm{d}\sigma(z)$$

for all multiindices $P = (p_1, ..., p_m), N = (n_1, ..., n_m)$.

From the definition of the scalar product on $H^2(\mu)$, we have also

$$[S^P 1, S^N 1] = \int_{\overline{D}^m \times \overline{D}^m} z^P w^N \mathrm{d}\mu(z, w)$$

for all multiindices $P = (p_1, ..., p_m), N = (n_1, ..., n_m))$. Because the polynomials in (z, w) are uniformly dense on compacta in $A^2(D^m \times D^m)$, we have also

$$\int_{\overline{D}^m \times \overline{D}^m} h(z, w) \mathrm{d}\mu(z, w) = \int_{\overline{D}^m} h(z, \overline{z}) \mathrm{d}\sigma(z),\ \forall h \in A^2(D^m \times D^m).$$

The above inequality happens in particular for $h_0 \in M - \overline{C}$. From the definition of M, we have $h_0(z, \overline{z}) \ge 0$, the scalar measure $\mathrm{d}\sigma$ is a positive one, we have than

$$0 > \int_{\overline{D}^m \times \overline{D}^m} h_0(z, w) \mathrm{d}\mu(z, w) = \int_{\overline{D}^m} h(z, \overline{z}) \mathrm{d}\sigma(z) \ge 0$$

which represents a contradiction; that is $M = \overline{C}$.

Conversely. Theorem 4.1 $\Rightarrow$ Proposition 3.1. Let $S = (S_1, ..., S_m)$ a commuting tuple of operators like in hypothesis of Proposition 3.1; the implication $(i) \Rightarrow (ii)$ is always true; we will prove that in conditions of Theorem 4.1, also $(ii) \Rightarrow (i)$ happens. In this case,

$$\prod_{i=1}^{m} (1 - z_i w_i)^{k_i} (S, S^*) \ge 0;$$

from this inequalities it results that all S_i operators are contractions and consequently for each $0 \leq r < 1$, we have also

$$\prod_{i=1}^{m}(1 - z_i wi)^{k_i}(rS, rS^*) \geq 0,$$

inequality that imply

$$p(rS)^* \prod_{i=1}^{m}(1 - z_i wi)^{k_i}(rS, rS^*)p(rS) \geq 0.$$

Let the holomorphic function $f(z, w) = \sum_{I,J} a_{IJ} z^I w^J,\ f \in A^2(D^m \times D^m)$ and the subnormal tuple of contractions $S = (S_1, ..., S_m)$ with the spectrum in $\overline{D}^m$. We define using the analytic calculus $f(rS, rS^*) = \sum_{I,J} a_{IJ}(rS)^I(rS^*)^J$. If $f \in C$ with

$$f(z, w) = p(z)\prod_{i=1}^{m}(1 - z_i w_i)^{k_i}\overline{p(\overline{w})},$$

we will also have $f(rS, rS^*) \geq 0$ for every such an element $f \in C$ and for every $0 \leq r < 1$. Because Theorem 4.1 is true, we also have $f(rS, rS^*) \geq 0$ for every $f \in M$. In particular, for any polynomial $p(z, w)$ with $p(z, \overline{z}) \geq 0$, also $p(rS, rS^*) \geq 0$ happens; subsequently we have:

$$\sum_{I,J} < r^{I+J} S^{I+J} x_I, r^{I+J} S^{I+J} x_J >_{H^2(\mu)} \geq 0.$$

Passing to the limit for $r \to 1$, we obtain

$$\sum_{I,J} < S^{I+J} x_I, S^{I+J} x_J >_{H^2(\mu)} \geq 0,$$

that is Lubin's condition for subnormality [12].

We prove that $S = (S_1, ...S_m)$ is a subnormal tuple (exactly condition (i) in Proposition 3.1) that is, Theorem 4.1 $\Leftrightarrow$ Proposition 3.1. From above, we have Theorem 4.1 $\Leftrightarrow$ Proposition 3.1. Because Proposition 3.1 is true, as it was shown in Section 3, Theorem 4.1 is also true.

5. Applications of Theorem 4.1 in solving an operator-valued complex moment problem

In this section, we give a useful application of Theorem 4.1, in the present paper, in solving the operator-valued complex moment problem on the complex unit polydisc $\overline{D}^m \subset \mathrm{C}^m,\ m > 1$.

We recall, that given a sequence of bounded operators $\Gamma = (\Gamma_{\alpha,\beta})_{\alpha,\beta \in \mathrm{Z}^m}, \Gamma_{\alpha,\beta} = \Gamma^*_{\beta,\alpha}$ acting on an arbitrary Hilbert space H, the operator-valued moment problem asks for necessary and sufficient conditions on Γ such that there exists a positive operator-valued measure F_Γ on $\overline{D}^m$ such that

$$\int_{\overline{D}^m} z^\alpha \overline{z}^\beta \mathrm{d}F_\Gamma(z),\ \forall\ \alpha, \beta \in \mathrm{Z}^m_+. \tag{5.1}$$

In [27], this multidimensional operator-valued moment problem is solved in Theorem 3.7 of the paper by passing, from the complex multidimensional moment problem on $\overline{D}^m \subset \mathrm{C}^m$, to a real multidimensional operator-valued problem on the set $\tau(\overline{D}^m) \subset \mathrm{C}^m$ with τ the mapping $\tau : \mathrm{C}^m \to \mathrm{R}^m$, $\tau(z_1, ..., z_m) = (|z_1|^2, ..., |z_m|^2) \in \mathrm{R}^m$. The multidimensional real operator-valued moment problem is solved in [27] by applying Cassier's results in [5] for solving the scalar real moment problem, naturally generated by the operator-valued one on the compact $\tau(\overline{D}^m)$. In the sequel, by a standard polarization argument for the obtained scalar representing measures, the operator-valued representing measure, solution for the real multidimensional operator-valued moment problem on $\tau(\overline{D}^m)$ is obtained. Via the inverse mapping τ^{-1}, the solution of the real multidimensional operator-valued moment problem furnished a solution of (5.1).

The enounce of the complex operator-valued moment problem from Theorem 5.1 in this note is quite the same with that of Theorem 3.7 in [27]. It adds, only, an additional condition on the sequence of operators $\Gamma = (\Gamma_{\alpha,\beta})$ in hypothesis, condition that ensure a direct proof of the operator-valued complex moment problem by using the Hahn-Banach theorem, Riesz representation theorem and Theorem 4.1 in this note.

Theorem 5.1. *Let $\Gamma = (\Gamma_{\alpha,\beta})_{\alpha,\beta\in \mathrm{Z}_+^m}$ be a sequence of bounded linear operators acting on an arbitrary, complex Hilbert space H, such that $\Gamma_{\alpha,\beta} = (\Gamma^*_{\alpha,\beta})$ for all $\alpha, \beta \in \mathrm{Z}_+^m$, with $\Gamma_{0,0} = Id_H$ and $\{\Gamma_{\alpha,\beta}(x)\}_{\alpha,\beta\in \mathrm{Z}_+^m}$ a bounded sequence in H for all $x \in H$. The following assertions are equivalent:*

(i) The following inequalities occur:

$$\sum_{\alpha,\beta\in \mathrm{Z}_+^m} \sum_{0\le\theta\le K} (-1)^{|\theta|} C_K^\theta c_\alpha \overline{c}_\beta \Gamma_{\alpha+\theta,\beta+\theta} \ge 0, \ \forall\ K \in \mathrm{Z}_+^m \tag{5.2}$$

and all sequences of complex numbers $\{c_\alpha\}_{\alpha\in \mathrm{Z}_+^m}$ with finite support.

(ii) There exists a positive operator-valued measure F_Γ on $\overline{D}^m$ such that

$$\Gamma_{\alpha,\beta} = \int_{\overline{D}^m} z^\alpha \overline{z}^\beta \mathrm{d}F_\Gamma(z) \ \text{ for all } \alpha, \beta \in \mathrm{Z}_+^m. \tag{5.3}$$

Proof. $(i) \Rightarrow (ii)$ For the data $\Gamma = (\Gamma_{\alpha,\beta})$ we set $L_\Gamma(z^\alpha \overline{z}^\beta) = \Gamma_{\alpha,\beta}$ for all $\alpha, \beta \in \mathrm{Z}_+^m$ and extend L_Γ on $P(\mathrm{C}^m)$ by linearity. For all $x \in H$, we denote with $L_\Gamma^x : P(\mathrm{C}^m) \to H$ the obtained linear functional $L_\Gamma^x(z^\alpha \overline{z}^\beta) =< \Gamma_{\alpha,\beta}x, x >_H$ from L_Γ, for all α, β in Z_+^m. From the hypothesis, the sequences $\{\Gamma_{\alpha,\beta}(x)\}_{\alpha,\beta\in \mathrm{Z}_+^n} \subset H$ are bounded $\forall\ x \in H$, it results that L_Γ^x are linear, continuous functionals on $P(\mathrm{C}^m)$. Because of (i), we have also

$$< \sum_{\alpha,\beta\in \mathrm{I}\subset \mathrm{Z}_+^m} \sum_{0\le\theta\le k} (-1)^{|\theta|} C_K^\theta c_\alpha \overline{c}_\beta \Gamma_{\alpha+\theta,\beta+\theta} x, x >_H \ge 0$$

with I a finite set; that is we have

$$L_\Gamma^x(|p(z)|^2 \prod_{i=1}^m (1 - z_i \overline{z}_i)^{k_i}) \ge 0, \ \text{for all } K = (k_1, ..., k_m) \in \mathrm{Z}_+^m \text{ and all } x \in H. \tag{5.4}$$

Let $f \in P(\mathrm{C}^m \times \mathrm{C}^m)$ be the function

$$f(z,w) = \sum_{\alpha,\beta \in \mathrm{I} \subset \mathrm{Z}_+^m} a_{\alpha,\beta} z^\alpha w^\beta,$$

I finite, such that $f(z,\overline{z}) \geq 0$,$\forall\ z \in D^m$; it results that $a_{\alpha,\beta} = \overline{a}_{\beta,\alpha}$. Because also $\Gamma_{\alpha,\beta} = \Gamma^*_{\beta,\alpha}$, it follows that $L_x^\Gamma(f(z,\overline{z})) \in \mathrm{R}$. From Theorem 4.1 in section 4, such an analytical function is uniformly approximate on compacta with polynomials of type $p(z)\prod_{i=1}^m(1-z_i w_i)^{k_i}\overline{p(\overline{w})}$ for arbitrary $k_i \in \mathrm{N}$ and analytic polynomials $p \in P(\mathrm{C}^m)$. In this case, from (5.4), we have also $L_x^\Gamma(f(z,\overline{z})) \geq 0$; that is, also from hypothesis and from Weierstrass approximation theorem, $L_x^\Gamma : P(\mathrm{C}^m) \to \mathrm{C}$ is a positive, continuous, linear functional. With the Hahn-Banach theorem, we extend L_x^Γ on $C(\overline{\mathrm{D}}^m)$ preserving the linearity, continuity and positivity. In this case, from Riesz representation theorem, there exists a positive Radon measure $\mu(x,.)$ on $\overline{\mathrm{D}}^m$ such that $L_x^\Gamma(z^\alpha\overline{z}^\beta) = \int_{\overline{\mathrm{D}}^m} z^\alpha\overline{z}^\beta \mathrm{d}\mu(x,z) =< \Gamma_{\alpha,\beta}x, x >_H$ for all multiindices $\alpha,\beta \in \mathrm{Z}_+^m$. Because $\mu(x,B)$ is positive for all $B \in Bor(\overline{\mathrm{D}}^m)$, for any couple of vectors $u,v \in$H, we associate by a standard polarization argument, the scalar measure

$$\mu(u,v,B) = \frac{1}{4}[\mu(u+v,B) - \mu(u-v,B) + i\mu(u+iv,B) - i\mu(u-iv,B)].$$

For this measure, we have also

$$\int_{\overline{\mathrm{D}}^m} z^\alpha\overline{z}^\beta \mathrm{d}\mu(u,v,z) = \frac{1}{4}[\int_{\overline{\mathrm{D}}^m} z^\alpha\overline{z}^\beta \mathrm{d}\mu(u+v,z)$$

$$-\int_{\overline{\mathrm{D}}^m} z^\alpha\overline{z}^\beta \mathrm{d}\mu(u-v,z) + i\int_{\overline{\mathrm{D}}^m} z^\alpha\overline{z}^\beta \mathrm{d}\mu(u+iv,z) - i\int_{\overline{\mathrm{D}}^m} z^\alpha\overline{z}^\beta \mathrm{d}\mu(u-iv,z)]$$

$$=< \Gamma_{\alpha,\beta}u, v >_H \text{ for all } \alpha,\beta \in \mathrm{Z}_+^m. \tag{5.5}$$

From (5.5) and from the definition of L_u^Γ, we obtain

$$\int_{\overline{\mathrm{D}}^m} z^\alpha\overline{z}^\beta \mathrm{d}\mu(u,z) = \int_{\overline{\mathrm{D}}^m} z^\alpha\overline{z}^\beta \mathrm{d}\mu(u,u,z) =< \Gamma_{\alpha,\beta}u, u >_H; \tag{5.6}$$

that is, because the scalar moment problem on compacta is determinate, $\mu(z,u,u) = \mu(z,u)$. The mapping $< \Gamma_{\alpha,\beta}u, v >_H$ is linear in the first argument, antilinear in the second one; the same is true for the measure $\mu(.,.,z) : H \times H \to \mathrm{C}$. In the same time, we have also

$$0 \leq \mu(B,u) = \int_B d\mu(u,u,z) \leq \int_{\overline{\mathrm{D}}^m} d\mu(u,v,z) =< \Gamma_{0,0}u, u >_H \leq ||u||^2 = 1$$

for u with $||u|| = 1$; it follows that the bilinear form $\mu(u,v,z)$ has on the unit sphere 0 and 1 bounds. In these conditions, there exists for $\forall\ B \in Bor(\overline{\mathrm{D}}^m)$ a bounded selfadjoint operator $F_\Gamma(B)$ such that $\mu(u,v,B) =< F(B)u, v >_H$. Because $0 \leq \mu(u,u,B) =< F_\Gamma(B)u, u >_H \leq 1$, it follows that $F_\Gamma(B) \geq 0$ (is a positive operator). In the given conditions, (5.6) can be written symbolically $\Gamma_{\alpha,\beta} = \int_{\overline{\mathrm{D}}^m} z^\alpha\overline{z}^\beta \mathrm{d}F_\Gamma(z)$ for all $\alpha,\beta \in\ \mathrm{Z}_+^m$, that is condition (ii).

Conversely. $(ii) \Rightarrow (i)$. If we have a positive operator-valued measure F_Γ defined on $Bor(\overline{\mathrm{D}}^m)$, such that $\Gamma_{\alpha,\beta} = \int_{\overline{\mathrm{D}}^m} z^\alpha \overline{z}^\beta \mathrm{d}F_\Gamma(z)$, it follows that

$$\sum_{\alpha,\beta\in \mathrm{Z}_+^m} \sum_{0\le\theta\le k} (-1)^{|\theta|} C_k^\theta c_\alpha \overline{c}_\beta \Gamma_{\alpha+\theta,\beta+\theta}$$

$$= \sum_{\alpha,\beta\in \mathrm{Z}_+^m} \sum_{0\le\theta\le k} (-1)^{|\theta|} C_k^\theta c_\alpha \overline{c}_\beta \int_{\overline{\mathrm{D}}^m} z^{\alpha+\theta} \overline{z}^{\beta+\theta} \mathrm{d}F_\Gamma(z)$$

$$= \int_{\mathrm{D}^m} (\sum_{\alpha,\beta\in \mathrm{Z}_+^m} \sum_{0\le\theta\le k} (-1)^{|\theta|} C_k^\theta c_\alpha \overline{c}_\beta z^{\alpha+\theta} \overline{z}^{\beta+\theta} \mathrm{d}F_\Gamma(z)$$

$$= \int_{\mathrm{D}^m} |p(z)|^2 \prod_{i=1}^m (1-|z_i|^2)^{k_i} \mathrm{d}F_\Gamma(z) \ge 0$$

for any multiindices $k = (k_1, ..., k_m) \in \mathrm{Z}_+^m$ and any arbitrary analytic polynomial

$$p(z) = \sum_{\alpha\in \mathrm{I}\subset \mathrm{Z}_+^m} c_\alpha z^\alpha,$$

I finite; that is condition (i).

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