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An application of generalized integral operator

Rabha M. El-Ashwah, Mohamed K. Aouf and Teodor Bulboacă

Abstract. In this paper the authors introduced a new certain integral operator for analytic univalent functions defined in the open unit disc U. The object of this paper is to give an application of this operator to the differential inequalities.

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1. Introduction

Let A denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disc $\mathrm{U} = \{z \in \mathbb{C} : |z| < 1\}$.

In [3], Cătaş extended the multiplier transformations and defined the operator $I^m(\lambda, l)$ on A by the following series

$$I^m(\lambda, l)f(z) = z + \sum_{n=2}^{\infty} \left[\frac{1 + l + \lambda(n-1)}{1+l}\right]^m a_n z^n,\ z \in \mathrm{U},$$

where $\lambda \geq 0$, $l \geq 0$, and $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We note that $I^0(1,0)f(z) = f(z)$ and $I^1(1,0)f(z) = zf'(z)$.

Now, we define the integral operator $J^m(\lambda, l) : A \to A$, with $\lambda > 0$, $l \geq 0$, and $m \in \mathbb{N}_0$ as follows:

$$\begin{aligned}
J^0(\lambda, l)f(z) &= f(z), \\
J^1(\lambda, l)f(z) &= \frac{1+l}{\lambda} z^{1-\frac{1+l}{\lambda}} \int_0^z t^{\frac{1+l}{\lambda}-2} f(t)\,\mathrm{d}\,t, \\
J^2(\lambda, l)f(z) &= \frac{1+l}{\lambda} z^{1-\frac{1+l}{\lambda}} \int_0^z t^{\frac{1+l}{\lambda}-2} J^1(\lambda, l)f(t)\,\mathrm{d}\,t,
\end{aligned}$$

and, in general,

$$J^m(\lambda,l)f(z)=\frac{1+l}{\lambda}z^{1-\frac{1+l}{\lambda}}\int_0^z t^{\frac{1+l}{\lambda}-2}J^{m-1}(\lambda,l)f(t)\,\mathrm{d}\,t$$

$$=\underbrace{J^1(\lambda,l)\left(\frac{z}{1-z}\right)*J^1(\lambda,l)\left(\frac{z}{1-z}\right)*\dots*J^1(\lambda,l)\left(\frac{z}{1-z}\right)*f(z)}_{m\text{ times}}. \tag{1.2}$$

We note that if $f\in A$, then from (1.1) and (1.2), we have

$$J^m(\lambda,l)f(z)=z+\sum_{n=2}^{\infty}\left[\frac{1+l}{1+l+\lambda(n-1)}\right]^m a_n z^n,\ z\in\mathrm{U}, \tag{1.3}$$

for $\lambda>0$, $l\geq 0$, and $m\in\mathbb{N}_0$. From (1.3), it is easy to verify that

$$\lambda z(J^{m+1}(\lambda,l)f(z))'=(1+l)J^m(\lambda,l)f(z)-(1+l-\lambda)J^{m+1}(\lambda,l)f(z), \tag{1.4}$$

whenever $\lambda>0$.

We note that:

(i) $J^m(1,1)f(z)=I^m f(z)$ (see Flett [4], and Uralegaddi and Somanatha [9]);

(ii) $J^m(1,0)f(z)=I^m f(z)$, $m\in\mathbb{N}_0$ (see Sălăgean [8]);

(iii) $J^\alpha(1,1)f(z)=I^\alpha f(z)$, $\alpha>0$ (see Jung et al. [5]);

(iv) $J^m(\lambda,0)f(z)=J_\lambda^{-m}f(z)$, $m\in\mathbb{N}_0$ (see Patel [7]).

For our purpose, we introduce the next definition:

Definition 1.1. *Let H be the set of complex-valued function $h(r,s,t):\mathbb{C}^3\to\mathbb{C}$ such that:*

(i) $h(r,s,t)$ is continuous in a domain $D\subset\mathbb{C}^3$;

(ii) $(1,1,1)\in D$ and $|h(1,1,1)|<1$;

(iii) $\Big|h\Big(e^{i\theta},\left(1-\frac{\lambda}{l+1}\right)e^{i\theta}+\frac{\lambda}{l+1}\zeta e^{i\theta},$

$$\left(1-\frac{\lambda}{l+1}\right)^2e^{i\theta}+\left(2\frac{\lambda}{l+1}-\left(\frac{\lambda}{l+1}\right)^2\right)\zeta e^{i\theta}+\left(\frac{\lambda}{l+1}\right)^2Le^{i\theta}\Big)\Big|\geq 1$$

whenever

$$\Big(e^{i\theta},\left(1-\frac{\lambda}{l+1}\right)e^{i\theta}+\frac{\lambda}{l+1}\zeta e^{i\theta},$$

$$\left(1-\frac{\lambda}{l+1}\right)^2e^{i\theta}+\left(2\frac{\lambda}{l+1}-\left(\frac{\lambda}{l+1}\right)^2\right)\zeta e^{i\theta}+\left(\frac{\lambda}{l+1}\right)^2Le^{i\theta}\Big)\in D,$$

with $\mathrm{Re}\left(e^{-i\theta}L\right)>\zeta(\zeta-1)$ *for all real θ, and for $\zeta\geq 1$.*

2. Main result

To prove our main result we shall need the following lemma due to Miller and Mocanu:

Lemma 2.1. [6] *Let* $w(z) = a + w_n z^n + \ldots$ *be analytic in* U, *with* $w(z) \not\equiv a$. *If* $z_0 = r_0 e^{i\theta}$ $(0 < r_0 < 1)$, *and* $|w(z_0)| = \max_{|z| \le r_0} |w(z)|$. *Then,*

$$z_0 w'(z_0) = \zeta w(z_0),$$

and

$$\operatorname{Re}\left[1 + \frac{z_0 w''(z_0)}{w'(z_0)}\right] \ge \zeta, \tag{2.1}$$

where ζ *is a real number, and* $\zeta \ge 1$.

Theorem 2.2. *Let* $h(r,s,t) \in H$, *and let* $f \in A$ *satisfying*

$$\left(J^m(\lambda,l)f(z), J^{m-1}(\lambda,l)f(z), J^{m-2}(\lambda,l)f(z)\right) \in D \subset \mathbb{C}^3 \tag{2.2}$$

and

$$\left|h\left(J^m(\lambda,l)f(z), J^{m-1}(\lambda,l)f(z), J^{m-2}(\lambda,l)f(z)\right)\right| < 1 \tag{2.3}$$

for all $z \in$ U, *and for some* $\lambda > 0$, $l \ge 0$, *and* $m \ge 2$. *Then, we have*

$$|J^m(\lambda,l)f(z)| < 1, \ z \in \mathrm{U}.$$

Proof. If we define the function w by

$$J^m(\lambda,l)f(z) = w(z), \ m \in \mathbb{N}_0,$$

with $f \in A$, then we have $w \in A$, and $w(z) \neq 0$ at least for one $z \in$ U. With the aid of the identity (1.4), we obtain

$$J^{m-1}(\lambda,l)f(z) = \left(1 - \frac{\lambda}{l+1}\right) w(z) + \frac{\lambda}{l+1} z w'(z)$$

and

$$J^{m-2}(\lambda,l)f(z) = \left(1 - \frac{\lambda}{l+1}\right)^2 w(z) + \left(2\frac{\lambda}{l+1} - \left(\frac{\lambda}{l+1}\right)^2\right) z w'(z) + \left(\frac{\lambda}{l+1}\right)^2 z^2 w''(z).$$

We claim that $|w(z)| < 1$ for all $z \in$ U. Otherwise, there exists a point $z_0 \in$ U such that $\max_{|z|<|z_0|} |w(z)| = |w(z)| = 1$. Letting $w(z_0) = e^{i\theta}$ and using Lemma 2.1 we deduce that

$$J^m(\lambda,l)f(z_0) = w(z_0) = e^{i\theta},$$

$$J^{m-1}(\lambda,l)f(z_0) = \left(1 - \frac{\lambda}{l+1}\right) e^{i\theta} + \left(\frac{\lambda}{l+1}\right) \zeta e^{i\theta},$$

and

$$J^{m-2}(\lambda,l)f(z_0) = \left(1 - \frac{\lambda}{l+1}\right)^2 e^{i\theta} + \left(2\frac{\lambda}{l+1} - \left(\frac{\lambda}{l+1}\right)^2\right) \zeta e^{i\theta} + \left(\frac{\lambda}{l+1}\right)^2 L e^{i\theta},$$

where $L = z_0^2 w''(z_0)$, and $\zeta \geq 1$.

Further, an application of (2.1) from Lemma 2.1 gives that

$$\operatorname{Re} \frac{z_0 w''(z_0)}{w'(z_0)} = \operatorname{Re} \frac{z_0^2 w''(z_0)}{\zeta e^{i\theta}} \geq \zeta - 1,$$

or

$$\operatorname{Re}\left(e^{-i\theta} L\right) \geq \zeta(\zeta - 1).$$

Since $h(r, s, t) \in H$, we have

$$\left| h\left(J^m(\lambda, l) f(z_0), J^{m-1}(\lambda, l) f(z_0), J^{m-2}(\lambda, l) f(z_0)\right) \right|$$
$$= \left| h\Big(e^{i\theta}, \left(1 - \frac{\lambda}{l+1}\right) e^{i\theta} + \frac{\lambda}{l+1} \zeta e^{i\theta},\right.$$
$$\left.\left(1 - \frac{\lambda}{l+1}\right)^2 e^{i\theta} + \left(2\frac{\lambda}{l+1} - \left(\frac{\lambda}{l+1}\right)^2\right) \zeta e^{i\theta} + \left(\frac{\lambda}{l+1}\right)^2 L e^{i\theta}\Big)\right| \geq 1,$$

which contradicts the condition (2.3) of the theorem, and therefore we conclude that

$$|J^m(\lambda, l) f(z)| < 1,\ z \in \mathrm{U}.$$

□

Corollary 2.3. *Let $h(r, s, t) = s$ and $f \in A$ satisfying the conditions* (2.2) *and* (2.3) *for $m \geq 2$. Then,*

$$\left| J^{m+j}(\lambda, l) f(z) \right| < 1,\ z \in \mathrm{U},$$

for $j \geq 0$, $\lambda > 0$, $l \geq 0$, $m \geq 2$.

Proof. Since $h(r, s, t) = s \in H$, with the aid of the above theorem we have that

$$\left| J^{m-1}(\lambda, l) f(z) \right| < 1,\ z \in \mathrm{U},$$

implies

$$|J^m(\lambda, l) f(z)| < 1,\ z \in \mathrm{U},\ (m \geq 2),$$

and from here it follows

$$\left| J^{m+j}(\lambda, l) f(z) \right| < 1,\ z \in \mathrm{U},\ (j \geq 0).$$

□

Remark 2.4. (i) Putting $l = 0$ and $\lambda = 1$ in the above results we obtain the results obtained by Aouf et al. [1];

(ii) Putting $\lambda = l = 1$ in the above results we obtain the results obtained by Aouf et al. [2, Theorem 1 and Corollary 1] respectively;

(iii) Putting $l = 0$ in the above results we obtain the corresponding results for the operator $J_\lambda^{-m} f(z)$.

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Rabha M. El-Ashwah
Department of Mathematics, Faculty of Science (Damietta Branch)
Mansoura University, New Damietta 34517, Egypt
e-mail: r_elashwah@yahoo.com

Mohamed K. Aouf
Department of Mathematics, Faculty of Science
Mansoura University, Mansoura 35516, Egypt
e-mail: mkaouf127@yahoo.com

Teodor Bulboacă
Faculty of Mathematics and Computer Science
Babeş-Bolyai University
400084 Cluj-Napoca, Romania
e-mail: bulboaca@math.ubbcluj.ro