Stud. Univ. Babeş-Bolyai Math. 57(2012), No. 2, 181–187

Subordination of certain subclass of convex function

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Abstract. In this paper we study the subordination of a certain subclass of convex functions with negative coefficients.

Mathematics Subject Classification (2010): 30C45.

Keywords: Analytic functions, positive coefficients, negative coefficients, subordination, convolution product, Pochammer symbol.

1. Introduction

Let $\mathcal{H}(U)$ be the set of functions which are regular in the unit disc U,

$$\mathcal{A} = \{f \in \mathcal{H}(U) : f(0) = f'(0) - 1 = 0\}$$

and $S = \{f \in \mathcal{A} : f \text{ is univalent in } U\}$.

In [11], the subfamily T of S consisting of functions f,

$$f(z) = z - \sum_{j=2}^{\infty} a_j z^j, \;\; a_j \geq 0, j = 2, 3, ..., \;\; z \in U, \tag{1.1}$$

was introduced.

Thus, we have the subfamily $S - T$ consisting of functions f of the form

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j, \;\; a_j \geq 0, j = 2, 3, ..., \;\; z \in U \tag{1.2}$$

Let consider N to be the class of all functions Φ which are analytic, convex, univalent in U and normalized by $\Phi(0) = 1$, $Re(\Phi(z)) > 0$ $(z \in U)$. Making use of the subordination principle of the analytic functions, many authors investigated the subclasses $S^\star(\Phi)$, $K(\Phi)$ and $C(\Phi, \psi)$ of the class $\mathcal{A}$, $\Phi, \psi \in N$ (see [4]), as follows:

$$S^\star(\Phi) := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \Phi(z) \in U \right\}$$

$$K(\Phi) := \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \Phi(z) \in U \right\} \tag{1.3}$$

$$C(\Phi, \psi) := \left\{ f \in \mathcal{A} : \exists g \in S^\star(\Phi)\ s.t.\ \frac{zf'(z)}{g(z)} \prec \psi(z) \in U \right\}.$$

Let $g(z) \in \mathcal{A}$, $g(z) = z + \sum_{j \geq 2} b_j z^j$. Then, the Hadamard product (or convolution) $f * g$ is defined by

$$f(z) * g(z) = (f * g)(z) = z + \sum_{j \geq 2} a_j b_j z^j.$$

If $g(z) \in \mathcal{A}$, $g(z) = z - \sum_{j \geq 2} b_j z^j$, the Hadamard product (or convolution) $f * g$ is defined by

$$f(z) * g(z) = (f * g)(z) = z - \sum_{j \geq 2} a_j b_j z^j.$$

Next, we have the basic idea of subordination as following: if f and g are analytic in U, then the function f is said to be subordinate to g, such as

$$f \prec g \text{ or } f(z) \prec g(z)\ (z \in U),$$

iff there exist the Schwarz function w, analytic in U, with $w(0) = 0$ and $|w(z)| < 1$, such that $f(z) = g(w(z))\ (z \in U)$.

Let $\psi : \mathbb{C}^2 \times U \to \mathbb{C}$ and h analytic in U. If p and $\psi(p(z), zp'(z); z)$ are univalent in U and satisfy the first-order differential superordination

$$h(z) \prec \psi(p(z), zp'(z); z), \quad for\ z \in U, \tag{1.4}$$

then p is considered to be a function of differential superordination. The analytic function q is a subordination of the differential superordination solutions, or more simple a subordination, if $q \prec p$ for all p that satisfy (1.4).

An univalent subordination $\tilde{q}$ that satisfies $q \prec \tilde{g}$ for all subordinations (1.4) is said to be the best subordination for (1.4). The best subordination is unique up to a rotation of U.

We continue our paper with already studied operators and known theories concerning the subordination principle that have to help us in our research.

2. Preliminary results

Let D^n be the Sălăgean differential operator (see [10]) $D^n : \mathcal{A} \to \mathcal{A}$, $n \in \mathbb{N}$, defined as:

$$D^0 f(z) = f(z),\ D^1 f(z) = Df(z) = zf'(z),\ D^n f(z) = D(D^{n-1} f(z)) \tag{2.1}$$

and $D^k,\ D^k : \mathcal{A} \to \mathcal{A},\ k \in \mathbb{N} \cup \{0\}$, of form:

$$D^0 f(z) = f(z),\ \ldots,\ D^k f(z) = D(D^{k-1} f(z)) = z + \sum_{n=2}^{\infty} n^k a_n z^n. \tag{2.2}$$

Definition 2.1. [5] *Let* $\beta, \lambda \in \mathbb{R}$, $\beta \geq 0$, $\lambda \geq 0$ *and* $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$. *We denote by* D_λ^β *the linear operator defined by*

$$D_\lambda^\beta : A \to A, \quad D_\lambda^\beta f(z) = z + \sum_{j=n+1}^{\infty} [1 + (j-1)\lambda]^\beta a_j z^j . \tag{2.3}$$

Remark 2.2. In [1], we have introduced the following operator concerning the functions of form (1.1):

$$D_\lambda^\beta : A \to A, \quad D_\lambda^\beta f(z) = z - \sum_{j=n+1}^{\infty} [1 + (j-1)\lambda]^\beta a_j z^j . \tag{2.4}$$

The neighborhoods concerning the class of functions defined using the operator (2.4) is studied in [3].

Definition 2.3. [13] *We denote by* Q *the set of functions that are analytic and injective on* $\overline{U} - E(f)$, *where* $E(f) = \{\zeta \in \partial U : \lim_{z \to \zeta} f(z) = \infty\}$, *and* $f'(\zeta) \neq 0$ *for* $\zeta \in \partial U - E(f)$. *The subclass of* Q *for which* $f(0) = a$ *is denoted by* $Q(a)$.

Lemma 2.4. [13] *Let* h *be a convex function with* $h(0) = a$, *and let* $\gamma \in \mathbb{C} - \{0\}$ *be a complex number with* $Re\gamma \geq 0$. *If* $p \in \mathcal{H}[a, n] \cap Q$, $p(z) + \frac{1}{\gamma} zp'(z)$ *is univalent in* U *and*

$$h(z) \prec p(z) + \frac{1}{\gamma} zp'(z), \ for \ z \in U,$$

then

$$q(z) \prec p(z), \ for \ z \in U,$$

where $q(z) = \dfrac{\gamma}{nz^{\gamma/n}} \displaystyle\int_0^z h(t) t^{\gamma/n - 1} dt$, *for* $z \in U$. *The function* q *is convex and it is the best subordination.*

Lemma 2.5. [13] *Let* q *be a convex function and let* $h(z) = q(z) + \frac{1}{\gamma} zq'(z)$, *for* $z \in U$, *where* $Re\gamma \geq 0$. *If* $p \in \mathcal{H}[a, n] \cap Q$, $p(z) + \frac{1}{\gamma} zp'(z)$ *is univalent in* U *and*

$$q(z) + \frac{1}{\gamma} zp'(z) \prec p(z) + \frac{1}{\gamma} zp'(z), \ for \ z \in U,$$

then

$$q(z) \prec p(z), \ for \ z \in U,$$

where $q(z) = \dfrac{\gamma}{nz^{\gamma/n}} \displaystyle\int_0^z h(t) t^{\gamma/n - 1} dt$, *for* $z \in U$. *The function* q *is the best subordination.*

Definition 2.6. *For* $f \in \mathcal{A}$, *the generalized derivative operator* $\mu_{\lambda_1, \lambda_2}^{n,m}$ *is defined by*

$$\mu_{\lambda_1, \lambda_2}^{n,m} : \mathcal{A} \to \mathcal{A}$$

$$\mu^{n,m}_{\lambda_1,\lambda_2} f(z) = z + \sum_{k\geq 2} \frac{[1+\lambda_1(k-1)]^{m-1}}{[1+\lambda_2(k-1)]^m} c(n,k) a_k z^k, \ (z \in U), \tag{2.5}$$

where $n, m \in \mathbb{N}$, $\lambda_2 \geq \lambda_1 \geq 0$ *and* $c(n,k) = \dfrac{(n+1)_{k-1}}{(1)_{k-1}}$, $(x)_k$ *is the Pochammer symbol (or the shifted factorial).*

Remark 2.7. If we denote by $(x)_k$ the Pochammer symbol, we define it as follows:

$$(x)_k = \begin{cases} 1 & \text{for } \ k = 0, \ x \in \mathbb{C} - \{0\} \\ x(x+1)(x+2)\cdot \ldots \cdot (x+k-1) & \text{for } \ k \in \mathbb{N}^\star \text{ and } x \in \mathbb{C}. \end{cases}$$

Lemma 2.8. [14] *Let* $0 < a \leq c$. *If* $c \geq 2$ *or* $a + c \geq 3$, *then the function*

$$h(a, c; z) = z + \sum_{k\geq 2} \frac{(a)_{k-1}}{(c)_{k-1}} z^k \ (z \in U),$$

belongs to the class K *of convex functions (defined in (*1.3*)).*

Lemma 2.9. [12] *Let* $\Phi \in \mathcal{A}$, *convex in* U, *with* $\Phi(0) = 1$ *and*

$$Re(\beta\Phi(z) + \gamma) > 0 \ (\beta, \gamma \in \mathbb{C}\,; \ z \in U).$$

If $p(z)$ *is analytic in* U, $p(0) = \Phi(0)$, *then*

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec \Phi(z) \Rightarrow p(z) \prec \Phi(z).$$

Next we study the subordination of a certain subclass of convex functions defined by using the Hadamard (convolution) product.

3. Main results

We consider the following operator (see [8], [9]):

$$\psi_1(z) = \sum_{k\geq 1} \frac{1+c}{k+c} z^k \ (Re\{c\} \geq 0\,; \ z \in U). \tag{3.1}$$

Let the operator $D^{n,\beta}_{\lambda_1,\lambda_2} f(z)$, $n \in \mathbb{N}, \beta \geq 0$, $\lambda_1, \lambda_2 \geq 0$ to be the following:

$$D^{n,\beta}_{\lambda_1,\lambda_{2_\ominus}} f(z) = \mu^{n,\beta}_{\lambda_1,\lambda_2} f(z) * \psi_1(z)$$

$$= z - \sum_{k\geq 2} \frac{[1-\lambda_1(k-1))]^{\beta-1}}{[1-\lambda_2(k-1))]^{\beta}} \cdot \frac{1+c}{k+c} \cdot c(n,k) \cdot a_k z^k, \tag{3.2}$$

where $f(z)$ is of form (1.1) and

$$D^{n,\beta}_{\lambda_1,\lambda_{2_\oplus}} f(z) = \mu^{n,\beta}_{\lambda_1,\lambda_2} f(z) * \psi_1(z)$$

$$= z + \sum_{k\geq 2} \frac{[1-\lambda_1(k-1))]^{\beta-1}}{[1-\lambda_2(k-1))]^{\beta}} \cdot \frac{1+c}{k+c} \cdot c(n,k) \cdot a_k z^k, \tag{3.3}$$

where $f(z)$ is of form (1.2).

Furthermore, we consider $D^{n,\beta}_{\lambda_1,\lambda_2} f(z)$ to be of form (3.2) or (3.3).

Definition 3.1. *Let $f(z)$ of form (1.2), $z \in U$. We say that f is in the class $K_\lambda^\beta(\Phi(z))$ if:*

$$1 + \frac{z(D_{\lambda_1,\lambda_2}^{n,\beta} f(z))''}{(D_{\lambda_1,\lambda_2}^{n,\beta} f(z))'} \prec \Phi(z), \quad n \in \mathbb{N}, \beta \geq 0, \ \lambda_1, \lambda_2 \geq 0, \quad z \in U,$$

where the function Φ is analytic, convex and univalent in U, normalized by

$$\Phi(0) = 1, \ Re(\Phi(z)) > 0 \ (z \in U).$$

Remark 3.2. From Definition 3.1, we have the class $K_\lambda^\beta(\Phi(z))$ as follows:

$$K_\lambda^\beta(\Phi(z)) = \left\{ f(z) \in S : 1 + \frac{z(D_{\lambda_1,\lambda_2}^{n,\beta} f(z))''}{(D_{\lambda_1,\lambda_2}^{n,\beta} f(z))'} \prec \Phi(z), \ \Phi(z) \in S, \ \Phi \text{ is convex}, z \in U \right\},$$

where $n \in \mathbb{N}, \beta \geq 0, \ \lambda_1, \lambda_2 \geq 0$.

Theorem 3.3. *Let the function $\Phi(z)$ to be analytic, convex and univalent in U, normalized by $\Phi(0) = 1$, $Re(\Phi(z)) > 0$ $(z \in U)$. Let $\lambda \geq 0$, $\gamma, \chi \in \mathbb{C}$, with $Re(\chi\Phi(z) + \gamma) > 0$, $n \in \mathbb{N}$, $f \in \mathcal{A}_n$ and suppose that*

$$[D_{\lambda_1,\lambda_2}^{n,\beta} f(z)]' + \frac{(n+1)[(D_{\lambda_1,\lambda_2}^{n+1,\beta} f(z) - D_{\lambda_1,\lambda_2}^{n,\beta} f(z)]'}{\chi[D_{\lambda_1,\lambda_2}^{n,\beta} f(z)]' + \gamma}, \ \beta \geq 0, \ \lambda_1, \lambda_2 \geq \ (z \in U),$$

is univalent and the operator $D_{\lambda_1,\lambda_2}^{n,\beta} f(z)$ is in $\mathcal{H}[1,n] \cap Q$. If

$$\Phi(z) \prec [D_{\lambda_1,\lambda_2}^{n,\beta} f(z)]' + \frac{(n+1)[(D_{\lambda_1,\lambda_2}^{n+1,\beta} f(z) - D_{\lambda_1,\lambda_2}^{n,\beta} f(z)]'}{\chi[D_{\lambda_1,\lambda_2}^{n,\beta} f(z)]' + \gamma}, \ z \in U, \tag{3.4}$$

then

$$q(z) \prec [D_{\lambda_1,\lambda_2}^{n,\beta} f(z)]' \ for \ z \in U,$$

where $q(z) = \dfrac{1}{nz^{\frac{u(z)}{n}}} \cdot \displaystyle\int_0^z \Phi(t) \cdot t^{\frac{u(t)}{n-1}} \, dt$, $u(z) = \chi\Phi(z) + \gamma, \ z \in U$.

Proof. We are going to prove the Theorem 3.3 by taking into account the operator $D_{\lambda_1,\lambda_{2\oplus}}^{n,\beta} f(z)$. We use the notation $D_{\lambda_1,\lambda_2}^{n,\beta} f(z) = D_{\lambda_1,\lambda_{2\oplus}}^{n,\beta} f(z)$ for simplification.

Let

$$\Phi(z) = \frac{z(D_{\lambda_1,\lambda_2}^{n,\beta} f(z))'}{(D_{\lambda_1,\lambda_2}^{n,\beta} f(z))} = \frac{[h_{\lambda_1,\lambda_2}^{n,\beta} * \psi_1 * zf'](z)}{[h_{\lambda_1,\lambda_2}^{n,\beta} * \psi_1 * f](z)},$$

where $h_{\lambda_1,\lambda_2}^{n,\beta}(z) = z + \displaystyle\sum_{k \geq 2} \frac{[1 - \lambda_1(k-1))]^{\beta-1}}{[1 - \lambda_2(k-1))]^{\beta}} \cdot c(n,k) \cdot a_k z^k, \ \beta \geq 0, \ \lambda_1, \lambda_2 \geq \ (z \in U)$.

Thus, we obtain from $Re(\chi\Phi(z) + \gamma) > 0$ that $|\chi| + |\gamma| \leq 2$.

We consider $p(z) = [D_{\lambda_1,\lambda_2}^{n,\beta} f(z)]'$ and we obtain the following:

$$p(z) + zp'(z) = [h_{\lambda_1,\lambda_2}^{n,\beta} * \psi_1 * zf'](z) = [D_{\lambda_1,\lambda_2}^{n,\beta} f(z)]' + (n+1)[[D_{\lambda_1,\lambda_2}^{n+1,\beta} f(z)]' - [D_{\lambda_1,\lambda_2}^{n,\beta} f(z)]']$$

and

$$p(z)+\frac{zp'(z)}{\chi p(z)+\gamma}=[D^{n,\beta}_{\lambda_1,\lambda_2}f(z)]'\cdot\left[1-\frac{n+1}{\chi[D^{n,\beta}_{\lambda_1,\lambda_2}f(z)]'+\gamma}\right]+\frac{(n+1)[D^{n+1,\beta}_{\lambda_1,\lambda_2}f(z)]'}{\chi[D^{n,\beta}_{\lambda_1,\lambda_2}f(z)]'+\gamma}$$

$$=[D^{n,\beta}_{\lambda_1,\lambda_2}f(z)]'+\frac{(n+1)[(D^{n+1,\beta}_{\lambda_1,\lambda_2}f(z)-D^{n,\beta}_{\lambda_1,\lambda_2}f(z)]'}{\chi[D^{n,\beta}_{\lambda_1,\lambda_2}f(z)]'+\gamma}.$$

It is obviously that $p\in\mathcal{H}[1,n]$.

Further, we see that (3.4) can be written as follows

$$\Phi(z)\prec p(z)+\frac{zp'(z)}{\chi p(z)+\gamma},\ z\in U.$$

Making use of Lemma 2.9, we obtain

$$q(z)\prec p(z),\ z\in U\ i.e.\ q(z)\prec[D^{n,\beta}_{\lambda_1,\lambda_2}f(z)]'\ for\ z\in U,$$

where $q(z)=\frac{1}{nz^{\frac{u(z)}{n}}}\cdot\int\limits_0^z\Phi(t)\cdot t^{\frac{u(t)}{n-1}}\,dt$. The function q is convex and it is the best subordinant.

Remark 3.4. The proof is similar for $D^{n,\beta}_{\lambda_1,\lambda_2}f(z)$ of form (3.2).

Example 3.5. If we consider $\beta\in\mathbb{N}$ and $\psi_1(z)=1_z$ we obtain the operator $D^{n,\beta}_{\lambda_1,\lambda_{2\oplus}}f(z)$. Therefore we have

$$D^{n,\beta}_{\lambda_1,\lambda_{2\oplus}}f(z)=z+\sum_{k\geq2}\frac{[1-\lambda_1(k-1))]^{\beta-1}}{[1-\lambda_2(k-1))]^{\beta}}\cdot c(n,k)\cdot a_kz^k=\mu^{n,\beta}_{\lambda_1,\lambda_2}f(z),$$

which is a particular case of the Theorem 3.3. Thus, the open problem from [5] concerning the subordination of the class of convex functions is solved.

Acknowledgment. This work was partially supported by the strategic project POSDRU 107/1.5/S/77265, inside POSDRU Romania 2007-2013 co-financed by the European Social Fund-Investing in People.

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