Stud. Univ. Babeş-Bolyai Math. 57(2012), No. 2, 175–180

On conditions for univalence of two integral operators

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Abstract. In this paper we consider two integral operators. These operators was made based on the fact that the number of functions from their composition is entire part of the complex number modulus. The complex number is equal with the sum of the powers related to the functions from the composition of the integral operator.

Mathematics Subject Classification (2010): 30C45.

Keywords: Integral operator, univalence, integer part, modulus.

1. Introduction

Consider $\mathcal{U}$ the open unit disk. Let consider $\mathcal{A}$ be the class of analytic functions defined by $f(z) = z + a_2z^2 + ...$ We denote $\mathcal{S}$ be the class of univalent functions.

Theorem 1.1. [4] *If the function f belongs to the class $\mathcal{S}$, then for any complex number γ, $|\gamma| \leq \frac{1}{4}$, the function*

$$F_\gamma(z) = \int_0^z \left(\frac{f(t)}{t}\right)^\gamma dt$$

is in the class $\mathcal{S}$.

Theorem 1.2. [2] *If the function f is regular in unit disk $\mathcal{U}$, $f(z) = z + a_2z^2 + ...$ and*

$$\left(1 - |z|^2\right)\left|\frac{zf''(z)}{f'(z)}\right| \leq 1,$$

for all $z \in \mathcal{U}$, then the function f is univalent in $\mathcal{U}$.

Theorem 1.3. [5] *Let α be a complex number,* $\mathrm{Re}\alpha > 0$ *and $f(z) = z + a_2z^2 + ...$ be a regular function in $\mathcal{U}$. If*

$$\frac{1 - |z|^{2\mathrm{Re}\alpha}}{\mathrm{Re}\alpha}\left|\frac{zf''(z)}{f'(z)}\right| \leq 1,$$

for all $z \in \mathcal{U}$*, then for any complex number* β, $\mathrm{Re}\beta \geq \mathrm{Re}\alpha$*, the function*

$$F_\beta(z) = \left[\beta \int_0^z t^{\beta-1} f'(t)dt\right]^{\frac{1}{\beta}}$$

is in the class $\mathcal{S}$*.*

Theorem 1.4. [3] *If the function* g *is regular in* $\mathcal{U}$ *and* $|g(z)| < 1$ *in* $\mathcal{U}$*, then for all* $\xi \in \mathcal{U}$*, the following inequalities hold*

$$\left|\frac{g(\xi) - g(z)}{1 - \overline{g(z)}g(\xi)}\right| \leq \left|\frac{\xi - z}{1 - \overline{z}\xi}\right| \tag{1.1}$$

and

$$|g'(z)| \leq \frac{1 - |g(z)|^2}{1 - |z|^2},$$

the equalities hold in the case $g(z) = \epsilon \frac{z+u}{1+\overline{u}z}$*, where* $|\epsilon| = 1$ *and* $|u| < 1$*.*

Remark 1.5. [3] For $z = 0$, from inequality (1.1) we obtain for every $\xi \in \mathcal{U}$,

$$\left|\frac{g(\xi) - g(0)}{1 - \overline{g(0)}g(\xi)}\right| \leq |\xi|$$

and hence,

$$|g(\xi)| \leq \frac{|\xi| + |g(0)|}{1 + \overline{g(0)}g(\xi)}.$$

Considering $g(0) = a$ and $\xi = z$, then

$$|g(z)| \leq \frac{|z| + |a|}{1 + |a|\,|z|},$$

for all $z \in \mathcal{U}$.

Theorem 1.6. [7] *Let* $\gamma \in \mathbb{C}$*,* $f \in \mathcal{S}$*,* $f(z) = z + a_2 z^2 + ...$

If

$$\left|\frac{zf'(z) - f(z)}{zf(z)}\right| \leq 1, \ \forall\, z \in \mathcal{U}$$

and

$$|\gamma| \leq \frac{1}{\max_{|z|\leq 1}\left[\left(1 - |z|^2\right) \cdot |z| \cdot \frac{|z|+|a_2|}{1+|a_2||z|}\right]},$$

then

$$F_\gamma(z) = \int_0^z \left(\frac{f(t)}{t}\right)^\gamma dt$$

is in the class $\mathcal{S}$*.*

Theorem 1.7. [7] *Let* $\alpha, \beta, \gamma \in \mathbb{C}$*,* $f \in \mathcal{S}$*,* $f(z) = z + a_2 z^2 + ...$

If

$$\left|\frac{zf'(z) - f(z)}{zf(z)}\right| \leq 1, \ \forall\, z \in \mathcal{U},$$

$$\mathrm{Re}\beta \geq \mathrm{Re}\alpha > 0$$

and

$$|\gamma| \le \frac{1}{\max_{|z|\le 1}\left[\frac{1-|z|^{2\mathrm{Re}\alpha}}{\mathrm{Re}\alpha}\cdot |z| \cdot \frac{|z|+|a_2|}{1+|a_2||z|}\right]},$$

then

$$G_{\beta,\gamma}(z) = \left[\beta \int_0^z t^{\beta-1}\left(\frac{f(t)}{t}\right)^{\gamma} dt\right]^{\frac{1}{\beta}}$$

is in the class $\mathcal{S}$.

We define the next two integral operators

$$F_{[|\delta|]}(z) = \int_0^z \left(\frac{f_1(t)}{t}\right)^{\alpha_1} \cdot \ldots \cdot \left(\frac{f_{[|\delta|]}(t)}{t}\right)^{\alpha_{[|\delta|]}} dt,$$

where $\delta \in \mathbb{C}$, $|\delta| \notin [0,1)$, $\alpha_i \in \mathbb{C}$, $f_i \in \mathcal{A}$, $i = \overline{1, [|\delta|]}$, $\alpha_1 + \ldots + \alpha_{[|\delta|]} = \delta$ and

$$G_{[|\gamma|]}(z) = \left[\gamma \int_0^z t^{\gamma-1}\left(\frac{f_1(t)}{t}\right)^{\alpha_1} \cdot \ldots \cdot \left(\frac{f_{[|\gamma|]}(t)}{t}\right)^{\alpha_{[|\gamma|]}} dt\right]^{\frac{1}{\gamma}},$$

$\gamma \in \mathbb{C}$, $|\gamma| \notin [0,1)$, $\alpha_i \in \mathbb{C}$, $f_i \in \mathcal{A}$, $i = \overline{1, [|\gamma|]}$, $\alpha_1 + \ldots + \alpha_{[|\gamma|]} = \gamma$.

2. Main results

Theorem 2.1. *Let $\delta \in \mathbb{C}$, $|\delta| \notin [0,1)$, $\alpha_i \in \mathbb{C}$, for $i = \overline{1, [|\delta|]}$ and $\alpha_1 + \ldots + \alpha_{[|\delta|]} = \delta$. If $f_i \in \mathcal{A}$, $f_i(z) = z + a_2^i z^2 + \ldots$, for $i = \overline{1, [|\delta|]}$ and*

$$\left|\frac{zf_i'(z) - f_i(z)}{zf_i(z)}\right| \le 1,\ \forall i = \overline{1, [|\delta|]},\ z \in \mathcal{U}, \tag{2.1}$$

$$\frac{|\alpha_1| + \ldots + \left|\alpha_{[|\delta|]}\right|}{\left|\alpha_1 \cdot \ldots \cdot \alpha_{[|\delta|]}\right|} \le 1, \tag{2.2}$$

$$\left|\alpha_1 \cdot \ldots \cdot \alpha_{[|\delta|]}\right| \le \frac{1}{\max_{|z|\le 1}\left[\left(1-|z|^2\right)\cdot |z| \cdot \frac{|z|+|c|}{1+|c||z|}\right]}, \tag{2.3}$$

where

$$|c| = \frac{\left|\alpha_1 a_2^1 + \ldots + \alpha_{[|\delta|]} a_2^{[|\delta|]}\right|}{\left|\alpha_1 \cdot \ldots \cdot \alpha_{[|\delta|]}\right|},$$

then

$$F_{[|\delta|]}(z) = \int_0^z \left(\frac{f_1(t)}{t}\right)^{\alpha_1} \cdot \ldots \cdot \left(\frac{f_{[|\delta|]}(t)}{t}\right)^{\alpha_{[|\delta|]}} dt$$

is in the class $\mathcal{S}$.

Proof. We have $f_i \in \mathcal{A}$, for all $i = \overline{1, [|\delta|]}$ and $\frac{f_i(z)}{z} \neq 0$, for all $i = \overline{1, [|\delta|]}$.

Let g be the function $g(z) = \left(\frac{f_1(z)}{z}\right)^{\alpha_1} \cdot \ldots \cdot \left(\frac{f_{[|\delta|]}(z)}{z}\right)^{\alpha_{[|\delta|]}}$, $z \in \mathcal{U}$. We have $g(0) = 1$.

Consider the function

$$h(z) = \frac{1}{\left|\alpha_1 \cdot ... \cdot \alpha_{[|\delta|]}\right|} \cdot \frac{F''_{[|\delta|]}(z)}{F'_{[|\delta|]}(z)}, z \in \mathcal{U}.$$

The function $h(z)$ has the form:

$$h(z) = \frac{1}{\left|\alpha_1 \cdot ... \cdot \alpha_{[|\delta|]}\right|} \sum_{i=1}^{[|\delta|]} \alpha_i \frac{zf'_i(z) - f_i(z)}{zf_i(z)}.$$

Also,

$$h(0) = \frac{1}{\left|\alpha_1 \cdot ... \cdot \alpha_{[|\delta|]}\right|} \sum_{i=1}^{[|\delta|]} \alpha_i a_2^i.$$

By using the relations (2.1) and (2.2) we obtain that $|h(z)| < 1$ and

$$|h(0)| = \frac{\left|\alpha_1 a_2^1 + ... + \alpha_{[|\delta|]} a_2^{[|\delta|]}\right|}{\left|\alpha_1 \cdot ... \cdot \alpha_{[|\delta|]}\right|} = |c|.$$

Applying Remark 1.5 for the function h we obtain

$$\frac{1}{\left|\alpha_1 \cdot ... \cdot \alpha_{[|\delta|]}\right|} \cdot \left|\frac{F''_{[|\delta|]}(z)}{F'_{[|\delta|]}(z)}\right| \leq \frac{|z| + |c|}{1 + |c|\,|z|}, \forall z \in \mathcal{U}$$

and

$$\left|\left(1 - |z|^2\right) \cdot z \cdot \frac{F''_{[|\delta|]}(z)}{F'_{[|\delta|]}(z)}\right| \leq \left|\alpha_1 \cdot ... \cdot \alpha_{[|\delta|]}\right| \left(1 - |z|^2\right) \cdot |z| \cdot \frac{|z| + |c|}{1 + |c|\,|z|}, \forall z \in \mathcal{U}. \quad (2.4)$$

Consider the function $H : [0,\ 1] \to \mathbb{R}$ defined by

$$H(x) = (1 - x^2)x \frac{x + |c|}{1 + |c|\,x};\ x = |z|.$$

We have

$$H\left(\frac{1}{2}\right) = \frac{3}{8} \cdot \frac{1 + 2|c|}{2 + |c|} > 0 \Rightarrow \max_{x \in [0,1]} H(x) > 0.$$

Using this result and from (2.4) we have:

$$\left|\left(1 - |z|^2\right) \cdot z \cdot \frac{F''_{[|\delta|]}(z)}{F'_{[|\delta|]}(z)}\right| \leq \left|\alpha_1 \cdot ... \cdot \alpha_{[|\delta|]}\right| \cdot \max_{|z|<1} \left[\left(1 - |z|^2\right) \cdot |z| \cdot \frac{|z| + |c|}{1 + |c|\,|z|}\right], \forall z \in \mathcal{U}. \quad (2.5)$$

Applying the condition (2.3) in the form (2.5) we obtain that

$$\left(1 - |z|^2\right) \cdot \left|z \cdot \frac{F''_{[|\delta|]}(z)}{F'_{[|\delta|]}(z)}\right| \leq 1, \forall z \in \mathcal{U},$$

and from Theorem 1.2 we obtain that $F_{[|\delta|]} \in \mathcal{S}$.

Theorem 2.2. *Let* $\gamma,\ \delta \in \mathbb{C},\ |\gamma| \notin [0,1),\ \alpha_i \in \mathbb{C}$, *for* $i = \overline{1,[|\gamma|]}$, $\alpha_1 + ... + \alpha_{[|\gamma|]} = \gamma$. *If* $f_i \in \mathcal{A},\ f_i(z) = z + a_2^i z^2 + ...$, *for* $i = \overline{1,[|\gamma|]}$ *and*

$$\left|\frac{zf_i'(z) - f_i(z)}{zf_i(z)}\right| \le 1,\ \forall i = \overline{1,[|\gamma|]},\ z \in \mathcal{U}, \tag{2.6}$$

$$\frac{|\alpha_1| + ... + \left|\alpha_{[|\gamma|]}\right|}{\left|\alpha_1 \cdot ... \cdot \alpha_{[|\gamma|]}\right|} \le 1, \tag{2.7}$$

$$\mathrm{Re}\gamma \ge \mathrm{Re}\delta > 0,$$

$$\left|\alpha_1 \cdot ... \cdot \alpha_{[|\gamma|]}\right| \le \frac{1}{\max_{|z|\le 1}\left[\left(1-|z|^2\right)\cdot |z| \cdot \frac{|z|+|c|}{1+|c||z|}\right]}, \tag{2.8}$$

where

$$|c| = \frac{\left|\alpha_1 a_2^1 + ... + \alpha_{[|\gamma|]} a_2^{[|\gamma|]}\right|}{\left|\alpha_1 \cdot ... \cdot \alpha_{[|\gamma|]}\right|},$$

then

$$G_{[|\gamma|]}(z) = \left[\gamma \int_0^z t^{\gamma-1}\left(\frac{f_1(t)}{t}\right)^{\alpha_1} \cdot ... \cdot \left(\frac{f_{[|\gamma|]}(t)}{t}\right)^{\alpha_{[|\gamma|]}} dt\right]^{\frac{1}{\gamma}}$$

is in the class $\mathcal{S}$.

Proof. We consider the function

$$h(z) = \int_0^z \left(\frac{f_1(t)}{t}\right)^{\alpha_1} \cdot ... \cdot \left(\frac{f_{[|\gamma|]}(t)}{t}\right)^{\alpha_{[|\gamma|]}} dt.$$

Let be the function

$$p(z) = \frac{1}{\left|\alpha_1 \cdot ... \cdot \alpha_{[|\gamma|]}\right|} \cdot \frac{h''(z)}{h'(z)},\ z \in \mathcal{U}.$$

The function $p(z)$ has the form:

$$p(z) = \frac{1}{\left|\alpha_1 \cdot ... \cdot \alpha_{[|\gamma|]}\right|} \sum_{i=1}^{[|\gamma|]} \alpha_i \frac{zf_i'(z) - f_i(z)}{zf_i(z)}.$$

By using the relations (2.6) and (2.7) we obtain $|p(z)| < 1$ and

$$|p(0)| = \frac{\left|\alpha_1 a_2^1 + ... + \alpha_{[|\gamma|]} a_2^{[|\gamma|]}\right|}{\left|\alpha_1 \cdot ... \cdot \alpha_{[|\gamma|]}\right|} = |c|\,.$$

Applying Remark 1.5 for the function h we obtain

$$\frac{1}{\left|\alpha_1 \cdot ... \cdot \alpha_{[|\gamma|]}\right|} \cdot \left|\frac{h''(z)}{h'(z)}\right| \le \frac{|z| + |c|}{1 + |c|\,|z|}, \forall z \in \mathcal{U}$$

and

$$\left|\frac{1-|z|^{2\mathrm{Re}\delta}}{\mathrm{Re}\delta} \cdot z \cdot \frac{h''(z)}{h'(z)}\right| \le \left|\alpha_1 \cdot ... \cdot \alpha_{[|\gamma|]}\right| \frac{1-|z|^{2\mathrm{Re}\delta}}{\mathrm{Re}\delta} \cdot |z| \cdot \frac{|z|+|c|}{1+|c|\,|z|}, \forall z \in \mathcal{U}. \tag{2.9}$$

Consider the function $Q : [0,\ 1] \to \mathbb{R}$ defined by

$$Q(x) = \frac{1 - x^{2\mathrm{Re}\delta}}{\mathrm{Re}\delta} \cdot x \cdot \frac{x + |c|}{1 + |c|\, x};\ x = |z|\,.$$

We have $Q\left(\frac{1}{2}\right) > 0 \Rightarrow \max_{x\in[0,1]} Q(x) > 0$.

Using this result in (2.9), we have:

$$\frac{1 - |z|^{2\mathrm{Re}\delta}}{\mathrm{Re}\delta} \left|\frac{zh''(z)}{h'(z)}\right| \leq \left|\alpha_1 \cdot \ldots \cdot \alpha_{[|\gamma|]}\right| \cdot \max_{|z|<1} \left[\frac{1 - |z|^{2\mathrm{Re}\delta}}{\mathrm{Re}\delta} \cdot |z| \cdot \frac{|z| + |c|}{1 + |c|\,|z|}\right], \forall z \in \mathcal{U}. \tag{2.10}$$

Applying the condition (2.8) in the relation (2.10), we obtain that

$$\frac{1 - |z|^{2\mathrm{Re}\delta}}{\mathrm{Re}\delta} \left|\frac{zh''(z)}{h'(z)}\right| \leq 1, \forall z \in \mathcal{U}$$

and from Theorem 1.3, we obtain that $G_{[|\gamma|]} \in \mathcal{S}$.

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