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On transformations groups of $N-$linear connections on the dual bundle of $k-$tangent bundle

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Abstract. In the present paper we study the transformations for the coefficients of an $N-$linear connection on dual bundle of $k-$tangent bundle, $T^{*k}M$, by a transformation of a nonlinear connection on $T^{*k}M$. We prove that the set $\mathcal{T}$ of these transformations together with the composition of mappings isn't a group. But we give some groups of transformations of $\mathcal{T}$, which keep invariant a part of components of the local coefficients of an $N-$linear connection.

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1. Introduction

The notion of Hamilton space was introduced by Acad. R. Miron in [7] , [8] . The Hamilton spaces appear as dual via Legendre transformation, of the Lagrange spaces.

The differential geometry of the dual bundle of $k-$osculator bundle was introduced and studied by Acad. R. Miron [13] .

The importance of Lagrange and Hamilton geometries consists in the fact that the variational problems for important Lagrangians or Hamiltonians have numerous applications in various fields, as: Mathematics, Mecanics, Theoretical Physics, Theory of Dynamical Systems, Optimal Control, Biology, Economy etc.

In the present section we keep the general setting from Acad. R. Miron [13] , and subsequently we recall only some needed notions. For more details see [13] .

Let M be a real $n-$dimensional C^{∞} $-$manifold and let $\left(T^{*k}M, \pi^{*k}, M\right)$, $(k \geq 2)$, $k \in \mathbb{N})$ be the dual bundle of $k-$osculator bundle (or $k-$cotangent bundle), where the total space is:

$$T^{*k}M = T^{*k-1}M \times T^{*}M. \qquad (1.1)$$

Let $\left(x^i, y^{(1)i}, ..., y^{(k-1)i}, p_i\right), (i=1,...,n),$ be the local coordinates of a point $u=\left(x, y^{(1)}, ..., y^{(k-1)}, p\right) \in T^{*k}M$ in a local chart on $T^{*k}M$.

The change of coordinates on the manifold $T^{*k}M$ is:

$$\begin{cases} \tilde{x}^i = \tilde{x}^i\left(x^1, ..., x^n\right), \det\left(\frac{\partial \tilde{x}^i}{\partial x^j}\right) \neq 0, \\ \tilde{y}^{(1)i} = \frac{\partial \tilde{x}^i}{\partial x^j} y^{(1)j}, \\ .... \\ (k-1)\,\tilde{y}^{(k-1)i} = \frac{\partial \tilde{y}^{(k-2)i}}{\partial x^j} y^{(1)j} + ... + (k-1)\,\frac{\partial \tilde{y}^{(k-2)i}}{\partial y^{(k-2)j}} y^{(k-1)j}, \\ \tilde{p}_i = \frac{\partial x^j}{\partial \tilde{x}^i} p_j, \end{cases} \tag{1.2}$$

where the following relations hold:

$$\frac{\partial \tilde{y}^{(\alpha)i}}{\partial x^j} = \frac{\partial \tilde{y}^{(\alpha+1)i}}{\partial y^{(1)j}} = ... = \frac{\partial \tilde{y}^{(k-1)i}}{\partial y^{(k-1-\alpha)j}}, \left(\alpha = 0, ..., k-2; y^{(0)} = x\right). \tag{1.3}$$

$T^{*k}M$ is a real differential manifold of dimension $(k+1)\,n$.

With respect to (1.1) the natural basis of the vector space $T_u\left(T^{*k}M\right)$ at the point $u \in T^{*k}M$:

$$\left\{ \left.\frac{\partial}{\partial x^i}\right|_u, \left.\frac{\partial}{\partial y^{(1)i}}\right|_u, ..., \left.\frac{\partial}{\partial y^{(k-1)i}}\right|_u, \left.\frac{\partial}{\partial p_i}\right|_u \right\} \tag{1.4}$$

is transformed as follows:

$$\begin{cases} \left.\frac{\partial}{\partial x^i}\right|_u = \frac{\partial \tilde{x}^j}{\partial x^i} \left.\frac{\partial}{\partial \tilde{x}^j}\right|_u + \frac{\partial \tilde{y}^{(1)j}}{\partial x^i} \left.\frac{\partial}{\partial \tilde{y}^{(1)j}}\right|_u + ... + \frac{\partial \tilde{y}^{(k-1)j}}{\partial x^i} \left.\frac{\partial}{\partial \tilde{y}^{(k-1)j}}\right|_u + \frac{\partial \tilde{p}_j}{\partial x^i} \left.\frac{\partial}{\partial \tilde{p}_j}\right|_u, \\ \left.\frac{\partial}{\partial y^{(1)i}}\right|_u = \frac{\partial \tilde{y}^{(1)j}}{\partial y^{(1)i}} \left.\frac{\partial}{\partial \tilde{y}^{(1)j}}\right|_u + ... + \frac{\partial \tilde{y}^{(k-1)j}}{\partial y^{(1)i}} \left.\frac{\partial}{\partial \tilde{y}^{(k-1)j}}\right|_u, \\ .... \\ \left.\frac{\partial}{\partial y^{(k-1)i}}\right|_u = \frac{\partial \tilde{y}^{(k-1)j}}{\partial y^{(k-1)i}} \left.\frac{\partial}{\partial \tilde{y}^{(k-1)j}}\right|_u, \\ \left.\frac{\partial}{\partial p_i}\right|_u = \frac{\partial x^j}{\partial \tilde{x}^i} \left.\frac{\partial}{\partial \tilde{p}_j}\right|_u, \end{cases} \tag{1.5}$$

the conditions (1.3) being satisfied.

The null section $0 : M \to T^{*k}M$ of the projection π^{*k} is defined by $0\,(x) \in M \to (x, 0, ..., 0) \in T^{*k}M$. We denote $\widetilde{T^{*k}M} = T^{*k}M \setminus \{0\}$.

Let us consider the tangent bundle of the differentiable manifold $T^{*k}M\left(TT^{*k}M, d\pi^{*k}, T^{*k}M\right)$, where $d\pi^{*k}$ is the canonical projection and the vertical distribution $V : u \in T^{*k}M \to V\,(u) \in T_uT^{*k}M$, locally generated by the vector fields: $\left\{\frac{\partial}{\partial y^{(1)i}}, ..., \frac{\partial}{\partial y^{(k-1)i}}, \frac{\partial}{\partial p_i}\right\}$ at every point $u \in T^{*k}M$.

The following $\mathcal{F}\left(T^{*k}M\right) -$ linear mapping:

$$J : \chi\left(T^{*k}M\right) \to \chi\left(T^{*k}M\right),$$

defined by:

$$J\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial y^{(1)i}}, J\left(\frac{\partial}{\partial y^{(1)i}}\right) = \frac{\partial}{\partial y^{(2)i}}, ..., J\left(\frac{\partial}{\partial y^{(k-2)i}}\right) =$$

$$= \frac{\partial}{\partial y^{(k-1)i}}, J\left(\frac{\partial}{\partial y^{(k-1)i}}\right) = 0, J\left(\frac{\partial}{\partial p_i}\right) = 0, \tag{1.6}$$

at every point $u \in \widetilde{T^{*k}M}$ is a tangent structure on $T^{*k}M$.

We denote with N a nonlinear connection on the manifold $T^{*k}M$, with the coefficients:

$$\left(\underset{(1)}{N}{}^{j}{}_{i}\left(x, y^{(1)}, ..., y^{(k-1)}, p\right), ..., \underset{(k-1)}{N}{}^{j}{}_{i}\left(x, y^{(1)}, ..., y^{(k-1)}, p\right),\right.$$

$$\left.N_{ij}\left(x, y^{(1)}, ..., y^{(k-1)}, p\right)\right), (i, j = 1, 2, ..., n).$$

The tangent space of $T^{*k}M$ in the point $u \in T^{*k}M$ is given by the direct sum of vector spaces:

$$T_u\left(T^{*k}M\right) = N_{0,u} \oplus N_{1,u} \oplus ... \oplus N_{k-2,u} \oplus V_{k-1,u} \oplus W_{k,u}, \ \forall u \in T^{*k}M \tag{1.7}$$

A local adapted basis to the direct decomposition (1.7) is given by:

$$\left\{\frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^{(1)i}}, ..., \frac{\delta}{\delta y^{(k-1)i}}, \frac{\delta}{\delta p_i}\right\}, (i = 1, 2, ..., n), \tag{1.8}$$

where:

$$\begin{cases} \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - \underset{(1)}{N}{}^{j}{}_{i}\frac{\partial}{\partial y^{(1)j}} - ... - \underset{(k-1)}{N}{}^{j}{}_{i}\frac{\partial}{\partial y^{(k-1)j}} + N_{ij}\frac{\partial}{\partial p_j}, \\ \frac{\delta}{\delta y^{(1)i}} = \frac{\partial}{\partial y^{(1)i}} - \underset{(1)}{N}{}^{j}{}_{i}\frac{\partial}{\partial y^{(2)j}} - ... - \underset{(k-2)}{N}{}^{j}{}_{i}\frac{\partial}{\partial y^{(k-1)j}}, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \frac{\delta}{\delta y^{(k-1)i}} = \frac{\partial}{\partial y^{(k-1)i}}, \\ \frac{\delta}{\delta p_i} = \frac{\partial}{\partial p_i}. \end{cases} \tag{1.9}$$

Under a change of local coordinates on $T^{*k}M$, the vector fields of the adapted basis transform by the rule:

$$\frac{\delta}{\delta x^i} = \frac{\partial \tilde{x}^j}{\partial x^i}\frac{\delta}{\delta \tilde{x}^j}, \frac{\delta}{\delta y^{(1)i}} = \frac{\partial \tilde{x}^j}{\partial x^i}\frac{\delta}{\delta \tilde{y}^{(1)j}}, ..., \frac{\delta}{\delta y^{(k-1)i}} = \frac{\partial \tilde{x}^j}{\partial x^i}\frac{\delta}{\delta \tilde{y}^{(k-1)j}}, \frac{\delta}{\delta p_i} = \frac{\delta x^j}{\delta \tilde{x}^i}\frac{\delta}{\delta \tilde{p}_j}. \tag{1.10}$$

The dual basis of the adapted basis (1.8) is given by:

$$\left\{\delta x^i, \delta y^{(1)i}, ..., \delta y^{(k-1)i}, \delta p_i\right\}, \tag{1.11}$$

where:

$$\begin{cases} dx^i = \delta x^i, \\ dy^{(1)i} = \delta y^{(1)i} - \underset{(1)}{N}{}^{i}{}_{j}\delta x^j, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ dy^{(k-1)i} = \delta y^{(k-1)i} - \underset{(1)}{N}{}^{i}{}_{j}\delta y^{(k-2)j} - ... - \underset{(k-2)}{N}{}^{i}{}_{j}\delta y^{(1)j} - \underset{(k-1)}{N}{}^{i}{}_{j}\delta x^j, \\ dp_i = \delta p_i + N_{ji}\delta x^j. \end{cases} \tag{1.12}$$

With respect to (1.2) the covector fields (1.11) are transformed by the rules:

$$\begin{matrix} \delta \tilde{x}^i = \frac{\partial \tilde{x}^i}{\partial x^j}\delta x^j, \delta \tilde{y}^{(1)i} = \frac{\partial \tilde{x}^i}{\partial x^j}\delta y^{(1)j}, ..., \delta \tilde{y}^{(k-1)i} = \frac{\partial \tilde{x}^i}{\partial x^j}\delta y^{(k-1)j}, \\ \delta \tilde{p}_i = \frac{\partial x^j}{\partial \tilde{x}^i}\delta p_j. \end{matrix} \tag{1.13}$$

Let D be an $N-$linear connection on $T^{*k}M$, with the local coefficients in the adapted basis (1.8) :

$$D\Gamma(N) = \left(H^i{}_{jh}, \underset{(\alpha)}{C}{}^i{}_{jh}, C_i^{jh} \right), (\alpha = 1, ..., k-1). \tag{1.14}$$

An $N-$linear connection D is uniquely represented in the adapted basis in the following form:

$$\begin{array}{l} D_{\frac{\delta}{\delta x^j}} \frac{\delta}{\delta x^i} = H^s{}_{ij} \frac{\delta}{\delta x^s}, D_{\frac{\delta}{\delta x^j}} \frac{\delta}{\delta y^{(\alpha)i}} = H^s{}_{ij} \frac{\delta}{\delta y^{(\alpha)s}}, (\alpha = 1, ..., k-1), \\ D_{\frac{\delta}{\delta x^j}} \frac{\delta}{\delta p_i} = -H^i_{sj} \frac{\delta}{\delta p_s}, \\ D_{\frac{\delta}{\delta y^{(\alpha)j}}} \frac{\delta}{\delta x^i} = \underset{(\alpha)}{C}{}^s{}_{ij} \frac{\delta}{\delta x^s}, D_{\frac{\delta}{\delta y^{(\alpha)j}}} \frac{\delta}{\delta y^{(\beta)i}} = \underset{(\alpha)}{C}{}^s{}_{ij} \frac{\delta}{\delta y^{(\beta)s}}, \\ D_{\frac{\delta}{\delta y^{(\alpha)j}}} \frac{\delta}{\delta p_i} = -\underset{(\alpha)}{C}{}^i{}_{sj} \frac{\delta}{\delta p_s}, (\alpha, \beta = 1, ..., k-1), \\ D_{\frac{\delta}{\delta p_j}} \frac{\delta}{\delta x^i} = C_i{}^{js} \frac{\delta}{\delta x^s}, D_{\frac{\delta}{\delta p_j}} \frac{\delta}{\delta y^{(\alpha)i}} = C_i{}^{js} \frac{\delta}{\delta y^{(\alpha)s}}, (\alpha = 1, ..., k-1), \\ D_{\frac{\delta}{\delta p_j}} \frac{\delta}{\delta p_i} = -C_s{}^{ij} \frac{\delta}{\delta p_s}. \end{array} \tag{1.15}$$

2. The set of the transformations of $N-$linear connections

Let $\bar{N}$ be another nonlinear connection on $T^{*k}M$, with the local coefficients

$$\left(\underset{(1)}{\bar{N}}{}^j{}_i \left(x, y^{(1)}, ..., y^{(k-1)}, p \right), ..., \underset{(k-1)}{\bar{N}}{}^j{}_i \left(x, y^{(1)}, ..., y^{(k-1)}, p \right), \ N_{ij} \left(x, y^{(1)}, ..., y^{(k-1)}, p \right) \right)$$

$(i, j = 1, 2, ..., n)$.

Then there exists the uniquely determined tensor fields

$$\underset{(\alpha)}{A}{}^j{}_i \in \tau_1^1 \left(T^{*k}M \right), (\alpha = 1, ..., k-1)$$

and $A_{ij} \in \tau_2^0 \left(T^{*k}M \right)$, such that:

$$\begin{cases} \underset{(\alpha)}{\bar{N}}{}^i{}_j = \underset{(\alpha)}{N}{}^i{}_j - \underset{(\alpha)}{A}{}^i{}_j, (\alpha = 1, 2, ..., k-1), \\ \bar{N}_{ij} = N_{ij} - A_{ij}, (i, j = 1, 2, ..., n). \end{cases} \tag{2.1}$$

Conversely, if $\underset{(\alpha)}{N}{}^i{}_j$ and $\underset{(\alpha)}{A}{}^i{}_j, (\alpha = 1, 2, ..., k-1)$, respectively N_{ij} and A_{ij} are given, then $\underset{(\alpha)}{\bar{N}}{}^i{}_j, (\alpha = 1, 2, ..., k-1)$, respectively $\bar{N}_{ij}$, given by (2.1) are the coefficients of a nonlinear connection.

Theorem 2.1. *Let N and $\bar{N}$ be two nonlinear connections on $T^{*k}M$, $(k \geq 2,\ k \in \mathbb{N})$ with local coefficients:*

$$\left(\underset{(1)}{N}{}^j{}_i \left(x, y^{(1)}, ..., y^{(k-1)}, p \right), ..., \underset{(k-1)}{N}{}^j{}_i \left(x, y^{(1)}, ..., y^{(k-1)}, p \right), N_{ij} \left(x, y^{(1)}, ..., y^{(k-1)}, p \right) \right),$$

$$\left(\underset{(1)}{\bar{N}}{}^j{}_i \left(x, y^{(1)}, ..., y^{(k-1)}, p \right), ..., \underset{(k-1)}{\bar{N}}{}^j{}_i \left(x, y^{(1)}, ..., y^{(k-1)}, p \right), \overline{N}_{ij} \left(x, y^{(1)}, ..., y^{(k-1)}, p \right) \right),$$

$(i, j = 1, 2, ..., n)$, *respectively.*

If D is an $N-$linear connection on $T^{*k}M$, with local coefficients

$$D\Gamma(N)=\left(H^{i}{}_{jh},\underset{(\alpha)}{C}{}^{i}{}_{jh},C_{i}^{jh}\right),\ (\alpha=1,...,k-1),$$

then the transformation: $N\longrightarrow\bar{N}$, given by (2.1) *of nonlinear connections implies for the coefficients*

$$D\Gamma(\bar{N})=\left(\bar{H}^{i}{}_{jh},\underset{(\alpha)}{\bar{C}}{}^{i}{}_{jh},\bar{C}_{i}^{jh}\right),\ (\alpha=1,...,k-1)$$

of the $\bar{N}-$linear connection D, the relations (2.2), *that is the transformation: $D\Gamma(N)\to D\Gamma(\overline{N})$ is given by:*

$$\left\{\begin{array}{l}
\bar{H}^{i}{}_{sj}=H^{i}{}_{sj}+\underset{(1)}{A}{}^{m}{}_{j}\left[\underset{(1)}{C}{}^{i}{}_{sm}+\underset{(1)}{N}{}^{l}{}_{m}\underset{(2)}{C}{}^{i}{}_{sl}+...+\underset{(k-2)}{N}{}^{l}{}_{m}\underset{(k-1)}{C}{}^{i}{}_{sl}+\underset{(1)}{N}{}^{l}{}_{m}\underset{(1)}{N}{}^{t}{}_{l}\underset{(3)}{C}{}^{i}{}_{st}+\right.\\
\left....+\left(\underset{(1)}{N}{}^{l}{}_{m}\underset{(k-3)}{N}{}^{t}{}_{l}+...+\underset{(k-3)}{N}{}^{l}{}_{m}\underset{(1)}{N}{}^{t}{}_{l}\right)\underset{(k-1)}{C}{}^{i}{}_{st}+...+\underbrace{\underset{(1)}{N}...\underset{(1)}{N}}_{(k-2)}\underset{(k-1)}{C}\right]+\\
+\underset{(2)}{A}{}^{m}{}_{j}\left[\underset{(2)}{C}{}^{i}{}_{sm}+\underset{(1)}{N}{}^{l}{}_{m}\underset{(3)}{C}{}^{i}{}_{sl}+...+\underset{(k-3)}{N}{}^{l}{}_{m}\underset{(k-1)}{C}{}^{i}{}_{sl}+...+\underbrace{\underset{(1)}{N}...\underset{(1)}{N}}_{(k-3)}\underset{(k-1)}{C}\right]+\\
+...+\underset{(k-2)}{A}{}^{m}{}_{j}\left(\underset{(k-2)}{C}{}^{i}{}_{sm}+\underset{(1)}{N}{}^{l}{}_{m}\underset{(k-1)}{C}{}^{i}{}_{sl}\right)+\underset{(k-1)}{A}{}^{m}{}_{j}\underset{(k-1)}{C}{}^{i}{}_{sm}-A_{jm}C_{s}{}^{im},\\
\underset{(1)}{\overline{C}}{}^{i}{}_{sj}=\underset{(1)}{C}{}^{i}{}_{sj}+\underset{(1)}{A}{}^{m}{}_{j}\left[\underset{(2)}{C}{}^{i}{}_{sm}+\underset{(1)}{N}{}^{r}{}_{m}\underset{(3)}{C}{}^{i}{}_{sr}+...+\underset{(k-3)}{N}{}^{r}{}_{m}\underset{(k-1)}{C}{}^{i}{}_{sr}+...+\right.\\
\left.+\underbrace{\underset{(1)}{N}\cdot...\cdot\underset{(1)}{N}}_{(k-3)}\underset{(k-1)}{C}\right]+...+\underset{(k-3)}{A}{}^{m}{}_{j}\left[\underset{(k-2)}{C}{}^{i}{}_{sm}+\underset{(1)}{N}{}^{r}{}_{m}\underset{(k-1)}{C}{}^{i}{}_{sr}\right]+\underset{(k-2)}{A}{}^{m}{}_{j}\underset{(k-1)}{C}{}^{i}{}_{sm},\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
\underset{(k-2)}{\overline{C}}{}^{i}{}_{sj}=\underset{(k-2)}{C}{}^{i}{}_{sj}+\underset{(1)}{A}{}^{l}{}_{j}\underset{(k-1)}{C}{}^{i}{}_{sl},\\
\underset{(k-1)}{\bar{C}}{}^{i}{}_{sj}=\underset{(k-1)}{C}{}^{i}{}_{sj},\\
\bar{C}_{s}{}^{ij}=C_{s}{}^{ij},\\
\underset{(1)}{A}{}^{h}{}_{i|j}=0,\\
A_{ih|j}=0,(i,j,h=1,2...,n),
\end{array}\right.\tag{2.2}$$

where $|$ denotes the $h-$covariant derivative with respect to $D\Gamma(N)$.

Proof. It follows first of all that the transformations (2.1) preserve the coefficients $\underset{(k-1)}{C}{}^{h}{}_{ij}$, C_i^{jh}.

Using the relations $(1.9), (1.15)$ and (2.1) we obtain:

$$\begin{cases} \frac{\bar{\delta}}{\delta x^i} = \frac{\delta}{\delta x^i} + \underset{(1)}{A}{}^j{}_i \frac{\partial}{\partial y^{(1)j}} + ... + \underset{(k-1)}{A}{}^j{}_i \frac{\partial}{\partial y^{(k-1)j}} - A_{ij} \frac{\partial}{\partial p_j}, \\ \frac{\bar{\delta}}{\delta y^{(1)i}} = \frac{\delta}{\delta y^{(1)i}} + \underset{(1)}{A}{}^j{}_i \frac{\partial}{\partial y^{(2)j}} + ... + \underset{(k-2)}{A}{}^j{}_i \frac{\partial}{\partial y^{(k-1)j}}, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \frac{\bar{\delta}}{\delta y^{(k-1)i}} = \frac{\delta}{\delta y^{(k-1)i}}, \\ \frac{\bar{\delta}}{\delta p_i} = \frac{\delta}{\delta p_i}. \end{cases} \tag{2.3}$$

Using $(1.15), (2.3)$ and (1.9) we get:

$$D_{\frac{\bar{\delta}}{\delta x^j}} \frac{\bar{\delta}}{\delta y^{(k-1)i}} = \bar{H}^s{}_{ij} \frac{\bar{\delta}}{\delta y^{(k-1)s}} = \bar{H}^s{}_{ij} \frac{\bar{\delta}}{\delta y^{(k-1)s}}.$$

$$D_{\frac{\bar{\delta}}{\delta x^j}} \frac{\bar{\delta}}{\delta y^{(k-1)i}} = D_{\left(\frac{\delta}{\delta x^j} + \underset{(1)}{A}{}^l{}_j \frac{\partial}{\partial y^{(1)l}} + \underset{(2)}{A}{}^l{}_j \frac{\partial}{\partial y^{(2)l}} + \dots + \underset{(k-1)}{A}{}^l{}_j \frac{\partial}{\partial y^{(k-1)l}} - A_{jl} \frac{\partial}{\partial p_l}\right)} \frac{\delta}{\delta y^{(k-1)i}} =$$

$$= H^s{}_{ij} \frac{\delta}{\delta y^{(k-1)s}} + \left(\underset{(1)}{A}{}^l{}_j \underset{(1)}{C}{}^s{}_{il} \frac{\delta}{\delta y^{(k-1)s}} + \underset{(2)}{A}{}^l{}_j \underset{(2)}{C}{}^s{}_{il} \frac{\delta}{\delta y^{(k-1)s}} + ... \right.$$

$$\left. ... + \underset{(k-1)}{A}{}^l{}_j \underset{(k-1)}{C}{}^s{}_{il} \frac{\delta}{\delta y^{(k-1)s}} - A_{jl} C_i^{ls} \frac{\delta}{\delta y^{(k-1)s}} \right) +$$

$$+ \underset{(1)}{A}{}^l{}_j \underset{(1)}{N}{}^r{}_l D_{\left(\frac{\delta}{\delta y^{(2)r}} + \underset{(1)}{N}{}^s{}_r \frac{\partial}{\partial y^{(3)s}} + \dots + \underset{(k-3)}{N}{}^s{}_r \frac{\delta}{\delta y^{(k-1)s}} \right)} \frac{\delta}{\delta y^{(k-1)i}} +$$

$$+ \underset{(1)}{A}{}^l{}_j \underset{(2)}{N}{}^r{}_l D_{\left(\frac{\delta}{\delta y^{(3)r}} + \underset{(1)}{N}{}^s{}_r \frac{\partial}{\partial y^{(4)s}} + \dots + \underset{(k-4)}{N}{}^s{}_r \frac{\delta}{\delta y^{(k-1)s}} \right)} \frac{\delta}{\delta y^{(k-1)i}} + ...$$

$$+ \underset{(1)}{A}{}^l{}_j \underset{(k-2)}{N}{}^r{}_l \underset{(k-1)}{C}{}^s{}_{ir} \frac{\delta}{\delta y^{(k-1)s}} + \underset{(2)}{A}{}^l{}_j \underset{(1)}{N}{}^r{}_l D_{\left(\frac{\delta}{\delta y^{(3)r}} + \underset{(1)}{N}{}^s{}_r \frac{\partial}{\partial y^{(4)s}} + \dots + \underset{(k-4)}{N}{}^s{}_r \frac{\delta}{\delta y^{(k-1)s}} \right)} \frac{\delta}{\delta y^{(k-1)i}} +$$

$$+ ... + \underset{(2)}{A}{}^l{}_j \underset{(k-3)}{N}{}^r{}_l \underset{(k-1)}{C}{}^s{}_{ir} \frac{\delta}{\delta y^{(k-1)s}} + ... =$$

$$= \left(H^s{}_{ij} + \underset{(1)}{A}{}^l{}_j \underset{(1)}{C}{}^s{}_{il} + \underset{(2)}{A}{}^l{}_j \underset{(2)}{C}{}^s{}_{il} + ... + \underset{(k-1)}{A}{}^l{}_j \underset{(k-1)}{C}{}^s{}_{il} - A_{jl} C_i^{ls} \right) \frac{\delta}{\delta y^{(k-1)s}} +$$

$$+ \left(\underset{(1)}{A}{}^l{}_j \underset{(1)}{N}{}^r{}_l \underset{(2)}{C}{}^s{}_{ir} \frac{\delta}{\delta y^{(k-1)s}} + \underset{(1)}{A}{}^l{}_j \underset{(2)}{N}{}^r{}_l \underset{(3)}{C}{}^s{}_{ir} \frac{\delta}{\delta y^{(k-1)s}} + ... + \underset{(1)}{A}{}^l{}_j \underset{(k-2)}{N}{}^r{}_l \underset{(k-1)}{C}{}^s{}_{ir} \right.$$

$$\left. \frac{\delta}{\delta y^{(k-1)s}} \right) + \left(\underset{(1)}{A}{}^l{}_j \underset{(1)}{N}{}^r{}_l \underset{(1)}{N}{}^s{}_r D_{\frac{\partial}{\partial y^{(3)s}}} \frac{\delta}{\delta y^{(k-1)i}} + ... + \underset{(1)}{A}{}^l{}_j \underset{(1)}{N}{}^r{}_l \underset{(k-3)}{N}{}^s{}_r \underset{(k-1)}{C}{}^m{}_{is} \right.$$

$$\left. \frac{\delta}{\delta y^{(k-1)m}} \right) + \left(\underset{(1)}{A}{}^l{}_j \underset{(2)}{N}{}^r{}_l \underset{(1)}{N}{}^s{}_r D_{\frac{\partial}{\partial y^{(4)s}}} \frac{\delta}{\delta y^{(k-1)i}} + ... + \underset{(1)}{A}{}^l{}_j \underset{(2)}{N}{}^r{}_l \underset{(k-4)}{N}{}^s{}_r \underset{(k-1)}{C}{}^m{}_{is} \right.$$

$$\frac{\delta}{\delta y^{(k-1)m}}\Big) + ... + \Big(\underset{(2)}{A}{}^{l}{}_{j}\underset{(1)}{N}{}^{r}{}_{l}\underset{(3)}{C}{}^{s}{}_{ir}\frac{\delta}{\delta y^{(k-1)s}} + ... + \underset{(2)}{A}{}^{l}{}_{j}\underset{(k-3)}{N}{}^{r}{}_{l}\underset{(k-1)}{C}{}^{s}{}_{ir}\frac{\delta}{\delta y^{(k-1)s}}\Big) +$$

$$+\Big(\underset{(2)}{A}{}^{l}{}_{j}\underset{(1)}{N}{}^{r}{}_{l}\underset{(1)}{N}{}^{s}{}_{r}D_{\frac{\partial}{\partial y^{(4)s}}}\frac{\delta}{\delta y^{(k-1)i}} + ... + \underset{(2)}{A}{}^{l}{}_{j}\underset{(1)}{N}{}^{r}{}_{l}\underset{(k-4)}{N}{}^{s}{}_{r}\underset{(k-1)}{C}{}^{m}{}_{is}\frac{\delta}{\delta y^{(k-1)m}}\Big) + ...$$

So, we have obtained (2.1_1).

$$D_{\frac{\bar\delta}{\delta y^{(1)j}}}\frac{\bar\delta}{\delta y^{(k-1)i}} = \underset{(1)}{\bar C}{}^{s}{}_{ij}\frac{\bar\delta}{\delta y^{(k-1)s}} = \underset{(1)}{\bar C}{}^{s}{}_{ij}\frac{\delta}{\delta y^{(k-1)s}}.$$

$$D_{\frac{\bar\delta}{\delta y^{(1)j}}}\frac{\bar\delta}{\delta y^{(k-1)i}} = D_{\left(\frac{\delta}{\delta y^{(1)j}} + \underset{(1)}{A}{}^{l}{}_{j}\frac{\partial}{\partial y^{(2)l}} + \underset{(2)}{A}{}^{l}{}_{j}\frac{\partial}{\partial y^{(3)l}} + ... + \underset{(k-2)}{A}{}^{l}{}_{j}\frac{\delta}{\delta y^{(k-1)l}}\right)}\frac{\delta}{\delta y^{(k-1)i}} =$$

$$= \Big(\underset{(1)}{C}{}^{s}{}_{ij} + \underset{(1)}{A}{}^{l}{}_{j}\underset{(2)}{C}{}^{s}{}_{il} + \underset{(2)}{A}{}^{l}{}_{j}\underset{(3)}{C}{}^{s}{}_{il} + ... + \underset{(k-2)}{A}{}^{l}{}_{j}\underset{(k-1)}{C}{}^{s}{}_{il}\Big)\frac{\delta}{\delta y^{(k-1)s}} +$$

$$+\underset{(1)}{A}{}^{l}{}_{j}\underset{(1)}{N}{}^{s}{}_{l}D_{\frac{\partial}{\partial y^{(3)s}}}\frac{\delta}{\delta y^{(k-1)i}} + ... + \underset{(1)}{A}{}^{l}{}_{j}\underset{(k-3)}{N}{}^{r}{}_{l}\underset{(k-1)}{C}{}^{s}{}_{ir}\frac{\delta}{\delta y^{(k-1)s}} +$$

$$+\underset{(2)}{A}{}^{l}{}_{j}\underset{(1)}{N}{}^{s}{}_{l}D_{\frac{\partial}{\partial y^{(4)s}}}\frac{\delta}{\delta y^{(k-1)i}} + ... + \underset{(2)}{A}{}^{l}{}_{j}\underset{(k-4)}{N}{}^{r}{}_{l}\underset{(k-1)}{C}{}^{s}{}_{ir}\frac{\delta}{\delta y^{(k-1)s}} + ...$$

So, we have obtained (2.2_2).

$$D_{\frac{\bar\delta}{\delta y^{(k-2)j}}}\frac{\bar\delta}{\delta y^{(k-1)i}} = \underset{(k-2)}{\bar C}{}^{s}{}_{ij}\frac{\bar\delta}{\delta y^{(k-1)s}} = \underset{(k-2)}{\bar C}{}^{s}{}_{ij}\frac{\delta}{\delta y^{(k-1)s}}.$$

$$D_{\frac{\bar\delta}{\delta y^{(k-2)j}}}\frac{\bar\delta}{\delta y^{(k-1)i}} = \underset{(k-2)}{\bar C}{}^{s}{}_{ij}\frac{\bar\delta}{\delta y^{(k-1)s}} + \underset{(1)}{A}{}^{l}{}_{j}\underset{(k-1)}{\bar C}{}^{s}{}_{il}\frac{\delta}{\delta y^{(k-1)s}}.$$

$$D_{\frac{\bar\delta}{\delta p_j}}\frac{\bar\delta}{\delta y^{(k-1)i}} = \bar C_i^{js}\frac{\bar\delta}{\delta y^{(k-1)s}} = \bar C_i^{js}\frac{\delta}{\delta y^{(k-1)s}};$$

$$D_{\frac{\bar\delta}{\delta p_j}}\frac{\bar\delta}{\delta y^{(k-1)i}} = D_{\frac{\bar\delta}{\delta p_j}}\frac{\delta}{\delta y^{(k-1)i}} = C_i^{js}\frac{\delta}{\delta y^{(k-1)s}};$$

So, we have obtained (2.2_{k-1}).

$$D_{\frac{\bar\delta}{\delta p_j}}\frac{\bar\delta}{\delta y^{(k-2)i}} = \bar C_i^{js}\frac{\bar\delta}{\delta y^{(k-2)s}} = \bar C_i^{js}\left(\frac{\delta}{\delta y^{(k-2)s}} + \underset{(1)}{A}{}^{l}{}_{s}\frac{\partial}{\partial y^{(k-1)l}}\right) =$$

$$= \bar C_i^{js}\frac{\delta}{\delta y^{(k-2)s}} + \bar C_i^{js}\underset{(1)}{A}{}^{l}{}_{s}\frac{\delta}{\delta y^{(k-1)l}}.$$

$$D_{\frac{\bar\delta}{\delta p_j}}\frac{\bar\delta}{\delta y^{(k-2)i}} = D_{\frac{\bar\delta}{\delta p_j}}\left(\frac{\delta}{\delta y^{(k-2)i}} + \underset{(1)}{A}{}^{l}{}_{i}\frac{\partial}{\partial y^{(k-1)l}}\right) =$$

$$= C_i^{js}\frac{\bar\delta}{\delta y^{(k-2)s}} + \left(\frac{\delta \underset{(1)}{A}{}^{s}{}_{i}}{\delta p_j} + \underset{(1)}{A}{}^{l}{}_{i}C_l^{js}\right)\frac{\bar\delta}{\delta y^{(k-1)s}}.$$

So, we have:

$$\bar C_i^{js} = C_i^{js} \tag{2.4}$$

$$\bar{C}_i^{\ jl} \underset{(1)}{A}{}^s{}_l = \frac{\delta \underset{(1)}{A}{}^s{}_i}{\delta p_j} + \underset{(1)}{A}{}^l{}_i C_l^{\ js}. \tag{2.5}$$

Analogous if we calculate $D_{\frac{\bar{\delta}}{\delta y^{(k-1)j}}} \frac{\bar{\delta}}{\delta y^{(k-2)i}}$ in two manner we obtain:

$$\underset{(k-1)}{\overline{C}}{}^s{}_{ij} = \underset{(k-1)}{C}{}^s{}_{ij}, \tag{2.6}$$

$$\underset{(k-1)}{\overline{C}}{}^l{}_{ij} \underset{(1)}{A}{}^s{}_l = \frac{\delta \underset{(1)}{A}{}^s{}_i}{\delta y^{(k-1)j}} + \underset{(1)}{A}{}^l{}_i \underset{(k-1)}{C}{}^s{}_{lj}. \tag{2.7}$$

We have:

$$\underset{(\alpha)}{A}{}^i{}_{j|k} = \frac{\delta \underset{(\alpha)}{A}{}^i{}_j}{\delta x_k} + \underset{(\alpha)}{A}{}^m{}_j H^i{}_{mk} - \underset{(\alpha)}{A}{}^i{}_m H^m{}_{jk}, \ (\alpha = 1, 2, ..., k-1). \tag{2.8}$$

Using $(2.8), (2.7), (2.6), (2.5), (2.4), (2.2_{k-1}), (2.2_2)$ in the relation obtained analogous from $D_{\frac{\bar{\delta}}{\delta x^j}} \frac{\bar{\delta}}{\delta y^i}$, we obtain: $\underset{(1)}{A}{}^h{}_{i|j} = 0$. In the same manner we get $A_{ih|j} = 0$. □

Theorem 2.2. *Let N and $\bar{N}$ be two nonlinear connections on $T^{*k}M$, $(k \geq 2,\ k \in \mathbb{N})$, with local coefficients*

$$\left(\underset{(1)}{N}{}^j{}_i\left(x, y^{(1)}, ..., y^{(k-1)}, p\right), ..., \underset{(k-1)}{N}{}^j{}_i\left(x, y^{(1)}, ..., y^{(k-1)}, p\right), N_{ij}\left(x, y^{(1)}, ..., y^{(k-1)}, p\right)\right),$$

$$\left(\underset{(1)}{\bar{N}}{}^j{}_i\left(x, y^{(1)}, ..., y^{(k-1)}, p\right), ..., \underset{(k-1)}{\bar{N}}{}^j{}_i\left(x, y^{(1)}, ..., y^{(k-1)}, p\right), \overline{N}_{ij}\left(x, y^{(1)}, ..., y^{(k-1)}, p\right)\right),$$

$(i, j = 1, 2, ..., n)$, *respectively.*

If

$$D\Gamma(N) = \left(H^i{}_{jh}, \underset{(\alpha)}{C}{}^i{}_{jh}, C_i^{\ jh}\right)$$

and

$$D\bar{\Gamma}(\bar{N}) = \left(\bar{H}^i{}_{jh}, \underset{(\alpha)}{\bar{C}}{}^i{}_{jh}, \bar{C}_i^{\ jh}\right),$$

$(\alpha = 1, ..., k-1)$ *are the local coefficients of two $N-$, respectively $\bar{N}$-linear connections, D, respectively $\bar{D}$ on the differentiable manifold $T^{*k}M, (k \geq 2, k \in \mathbb{N})$, then there exists only one system of tensor fields*

$$\left(\underset{(1)}{A}{}^i{}_j, ..., \underset{(k-1)}{A}{}^i{}_j, A_{ij}, B^i{}_{jh}, \underset{(1)}{D}{}^i{}_{jh}, ..., \underset{(k-1)}{D}{}^i{}_{jh}, D_i^{\ jh}\right)$$

such that:

$$\begin{cases}
\underset{(\alpha)}{\bar{N}}{}^{i}{}_{j} = \underset{(\alpha)}{N}{}^{i}{}_{j} - \underset{(\alpha)}{A}{}^{i}{}_{j}, (\alpha = 1, ..., k-1), \\
\bar{N}_{ij} = N_{ij} - A_{ij}, \\
\bar{H}^{i}{}_{sj} = H^{i}{}_{sj} + \underset{(1)}{A}{}^{m}{}_{j} \left[\underset{(1)}{C}{}^{i}{}_{sm} + \underset{(1)}{N}{}^{l}{}_{m} \underset{(2)}{C}{}^{i}{}_{sl} + ... + \underset{(k-2)}{N}{}^{l}{}_{m} \underset{(k-1)}{C}{}^{i}{}_{sl} + \underset{(1)}{N}{}^{l}{}_{m} \underset{(1)}{N}{}^{t}{}_{l} \underset{(3)}{C}{}^{i}{}_{st} + \right. \\
\left. + ... + \left(\underset{(1)}{N}{}^{l}{}_{m} \underset{(k-3)}{N}{}^{t}{}_{l} + ... + \underset{(k-3)}{N}{}^{l}{}_{m} \underset{(1)}{N}{}^{t}{}_{l} \right) \underset{(k-1)}{C}{}^{i}{}_{st} + ... + \underbrace{\underset{(1)}{N} ... \underset{(1)}{N}}_{(k-2)} \underset{(k-1)}{C} \right] + \\
+ \underset{(2)}{A}{}^{m}{}_{j} \left[\underset{(2)}{C}{}^{i}{}_{sm} + \underset{(1)}{N}{}^{l}{}_{m} \underset{(3)}{C}{}^{i}{}_{sl} + ... + \underset{(k-3)}{N}{}^{l}{}_{m} \underset{(k-1)}{C}{}^{i}{}_{sl} + ... + \underbrace{\underset{(1)}{N} ... \underset{(1)}{N}}_{(k-3)} \underset{(k-1)}{C} \right] + \\
+ ... + \underset{(k-2)}{A}{}^{m}{}_{j} \left(\underset{(k-2)}{C}{}^{i}{}_{sm} + \underset{(1)}{N}{}^{l}{}_{m} \underset{(k-1)}{C}{}^{i}{}_{sl} \right) + \underset{(k-1)}{A}{}^{m}{}_{j} \underset{(k-1)}{C}{}^{i}{}_{sm} - A_{jm} C_{s}{}^{im} - B^{i}{}_{sj}, \\
\underset{(1)}{\overline{C}}{}^{i}{}_{sj} = \underset{(1)}{C}{}^{i}{}_{sj} + \underset{(1)}{A}{}^{m}{}_{j} \left[\underset{(2)}{C}{}^{i}{}_{sm} + \underset{(1)}{N}{}^{r}{}_{m} \underset{(3)}{C}{}^{i}{}_{sr} + ... + \underset{(k-3)}{N}{}^{r}{}_{m} \underset{(k-1)}{C}{}^{i}{}_{sr} + ... + \right. \\
\left. + \underbrace{\underset{(1)}{N} \cdot ... \cdot \underset{(1)}{N}}_{(k-3)} \underset{(k-1)}{C} \right] + ... + \underset{(k-3)}{A}{}^{m}{}_{j} \left[\underset{(k-2)}{C}{}^{i}{}_{sm} + \underset{(1)}{N}{}^{r}{}_{m} \underset{(k-1)}{C}{}^{i}{}_{sr} \right] + \\
+ \underset{(k-2)}{A}{}^{m}{}_{j} \underset{(k-1)}{C}{}^{i}{}_{sm} - \underset{(1)}{D}{}^{i}{}_{sj}, \\
\underset{(k-2)}{\overline{C}}{}^{i}{}_{sj} = \underset{(k-2)}{C}{}^{i}{}_{sj} + \underset{(1)}{A}{}^{l}{}_{j} \underset{(k-1)}{C}{}^{i}{}_{sl} - \underset{(k-2)}{D}{}^{i}{}_{sj}, \\
\underset{(k-1)}{\bar{C}}{}^{i}{}_{sj} = \underset{(k-1)}{C}{}^{i}{}_{sj} - \underset{(k-1)}{D}{}^{i}{}_{sj}, \\
\bar{C}_{s}{}^{ij} = C_{s}{}^{ij} - D_{s}{}^{ij},
\end{cases} \tag{2.9}$$

with:

$$\begin{cases}
\underset{(1)}{A}{}^{h}{}_{i|j} = 0, \\
A_{ih|j} = 0, (i, j, h = 1, 2..., n),
\end{cases} \tag{2.10}$$

where "$|$" denotes the $h-$covariant derivative with respect to $D\Gamma(N)$.

Proof. The first equality (2.9) determines uniquely the tensor fields: $\underset{(\alpha)}{A}{}^{i}{}_{j}, (\alpha = 1, ..., k-1)$. The second equality (2.9) determines uniquely the tensor field A_{ij}. Since $\underset{(\alpha)}{C}{}^{i}{}_{jh}, (\alpha = 1, ..., k-1)$ and C_{i}^{jh} are $d-$tensor fields, the third equation (2.9) determines uniquely the tensor field $B^{i}{}_{jh}$. Similarly the fourth,... and the last equation (2.9) determines the tensor field $D_{i}{}^{jh}$ respectively. □

We have immediately:

Theorem 2.3. *If* $D\Gamma(N) = \left(H^i{}_{jh}, \underset{(\alpha)}{C}{}^i{}_{jh}, C_i{}^{jh}\right)(\alpha = 1, ..., k-1)$, *are the local coefficients of an* $N-$*linear connection* D *on* $T^{*k}M$ *and*

$$\left(\underset{(1)}{A}{}^i{}_j, ..., \underset{(k-1)}{A}{}^i{}_j, A_{ij}, B^i{}_{jh}, \underset{(1)}{D}{}^i{}_{jh}, ..., \underset{(k-1)}{D}{}^i{}_{jh}, D_i{}^{jh}\right),$$

is a system of tensor fields on $T^{*k}M$, *then* $D\bar{\Gamma}\left(\bar{N}\right) = \left(\bar{H}^i{}_{jh}, \underset{(\alpha)}{\bar{C}}{}^i{}_{jh}, \bar{C}_i{}^{jh}\right)$, $(\alpha = 1, ..., k-1)$, *given by* $(2.9)-(2.10)$ *are the local coefficients of an* $\bar{N}-$*linear connection,* $\bar{D}$, *on* $T^{*k}M, (k \geq 2, k \in \mathbb{N})$.

Following the definition given by M. Matsumoto [4, 5] in the case of Finsler spaces, we have:

Definition 2.1. i) The system of tensor fields:

$$\left(\underset{(1)}{A}{}^i{}_j, ..., \underset{(k-1)}{A}{}^i{}_j, A_{ij}, B^i{}_{jh}, \underset{(1)}{D}{}^i{}_{jh}, ..., \underset{(k-1)}{D}{}^i{}_{jh}, D_i{}^{jh}\right), \ (k \geq 2, k \in \mathbb{N})$$

is called the difference tensor fields of $D\Gamma(N)$ to $D\bar{\Gamma}\left(\bar{N}\right)$.

ii) The mapping: $D\Gamma(N) \longrightarrow D\bar{\Gamma}\left(\bar{N}\right)$ given by $(2.9)-(2.10)$ is called a transformation of $N-$linear connection to $\bar{N}-$linear connection on $T^{*k}M$, and it is noted by:

$$t\left(\underset{(1)}{A}{}^i{}_j, ..., \underset{(k-1)}{A}{}^i{}_j, A_{ij}, B^i{}_{jh}, \underset{(1)}{D}{}^i{}_{jh}, ..., \underset{(k-1)}{D}{}^i{}_{jh}, D_i{}^{jh}\right).$$

Theorem 2.4. *The set* $\mathcal{T}$ *of the transformations of* $N-$*linear connections to* $\bar{N}-$*linear connections on* $T^{*k}M, (k \geq 2, k \in \mathbb{N})$ *together with the composition of mappings isn't a group.*

Proof. Let

$$t\left(\underset{(1)}{A}{}^i{}_j, ..., \underset{(k-1)}{A}{}^i{}_j, A_{ij}, B^i{}_{jh}, \underset{(1)}{D}{}^i{}_{jh}, ..., \underset{(k-1)}{D}{}^i{}_{jh}, D_i{}^{jh}\right) : D\Gamma(N) \longrightarrow D\bar{\Gamma}\left(\bar{N}\right)$$

and

$$t\left(\underset{(1)}{\bar{A}}{}^i{}_j, ..., \underset{(k-1)}{\bar{A}}{}^i{}_j, \bar{A}_{ij}, \bar{B}^i{}_{jh}, \underset{(1)}{\bar{D}}{}^i{}_{jh}, ..., \underset{(k-1)}{\bar{D}}{}^i{}_{jh}, \bar{D}_i{}^{jh}\right) : D\bar{\Gamma}\left(\bar{N}\right) \longrightarrow D\bar{\bar{\Gamma}}\left(\bar{\bar{N}}\right),$$

be two transformations from $\mathcal{T}$, given by $(2.9)-(2.10)$.

From (2.9) we have:

$$\underset{(\alpha)}{\bar{\bar{N}}}{}^i{}_j = \underset{(\alpha)}{N}{}^i{}_j - \left(\underset{(\alpha)}{A}{}^i{}_j + \underset{(\alpha)}{\bar{A}}{}^i{}_j\right), \ (\alpha = 1, ..., k-1), \ \bar{\bar{N}}_{ij} = N_{ij} - \left(A_{ij} + \bar{\bar{A}}_{ij}\right).$$

We obtain for example:

$$\underset{(k-2)}{\bar{\bar{C}}}{}^i{}_{jh} = \underset{(k-2)}{C}{}^i{}_{jh} + \left(\underset{(1)}{A}{}^l{}_h + \underset{(1)}{\bar{A}}{}^l{}_h\right) \cdot \underset{(k-1)}{C}{}^i{}_{jl} - \left(\underset{(k-2)}{D}{}^i{}_{jh} + \underset{(k-2)}{\bar{D}}{}^i{}_{jh} + \underset{(k-1)}{D}{}^i{}_{jl}\underset{(1)}{\bar{A}}{}^l{}_h\right).$$

So $\underset{(k-2)}{\bar{\bar{C}}}{}^i{}_{jh}$ hasn't the form (2.9). It follows that the composition of two transformations from $\mathcal{T}$ isn't a transformation from $\mathcal{T}$, that is $\mathcal{T}$, together with the composition of mappings isn't a group. □

Remark 2.1. If we consider $\underset{(\alpha)}{A}{}^i{}_j = 0, (\alpha = 1, ..., k-1)$ and $A_{ij} = 0$ in (2.10) we obtain the set $\mathcal{T}_N$ of transformations of $N-$linear connections corresponding to the same nonlinear connection N :

$$\mathcal{T}_N = \left\{ t\left(\underbrace{0, ..., 0}_{(k)}, B^i{}_{jh}, \underset{(1)}{D}{}^i{}_{jh}, ..., \underset{(k-1)}{D}{}^i{}_{jh}, D_i{}^{jh} \right) \in \mathcal{T} \right\}.$$

We have:

Theorem 2.5. *The set $\mathcal{T}_N$ of the transformations of $N-$linear connections to $N-$linear connections on $T^{*k}M, (k \geq 2, k \in \mathbb{N})$, together with the composition of mappings is a group. This group, acts effectively and transitively on the set of $N-$linear connections.*

Proof. Let $t\left(\underbrace{0, ..., 0}_{(k)}, B^i{}_{jh}, \underset{(1)}{D}{}^i{}_{jh}, ..., \underset{(k-1)}{D}{}^i{}_{jh}, D_i{}^{jh} \right) : D\Gamma(N) \longrightarrow D\bar{\Gamma}(N)$ be a transformation from $\mathcal{T}_N$,given by (2.11) :

$$\begin{cases} \underset{(\alpha)}{\bar{N}}{}^i{}_j = \underset{(\alpha)}{N}{}^i{}_j, (\alpha = 1, ..., k-1), \\ \bar{N}_{ij} = N_{ij}, \\ \bar{H}^i{}_{jh} = H^i{}_{jh} - B^i{}_{jh}, \\ \underset{(\alpha)}{\bar{C}}{}^i{}_{jh} = \underset{(\alpha)}{C}{}^i{}_{jh} - \underset{(\alpha)}{D}{}^i{}_{jh}, (\alpha = 1, ..., k-1), \\ \bar{C}_i{}^{jh} = C_i{}^{jh} - D_i{}^{jh}, (i, j, h = 1, 2, ..., n). \end{cases} \tag{2.11}$$

The composition of two transformations from $\mathcal{T}_N$ is a transformation from $\mathcal{T}_N$, given by:

$$t\left(\underbrace{0, ..., 0}_{(k)}, \bar{B}^i{}_{jh}, \underset{(1)}{\bar{D}}{}^i{}_{jh}, ..., \underset{(k-1)}{\bar{D}}{}^i{}_{jh}, \bar{D}_i{}^{jh} \right) \circ t\left(\underbrace{0, ..., 0}_{(k)}, B^i{}_{jh}, \underset{(1)}{D}{}^i{}_{jh}, ..., \underset{(k-1)}{D}{}^i{}_{jh}, D_i{}^{jh} \right)$$

$$= t\left(\underbrace{0, ..., 0}_{(k)}, B^i{}_{jh}, +\bar{B}^i{}_{jh}, \underset{(1)}{D}{}^i{}_{jh} + \underset{(1)}{\bar{D}}{}^i{}_{jh}, ..., \underset{(k-1)}{D}{}^i{}_{jh} + \underset{(k-1)}{\bar{D}}{}^i{}_{jh}, D_i{}^{jh} + \bar{D}_i{}^{jh} \right).$$

The inverse of a transformation from $\mathcal{T}_N$ is the following transformation from $\mathcal{T}_N$:

$$t\left(0, 0, 0, -B^i{}_{jh}, -\underset{(1)}{D}{}^i{}_{jh}, ..., -\underset{(k-1)}{D}{}^i{}_{jh}, -D_i{}^{jh} \right) : D\Gamma(N) \longrightarrow D\bar{\Gamma}(N).$$

The transformations (2.11) preserve all $N-$linear connections D if:

$$B^i{}_{jh} = \underset{(1)}{D}{}^i{}_{jh} = ... = \underset{(k-1)}{D}{}^i{}_{jh} = D_i{}^{jh} = 0, (i, j, h = 1, 2, ..., n).$$

Therefore $\mathcal{T}_N$ acts effectively on the set of $N-$linear connections. From the Theorem 2.2 results that $\mathcal{T}_N$ acts transitively on this set. □

Let us consider:

$$\mathcal{T}_{NH}=\left\{t\left(\underbrace{0,...,0}_{(k+1)},\underset{(1)}{D}{}^{i}{}_{jh},...,\underset{(k-1)}{D}{}^{i}{}_{jh},D_i{}^{jh}\right)\in\mathcal{T}_N\right\},$$

$$\mathcal{T}_{N\underset{(1)}{C}}=\left\{t\left(\underbrace{0,...,0}_{(k)},B^{i}{}_{jh},0,\underset{(2)}{D}{}^{i}{}_{jh},...,\underset{(k-1)}{D}{}^{i}{}_{jh},D_i{}^{jh}\right)\in\mathcal{T}_N\right\},$$

....

$$\mathcal{T}_{N\underset{(k-1)}{C}}=\left\{t\left(\underbrace{0,...,0}_{(k)},B^{i}{}_{jh},\underset{(1)}{D}{}^{i}{}_{jh},...,\underset{(k-2)}{D}{}^{i}{}_{jh},0,D_i{}^{jh}\right)\in\mathcal{T}_N\right\},$$

$$\mathcal{T}_{NC}=\left\{t\left(\underbrace{0,...,0}_{(k)},B^{i}{}_{jh},\underset{(1)}{D}{}^{i}{}_{jh},...,\underset{(k-1)}{D}{}^{i}{}_{jh},0\right)\in\mathcal{T}_N\right\},$$

$$\mathcal{T}_{N\underset{(1)}{C}...\underset{(k-1)}{C}C}=\left\{t\left(\underbrace{0,...,0}_{(k)},B^{i}{}_{jh},\underbrace{0,...,0}_{(k)}\right)\in\mathcal{T}_N\right\},(k\geq 2,k\in\mathbb{N}).$$

Proposition 2.1. *The sets:*$\mathcal{T}_{NH},\mathcal{T}_{N\underset{(1)}{C}},...,\mathcal{T}_{N\underset{(k-1)}{C}},...,\mathcal{T}_{NC},\mathcal{T}_{N\underset{(1)}{C}...\underset{(k-1)}{C}C}$ *are Abelian subgroups of* $\mathcal{T}_N$.

Proposition 2.2. *The group* $\mathcal{T}_N$ *preserves the nonlinear connection* N, $\mathcal{T}_{NH}$ *preserves the nonlinear connection* N *and the component* $H^{i}{}_{jh}$ *of the local coefficients* $D\Gamma(N)$; $\mathcal{T}_{N\underset{(1)}{C}}$ *preserves the nonlinear connection* N *and the component* $\underset{(1)}{C}{}^{i}{}_{jh}$ *of the local coefficients* $D\Gamma(N)$, ..., $\mathcal{T}_{N\underset{(k-1)}{C}}$ *preserves the nonlinear connection* N *and the component* $\underset{(k-1)}{C}{}^{i}{}_{jh}$ *of the local coefficients* $D\Gamma(N)$, $\mathcal{T}_{NC}$ *preserves the nonlinear connection* N *and the component* $C_i{}^{jh}$ *of the local coefficients* $D\Gamma(N)$ *and* $\mathcal{T}_{N\underset{(1)}{C}...\underset{(k-1)}{C}C}$ *preserves the nonlinear connection* N *and the components* $\underset{(1)}{C}{}^{i}{}_{jh},,...,\underset{(k-1)}{C}{}^{i}{}_{jh},C_i{}^{jh}$ *of the local coefficients* $D\Gamma(N)$.

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