Stud. Univ. Babeş-Bolyai Math. 57(2012), No. 1, 89–105

Kantorovich type q-Bernstein-Stancu operators

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Abstract. In this paper, we construct a Kantorovich type generalization of q-Bernstein-Stancu operators by means of the Riemann type q-integral. We investigate some approximation properties and also establish a local approximation theorem for these operators.

Mathematics Subject Classification (2010): 41A25, 41A36.

Keywords: Kantorovich type operator, Riemann type q-integral, Lipschitz class, local approximation.

1. Introduction

Let $q > 0$ be a fixed real number. For any nonnegative integer n, the q-integer $[n]_q$ and the q-factorial $[n]_q!$ are respectively defined by (see [2])

$$[n]_q = \begin{cases} \dfrac{1-q^n}{1-q} & \text{if } q \neq 1 \\ n & \text{if } q = 1 \end{cases},$$

and

$$[n]_q! = \begin{cases} [1]_q[2]_q \cdots [n]_q & \text{if } n \geq 1 \\ 1 & \text{if } n = 0 \end{cases}.$$

For the integers $n \geq k \geq 0$, the q-binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

Now suppose that $0 < a < b$, $0 < q < 1$ and f is a real-valued function. The q-Jackson integral of f over the interval $[0, b]$ and a general interval $[a, b]$ are defined by (see [11])

$$\int_0^b f(x) d_q x = (1-q) b \sum_{j=0}^{\infty} f\left(bq^j\right) q^j$$

and

$$\int_a^b f(x)d_qx = \int_0^b f(x)d_qx - \int_0^a f(x)d_qx,$$

respectively, provided the series converge.

It is clear that q-Jackson integral of f over an interval $[a, b]$ contains two infinite sums, so some problems are encountered in deriving the q-analogues of some well-known integral inequalities which are used to compute order of approximation of linear positive operators containing q-Jackson integral. To solve this problem Marinković et.al. (see [12]) defined the Riemann type q-integral as

$$R_q(f; a, b) = \int_a^b f(x)d_q^R x = (1-q)(b-a)\sum_{j=0}^{\infty} f\left(a + (b-a)q^j\right) q^j$$

which contains only points within the interval of integral.
Dalmanoğlu and Doğru [4] proved that Riemann type q-integral is a linear positive operator and satisfies the Hölder inequality

$$R_q(|fg|; a, b) \leq \left(R_q(|f|^{m_1}; a, b)\right)^{\frac{1}{m_1}} \left(R_q(|g|^{m_2}; a, b)\right)^{\frac{1}{m_2}}$$

where $\frac{1}{m_1} + \frac{1}{m_2} = 1$.

In 2009 Nowak [13], for $f \in C[0, 1]$, $q > 0$, $\alpha \geq 0$ and each $n \in \mathbb{N}$ defined the q-Bernstein-Stancu operators

$$B_n^{q,\alpha}(f; x) = \sum_{k=0}^{n} P_{n,k}^{q,\alpha}(x) f\left(\frac{[k]_q}{[n]_q}\right), \quad x \in [0, 1] \tag{1.1}$$

with

$$P_{n,k}^{q,\alpha}(x) = \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{\prod_{i=0}^{k-1}(x + \alpha[i]_q)\prod_{s=0}^{n-k-1}(1 - q^s x + \alpha[s]_q)}{\prod_{i=0}^{n-1}(1 + \alpha[i]_q)} \tag{1.2}$$

and investigated Korovkin type approximation properties of these operators. Note that in (1.2) an empty product is taken to be equal to 1. In [10], the authors studied the rate of convergence and proved a Voronovskaya type theorem for the operator defined by (1.1). After that Agratini [1] introduced some estimates for the rate of convergence to the sequence $B_n^{q,\alpha}(f; x)$ by means of the modulus of continuity and Lipschitz type maximal function and also explored a probabilistic approach.

It is clear that for $\alpha = 0$, $B_n^{q,\alpha}(f; x)$ reduces to q-Bernstein polynomials defined by Phillips [15]

$$B_{n,q}(f; x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \prod_{s=0}^{n-k-1}(1 - q^s x) f\left(\frac{[k]_q}{[n]_q}\right), \quad x \in [0, 1].$$

For $q = 1$, $B_n^{q,\alpha}(f; x)$ turns out to be the Bernstein- Stancu polynomials proposed by Stancu in [16]

$$S_n(f; x) = \sum_{k=0}^{n} \binom{n}{k} \frac{\prod_{i=0}^{k-1}(x + \alpha i)\prod_{s=0}^{n-k-1}(1 - x + \alpha s)}{\prod_{i=0}^{n-1}(1 + \alpha i)} f\left(\frac{k}{n}\right), \quad x \in [0, 1].$$

For $\alpha = 0$ and $q = 1$, $B_n^{q,\alpha}(f;x)$ represents the classical Bernstein polynomials given by

$$B_n(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}x^k(1-x)^{n-k}, \quad x \in [0,1].$$

The following identities hold [13]

$$B_n^{q,\alpha}(1;x) = 1 \tag{1.3}$$

$$B_n^{q,\alpha}(t;x) = x \tag{1.4}$$

$$B_n^{q,\alpha}(t^2;x) = \frac{1}{1+\alpha}\left(x(x+\alpha) + \frac{x(1-x)}{[n]_q}\right) \tag{1.5}$$

for all $x \in [0,1]$ and $n \in \mathbb{N}$.

Generalization of Bernstein polynomials based on q-integers was studied by a number of authors. We now mention some papers related to integral modification of the q-Bernstein polynomials. Gupta [7] constructed Durrmeyer type modification of the q-Bernstein polynomials by means of the q-Jackson integral and studied their some approximation properties. Thereafter, Finta and Gupta [6] obtained some local and global direct results and also established a simultaneous approximation theorem for these operators. In [8], Gupta and Heping defined another Durrmeyer type q-Bernstein polynomials and obtained some approximation properties of such operators. Later in [9], Gupta and Finta proved some direct local and global approximation theorems for the operators given in [8]. Dalmanoğlu [3] presented Kantorovich type q-Bernstein polynomials via q-Jackson integral and investigated their approximation properties and the rate of convergence. Very recently, by introducing the following Kantorovich type generalization of q-Bernstein polynomials by means of the q-Riemann type integral

$$B_n^*(f;q;x) = [n+1]_q \sum_{k=0}^{n} q^{-k}\begin{bmatrix} n \\ k \end{bmatrix}_q x^k \prod_{s=0}^{n-k-1}(1-q^s x)\int_{\frac{[k]_q}{[n+1]_q}}^{\frac{[k+1]_q}{[n+1]_q}} f(t)d_q^R t, \tag{1.6}$$

where $x \in [0,1]$ and $0 < q < 1$, Dalmanoğlu and Doğru [4] studied statistical Korovkin type approximation properties of these operators. The authors derived the formulas

$$B_n^*(1;q;x) = 1,$$

$$B_n^*(t;q;x) = \frac{2q}{1+q}\frac{[n]_q}{[n+1]_q}x + \frac{1}{1+q}\frac{1}{[n+1]_q},$$

$$\begin{aligned} B_n^*(t^2;q;x) &= \left(\frac{q^2}{1+q} + \frac{3q^4}{(1+q)(1+q+q^2)}\right)\frac{[n]_q[n-1]_q}{[n+1]_q^2}x^2 \\ &+ \left(1 + \frac{2q}{1+q} + \frac{q^2-1}{1+q+q^2}\right)\frac{[n]_q}{[n+1]_q^2}x + \frac{1}{1+q+q^2}\frac{1}{[n+1]_q^2}. \end{aligned}$$

In this paper, for $f \in C[0,1]$, $0 < q < 1$ and each $n \in \mathbb{N}$, we consider the Kantorovich type generalization of the q-Bernstein Stancu operators defined by (1.1) with the help of the Riemann type q-integral as follows:

$$B_n^{\alpha}(f;q;x) = \sum_{k=0}^{n} P_{n,k}^{q,\alpha}(x) \frac{[n+1]_q}{q^k} \int_{\frac{[k]_q}{[n+1]_q}}^{\frac{[k+1]_q}{[n+1]_q}} f(t) d_q^R t, \quad x \in [0,1] \tag{1.7}$$

where $P_{n,k}^{q,\alpha}(x)$ is given by (1.2).

In the case $\alpha = 0$ the operator $B_n^{\alpha}(f;q;x)$ turns into the operator $B_n^{*}(f;q;x)$ defined by (1.6).

2. Estimation of moments

Lemma 2.1. *Let m be a nonnegative integer. Then we have*

$$\begin{aligned} I_{n,k}(t^m) &:= \int_{\frac{[k]_q}{[n+1]_q}}^{\frac{[k+1]_q}{[n+1]_q}} t^m d_q^R t \\ &= \frac{q^k}{[n+1]_q} \left(\frac{[n]_q}{[n+1]_q} \right)^m \sum_{l=0}^{m} \binom{m}{l} \left(\frac{[k]_q}{[n]_q} \right)^{m-l} C_{m,l}(q,n), \end{aligned}$$

where

$$C_{m,l}(q,n) = \frac{1}{([n]_q)^l} \sum_{s=0}^{m-l} \binom{m-l}{s} (-1)^s \frac{(1-q)^s}{[l+s+1]_q}.$$

Proof. By definition of Riemann type q-integral and Binomial formula, we get

$$\begin{aligned} &I_{n,k}(t^m) \\ =&(1-q)\frac{q^k}{[n+1]_q} \sum_{j=0}^{\infty} \left(\frac{[k]_q}{[n+1]_q} + \frac{q^k}{[n+1]_q} q^j \right)^m q^j \\ =&(1-q)\frac{q^k}{([n+1]_q)^{m+1}} \sum_{j=0}^{\infty} \left([k]_q + (1-(1-q)[k]_q) q^j \right)^m q^j \\ =&(1-q)\frac{q^k}{([n+1]_q)^{m+1}} \sum_{i=0}^{m} \sum_{j=0}^{\infty} (q^j)^{i+1} \binom{m}{i} (1-(1-q)[k]_q)^i ([k]_q)^{m-i}. \end{aligned}$$

Using the following fact

$$\sum_{j=0}^{\infty} (q^j)^{i+1} = \frac{1}{1-q^{i+1}} = \frac{1}{(1-q)} \frac{1}{[i+1]_q}$$

and Binomial formula again, we can write

$$\begin{aligned}
&I_{n,k}(t^m)\\
=&\frac{q^k}{([n+1]_q)^{m+1}}\sum_{i=0}^{m}\binom{m}{i}\left(1-(1-q)[k]_q\right)^i\frac{([k]_q)^{m-i}}{[i+1]_q}\\
=&\frac{q^k}{([n+1]_q)^{m+1}}\sum_{i=0}^{m}\binom{m}{i}\frac{([k]_q)^{m-i}}{[i+1]_q}\sum_{l=0}^{i}\binom{i}{l}(-1)^{i-l}\left((1-q)[k]_q\right)^{i-l}\\
=&\frac{q^k}{([n+1]_q)^{m+1}}\sum_{i=0}^{m}\binom{m}{i}\frac{1}{[i+1]_q}\sum_{l=0}^{i}\binom{i}{l}([k]_q)^{m-l}(-1)^{i-l}(1-q)^{i-l}\\
=&\frac{q^k}{([n+1]_q)^{m+1}}\sum_{l=0}^{m}\sum_{i=l}^{m}\binom{m}{i}\binom{i}{l}([k]_q)^{m-l}(-1)^{i-l}\frac{(1-q)^{i-l}}{[i+1]_q}\\
=&\frac{q^k}{([n+1]_q)^{m+1}}\sum_{l=0}^{m}\sum_{i=l}^{m}\frac{m!}{(m-i)!}\frac{1}{l!(i-l)!}([k]_q)^{m-l}(-1)^{i-l}\frac{(1-q)^{i-l}}{[i+1]_q}\\
=&\frac{q^k}{([n+1]_q)^{m+1}}\sum_{l=0}^{m}\sum_{s=0}^{m-l}\frac{m!}{(m-l-s)!}\frac{1}{l!s!}([k]_q)^{m-l}(-1)^{s}\frac{(1-q)^{s}}{[l+s+1]_q}\\
=&\frac{q^k}{([n+1]_q)^{m+1}}\sum_{l=0}^{m}\sum_{s=0}^{m-l}\frac{m!}{l!(m-l)!}\frac{(m-l)!}{s!(m-l-s)!}([k]_q)^{m-l}\\
&\times(-1)^{s}\frac{(1-q)^{s}}{[l+s+1]_q}\\
=&\frac{q^k}{([n+1]_q)^{m+1}}\sum_{l=0}^{m}\sum_{s=0}^{m-l}\binom{m}{l}\binom{m-l}{s}([k]_q)^{m-l}(-1)^{s}\frac{(1-q)^{s}}{[l+s+1]_q}\\
=&\frac{q^k}{([n+1]_q)}\left(\frac{[n]_q}{[n+1]_q}\right)^m\sum_{l=0}^{m}\sum_{s=0}^{m-l}\binom{m}{l}\binom{m-l}{s}\left(\frac{[k]_q}{[n]_q}\right)^{m-l}\frac{1}{([n]_q)^l}\\
&\times(-1)^{s}\frac{(1-q)^{s}}{[l+s+1]_q}\\
=&\frac{q^k}{([n+1]_q)}\left(\frac{[n]_q}{[n+1]_q}\right)^m\sum_{l=0}^{m}\binom{m}{l}\left(\frac{[k]_q}{[n]_q}\right)^{m-l}\frac{1}{([n]_q)^l}\sum_{s=0}^{m-l}\binom{m-l}{s}\\
&\times(-1)^{s}\frac{(1-q)^{s}}{[l+s+1]_q}.
\end{aligned}$$

Thus, if we take

$$\frac{1}{([n]_q)^l}\sum_{s=0}^{m-l}\binom{m-l}{s}(-1)^s\frac{(1-q)^s}{[l+s+1]_q}=C_{m,l}(q,n),$$

then the proof is completed. □

In the light of Lemma 2.1, we can state the following lemma.

Lemma 2.2. *Let m be a nonnegative integer. Then for the operator $B_n^{\alpha}(f;q;x)$ defined by* (1.7)*, we have*

$$B_n^{\alpha}(t^m;q;x) = \left(\frac{[n]_q}{[n+1]_q}\right)^m \sum_{l=0}^{m} \binom{m}{l} C_{m,l}(q,n) B_n^{q,\alpha}(t^{m-l};x),$$

*where $B_n^{q,\alpha}$ is given by (*1.1*) and $C_{m,l}(q,n)$ is defined as in Lemma 2.1.*

Proof. Indeed, by using Lemma 2.1 we can write

$$\begin{aligned} B_n^{\alpha}(t^m;q;x) &= \sum_{k=0}^{n} P_{n,k}^{q,\alpha}(x) \frac{[n+1]_q}{q^k} I_{n,k}(t^m) \\ &= \left(\frac{[n]_q}{[n+1]_q}\right)^m \sum_{k=0}^{n} P_{n,k}^{q,\alpha}(x) \sum_{l=0}^{m} \binom{m}{l} \left(\frac{[k]_q}{[n]_q}\right)^{m-l} C_{m,l}(q,n) \\ &= \left(\frac{[n]_q}{[n+1]_q}\right)^m \sum_{l=0}^{m} \binom{m}{l} C_{m,l}(q,n) \sum_{k=0}^{n} P_{n,k}^{q,\alpha}(x) \left(\frac{[k]_q}{[n]_q}\right)^{m-l} \\ &= \left(\frac{[n]_q}{[n+1]_q}\right)^m \sum_{l=0}^{m} \binom{m}{l} C_{m,l}(q,n) B_n^{q,\alpha}(t^{m-l};x). \end{aligned}$$

□

Corollary 2.3. *The operator $B_n^{\alpha}(f;q;x)$ defined by* (1.7) *satisfies*

$$B_n^{\alpha}(1;q;x) = 1 \tag{2.1}$$

$$B_n^{\alpha}(t;q;x) = \frac{2q}{1+q}\frac{[n]_q}{[n+1]_q}x + \frac{1}{1+q}\frac{1}{[n+1]_q} \tag{2.2}$$

$$\begin{aligned} &B_n^{\alpha}(t^2;q;x) \\ &= \frac{1}{1+\alpha}\frac{4q^4+q^3+q^2}{(1+q)(1+q+q^2)}\frac{[n]_q[n-1]_q}{[n+1]_q^2}x^2 \\ &\quad + \left\{\frac{\alpha}{1+\alpha}\frac{4q^3+q^2+q}{(1+q)(1+q+q^2)}\frac{[n]_q^2}{[n+1]_q^2} + \left(\frac{1}{1+\alpha}\frac{4q^3+q^2+q}{(1+q)(1+q+q^2)}\right.\right. \\ &\quad \left.\left. + \frac{4q^2+2q}{(1+q)(1+q+q^2)}\right)\frac{[n]_q}{[n+1]_q^2}\right\}x + \frac{1}{1+q+q^2}\frac{1}{[n+1]_q^2}. \end{aligned} \tag{2.3}$$

With the help of Lemma 2.2 and identities (1.3)- (1.5) it can be easily proved. So we omit it.

Lemma 2.4. *For the operator $B_n^\alpha(f;q;x)$ defined by (1.7), we have*

$$B_n^\alpha((t-x)^2;q;x) \le \left(\frac{1}{1+\alpha}\frac{4q^2}{(1+q)^2}\frac{[n]_q^2}{[n+1]_q^2} - \frac{4q}{1+q}\frac{[n]_q}{[n+1]_q} + 1\right)x^2 + \left(\frac{\alpha}{1+\alpha}\frac{[n]_q^2}{[n+1]_q^2} + \frac{1}{1+\alpha}\frac{1}{[n+1]_q}\right)x + \frac{1}{1+q+q^2}\frac{1}{[n+1]_q^2}. \tag{2.4}$$

Proof. From the linearity of B_n^α and the equalities (2.1)- (2.3), we may write

$$\begin{aligned} &B_n^\alpha((t-x)^2;q;x) \\ &= \left\{\frac{1}{1+\alpha}\frac{4q^4+q^3+q^2}{(1+q)(1+q+q^2)}\frac{[n]_q[n-1]_q}{[n+1]_q^2} - \frac{4q}{1+q}\frac{[n]_q}{[n+1]_q} + 1\right\}x^2 \\ &+ \left\{\frac{\alpha}{1+\alpha}\frac{4q^3+q^2+q}{(1+q)(1+q+q^2)}\frac{[n]_q^2}{[n+1]_q^2}\right. \\ &+ \left(\frac{1}{1+\alpha}\frac{4q^3+q^2+q}{(1+q)(1+q+q^2)} + \frac{4q^2+2q}{(1+q)(1+q+q^2)}\right)\frac{[n]_q}{[n+1]_q^2} \\ &\left. - \frac{2}{1+q}\frac{1}{[n+1]_q}\right\}x + \frac{1}{1+q+q^2}\frac{1}{[n+1]^2}. \end{aligned} \tag{2.5}$$

In [4], for $0<q<1$ and $n\in\mathbb{N}$ it was showed that

$$\frac{q^2}{1+q} + \frac{3q^4}{(1+q)(1+q+q^2)} = \frac{4q^4+q^3+q^2}{(1+q)(1+q+q^2)} < \frac{4q^2}{(1+q)^2}.$$

Since $[n-1]_q < [n]_q$ this leads to

$$\left(\frac{4q^4+q^3+q^2}{(1+q)(1+q+q^2)}\right)\frac{[n]_q[n-1]_q}{[n+1]_q^2} < \frac{4q^2}{(1+q)^2}\frac{[n]_q^2}{[n+1]_q^2}. \tag{2.6}$$

On the other hand, for $0<q<1$ we have

$$\frac{4q^3+q^2+q}{(1+q)(1+q+q^2)} - 1 = \frac{(3q^2+2q+1)(q-1)}{(1+q)(1+q+q^2)} < 0$$

which gives

$$\frac{4q^3+q^2+q}{(1+q)(1+q+q^2)} < 1. \tag{2.7}$$

Hence using (2.6), (2.7) and the inequality $\frac{[n]_q}{[n+1]_q^2} < \frac{1}{[n+1]_q}$ into (2.5), one gets

$$\begin{aligned}
&B_n^{\alpha}((t-x)^2;q;x) \\
&\leq \left(\frac{1}{1+\alpha} \frac{4q^2}{(1+q)^2} \frac{[n]_q^2}{[n+1]_q^2} - \frac{4q}{1+q} \frac{[n]_q}{[n+1]_q} + 1 \right) x^2 \\
&\quad + \left\{ \frac{\alpha}{1+\alpha} \frac{[n]_q^2}{[n+1]_q^2} + \left(\frac{1}{1+\alpha} + \frac{4q^2+2q}{(1+q)(1+q+q^2)} - \frac{2}{1+q} \right) \frac{1}{[n+1]_q} \right\} x \\
&\quad + \frac{1}{1+q+q^2} \frac{1}{[n+1]^2}.
\end{aligned}$$

Finally, for $0 < q < 1$ by means of the fact

$$\frac{4q^2+2q}{(1+q)(1+q+q^2)} - \frac{2}{1+q} = \frac{2(q^2-1)}{(1+q)(1+q+q^2)} < 0$$

we get

$$\begin{aligned}
B_n^{\alpha}((t-x)^2;q;x) \leq &\left(\frac{1}{1+\alpha} \frac{4q^2}{(1+q)^2} \frac{[n]_q^2}{[n+1]_q^2} - \frac{4q}{1+q} \frac{[n]_q}{[n+1]_q} + 1 \right) x^2 \\
&+ \left(\frac{\alpha}{1+\alpha} \frac{[n]_q^2}{[n+1]_q^2} + \frac{1}{1+\alpha} \frac{1}{[n+1]_q} \right) x \\
&+ \frac{1}{1+q+q^2} \frac{1}{[n+1]_q^2}
\end{aligned}$$

which is the required result. □

3. Main results

In this part, we study some approximation properties of the operator $B_n^{\alpha}(f;q;x)$ defined by (1.7).

Theorem 3.1. *Let $q = q_n \in (0,1)$ and $\alpha = \alpha_n \geq 0$ such that $\lim_{n\to\infty} q_n = 1$ and $\lim_{n\to\infty} \alpha_n = 0$. Then for each $f \in C[0,1]$, $B_n^{\alpha_n}(f;q_n;x)$ converges uniformly to f on $[0,1]$.*

Proof. By the Bohman-Korovkin Theorem it is sufficient to show that

$$\lim_{n\to\infty} ||B_n^{\alpha_n}(t^m;q_n;x) - x^m||_{C[0,1]} = 0, \quad m = 0,1,2.$$

By (2.1), it is clear that

$$\lim_{n\to\infty} ||B_n^{\alpha_n}(1;q_n;x) - 1||_{C[0,1]} = 0$$

Since $B_n^{\alpha_n}(t;q_n;x) = B_n^*(t;q_n;x)$, where B_n^* is defined by (1.6), from the formula (22) in [4] we have

$$||B_n^{\alpha_n}(t;q_n;x) - x||_{C[0,1]} \leq \frac{1-q_n}{1+q_n} + \frac{3}{1+q_n} \frac{1}{[n+1]_{q_n}}$$

which implies that

$$\lim_{n\to\infty} ||B_n^{\alpha_n}(t;q_n;x)-x||_{C[0,1]}=0.$$

Now using (2.3), (2.7) and the inequality $\frac{[n]_{q_n}}{[n+1]_{q_n}^2}<\frac{1}{[n+1]_{q_n}}$ we get

$$\begin{aligned}
&\left|B_n^{\alpha_n}(t^2;q_n;x)-x^2\right| \\
\leq&\left|\frac{1}{1+\alpha_n}\frac{4q_n^4+q_n^3+q_n^2}{(1+q_n)(1+q_n+q_n^2)}\frac{[n]_{q_n}[n-1]_{q_n}}{[n+1]_{q_n}^2}-1\right|x^2 \\
&+\left\{\frac{\alpha_n}{1+\alpha_n}\frac{4q_n^3+q_n^2+q_n}{(1+q_n)(1+q_n+q_n^2)}\frac{[n]_{q_n}^2}{[n+1]_{q_n}^2}\right. \\
&\left.+\left(\frac{1}{1+\alpha_n}\frac{4q_n^3+q_n^2+q_n}{(1+q_n)(1+q_n+q_n^2)}+\frac{4q_n^2+2q_n}{(1+q_n)(1+q_n+q_n^2)}\right)\frac{[n]_{q_n}}{[n+1]_{q_n}^2}\right\}x \\
&+\frac{1}{1+q_n+q_n^2}\frac{1}{[n+1]_{q_n}^2} \qquad (3.1)\\
\leq&\left|\frac{1}{1+\alpha_n}\frac{4q_n^4+q_n^3+q_n^2}{(1+q_n)(1+q_n+q_n^2)}\frac{[n]_{q_n}[n-1]_{q_n}}{[n+1]_{q_n}^2}-1\right|x^2 \\
&+\left\{\frac{\alpha_n}{1+\alpha_n}\frac{[n]_{q_n}^2}{[n+1]_{q_n}^2}+\left(\frac{1}{1+\alpha_n}+\frac{4q_n^2+2q_n}{(1+q_n)(1+q_n+q_n^2)}\right)\frac{1}{[n+1]_{q_n}}\right\}x \\
&+\frac{1}{1+q_n+q_n^2}\frac{1}{[n+1]_{q_n}^2}.
\end{aligned}$$

Since (see [4]),

$$\frac{[n]_{q_n}[n-1]_{q_n}}{[n+1]_{q_n}^2}=\frac{1}{q_n^3}\left(1-\frac{2+q_n}{[n+1]_{q_n}}+\frac{1+q_n}{[n+1]_{q_n}^2}\right)$$

the inequality (3.1) takes the form

$$\begin{aligned}
&\left|B_n^{\alpha_n}(t^2;q_n;x)-x^2\right| \\
\leq&\left\{\left|\frac{1}{1+\alpha_n}\frac{4q_n^2+q_n+1}{q_n(1+q_n)(1+q_n+q_n^2)}-1\right|\right. \\
&\left.+\frac{1}{1+\alpha_n}\frac{4q_n^2+q_n+1}{q_n(1+q_n)(1+q_n+q_n^2)}\left|\frac{1+q_n}{[n+1]_{q_n}^2}-\frac{2+q_n}{[n+1]_{q_n}}\right|\right\}x^2 \qquad (3.2)\\
&+\left\{\frac{\alpha_n}{1+\alpha_n}\frac{[n]_{q_n}^2}{[n+1]_{q_n}^2}+\left(\frac{1}{1+\alpha_n}+\frac{4q_n^2+2q_n}{(1+q_n)(1+q_n+q_n^2)}\right)\frac{1}{[n+1]_{q_n}}\right\}x \\
&+\frac{1}{1+q_n+q_n^2}\frac{1}{[n+1]_{q_n}^2}.
\end{aligned}$$

Taking maximum of both sides of (3.2) on $[0,1]$, we find

$$\begin{aligned} &\left\| B_n^{\alpha_n}(t^2;q_n;x) - x^2 \right\|_{C[0,1]} \\ \leq &\left| \frac{1}{1+\alpha_n} \frac{4q_n^2+q_n+1}{q_n(1+q_n)(1+q_n+q_n^2)} - 1 \right| \\ &+ \frac{1}{1+\alpha_n} \frac{4q_n^2+q_n+1}{q_n(1+q_n)(1+q_n+q_n^2)} \left| \frac{1+q_n}{[n+1]_{q_n}^2} - \frac{2+q_n}{[n+1]_{q_n}} \right| \\ &+ \frac{\alpha_n}{1+\alpha_n} \frac{[n]_{q_n}^2}{[n+1]_{q_n}^2} + \left(\frac{1}{1+\alpha_n} + \frac{4q_n^2+2q_n}{(1+q_n)(1+q_n+q_n^2)} \right) \frac{1}{[n+1]_{q_n}} \\ &+ \frac{1}{1+q_n+q_n^2} \frac{1}{[n+1]_{q_n}^2} \end{aligned}$$

which yields

$$\lim_{n\to\infty} ||B_n^{\alpha_n}(t^2;q_n;x) - x^2||_{C[0,1]} = 0.$$

Thus the proof is completed. □

Remark 3.2. If we choose $q_n = \frac{n}{n+1}$, it is easily seen that $\lim\limits_{n\to\infty} q_n = 1$ and $\lim\limits_{n\to\infty} q_n^n = e^{-1}$. Hence we guarantee that $\lim\limits_{n\to\infty} [n]_{q_n} = \infty$. Since $[n+1]_{q_n} = q_n[n]_{q_n} + 1$ and $\frac{[n]_{q_n}}{[n+1]_{q_n}} = \frac{1}{q_n + \frac{1}{[n]_{q_n}}}$ we have $\lim\limits_{n\to\infty} \dfrac{1}{[n+1]_{q_n}} = 0$ and $\lim\limits_{n\to\infty} \dfrac{[n]_{q_n}}{[n+1]_{q_n}} = 1$.

For $q \in (0,1)$ it is obvious that $\lim\limits_{n\to\infty} [n]_q = \dfrac{1}{1-q}$. In order to reach to convergence results of the operator $B_n^{\alpha_n}$ we take a sequence $q_n \in (0,1)$ such that $\lim\limits_{n\to\infty} q_n = 1$. So we get that $\lim\limits_{n\to\infty} [n]_{q_n} = \infty$.
By the above explanation, Remark 3.2 provides an example that such a sequence can always be found.

Next, we compute the approximation order of the operator $B_n^{\alpha}(f;q;x)$ in terms of the elements of the usual Lipschitz class.

Let $f \in C[0,1]$, $M > 0$ and $0 < \beta \leq 1$. We recall that f belongs to the class $Lip_M(\beta)$ if the inequality

$$|f(t) - f(x)| \leq M\,|t-x|^{\beta}\,; x,t \in [0,1]$$

holds.

Theorem 3.3. *Let $q = q_n \in (0,1)$ and $\alpha = \alpha_n \geq 0$ such that $\lim\limits_{n\to\infty} q_n = 1$ and $\lim\limits_{n\to\infty} \alpha_n = 0$. Then for each $f \in Lip_M(\beta)$ we have*

$$||B_n^{\alpha_n}(f;q_n;x) - f(x)||_{C[0,1]} \leq M\delta_n^{\beta}$$

where

$$\delta_n = \Bigg\{ \left(\frac{1}{1+\alpha_n} \frac{4q_n^2}{(1+q_n)^2} + \frac{\alpha_n}{1+\alpha_n} \right) \frac{[n]_{q_n}^2}{[n+1]_{q_n}^2} - \frac{4q_n}{1+q_n} \frac{[n]_{q_n}}{[n+1]_{q_n}} + \frac{1}{1+\alpha_n} \frac{1}{[n+1]_{q_n}} + \frac{1}{1+q_n+q_n^2} \frac{1}{[n+1]_{q_n}^2} + 1 \Bigg\}^{\frac{1}{2}}.$$

Proof. By the monotonicity of $B_n^{\alpha_n}$, we can write

$$|B_n^{\alpha_n}(f;q_n;x) - f(x)| \le B_n^{\alpha_n}(|f(t)-f(x)|;q_n;x) \le M \sum_{k=0}^{n} P_{n,k}^{q_n,\alpha_n}(x) \frac{[n+1]_{q_n}}{q_n^k} \int_{\frac{[k]_{q_n}}{[n+1]_{q_n}}}^{\frac{[k+1]_{q_n}}{[n+1]_{q_n}}} |t-x|^{\beta} \, d_{q_n}^R t.$$

On the other hand, by using the Hölder inequality for the Riemann type q-integral with $m_1 = \frac{2}{\beta}$ and $m_2 = \frac{2}{2-\beta}$, we have

$$|B_n^{\alpha_n}(f;q_n;x) - f(x)| \le M \sum_{k=0}^{n} P_{n,k}^{q_n,\alpha_n}(x) \left(\frac{[n+1]_{q_n}}{q_n^k} \int_{\frac{[k]_{q_n}}{[n+1]_{q_n}}}^{\frac{[k+1]_{q_n}}{[n+1]_{q_n}}} (t-x)^2 d_{q_n}^R t \right)^{\frac{\beta}{2}}.$$

Now applying the Hölder inequality for the sum with $p_1 = \frac{2}{\beta}$ and $p_2 = \frac{2}{2-\beta}$ and taking into consideration (1.3) and (2.4), one may write

$$\begin{aligned}
&|B_n^{\alpha_n}(f;q_n;x) - f(x)| \\
\le &M \left(\sum_{k=0}^{n} P_{n,k}^{q_n,\alpha_n}(x) \frac{[n+1]_{q_n}}{q_n^k} \int_{\frac{[k]_{q_n}}{[n+1]_{q_n}}}^{\frac{[k+1]_{q_n}}{[n+1]_{q_n}}} (t-x)^2 d_{q_n}^R t \right)^{\frac{\beta}{2}} \left(\sum_{k=0}^{n} P_{n,k}^{q_n,\alpha_n}(x) \right)^{\frac{2-\beta}{2}} \\
= &M \left(B_n^{\alpha_n}((t-x)^2;q_n;x) \right)^{\frac{\beta}{2}} \left(B_n^{q_n,\alpha_n}(1;x) \right)^{\frac{2-\beta}{2}} \\
\le &M \Bigg\{ \left(\frac{1}{1+\alpha_n} \frac{4q_n^2}{(1+q_n)^2} \frac{[n]_{q_n}^2}{[n+1]_{q_n}^2} - \frac{4q_n}{1+q_n} \frac{[n]_{q_n}}{[n+1]_{q_n}} + 1 \right) x^2 \\
&+ \left(\frac{\alpha_n}{1+\alpha_n} \frac{[n]_{q_n}^2}{[n+1]_{q_n}^2} + \frac{1}{1+\alpha_n} \frac{1}{[n+1]_{q_n}} \right) x + \frac{1}{1+q_n+q_n^2} \frac{1}{[n+1]_{q_n}^2} \Bigg\}^{\frac{\beta}{2}}.
\end{aligned}$$

This implies that

$$\|B_n^{\alpha_n}(f;q_n;x) - f(x)\|_{C[0,1]} \le M \Bigg\{ \left(\frac{1}{1+\alpha_n} \frac{4q_n^2}{(1+q_n)^2} + \frac{\alpha_n}{1+\alpha_n} \right) \frac{[n]_{q_n}^2}{[n+1]_{q_n}^2} - \frac{4q_n}{1+q_n} \frac{[n]_{q_n}}{[n+1]_{q_n}} + \frac{1}{1+\alpha_n} \frac{1}{[n+1]_{q_n}} + \frac{1}{1+q_n+q_n^2} \frac{1}{[n+1]_{q_n}^2} + 1 \Bigg\}^{\frac{\beta}{2}}.$$

Hence if we choose $\delta := \delta_n$, then we arrive at the desired result. $\square$

Finally, we establish a local approximation theorem for the operator $B_n^{\alpha}(f;q;x)$ defined by (1.7).

Let $W^2 = \{g \in C[0,1] : g', g'' \in C[0,1]\}$. For any $\delta > 0$, Peetre's K-functional is defined by

$$K_2(f;\delta) = \inf_{g \in W^2} \{\|f-g\| + \delta \|g''\|\}$$

where $\|.\|$ is the uniform norm on $C[0,1]$ (see [14]). From ([5], p.177, Theorem 2.4) there exists an absolute constant $C > 0$ such that

$$K_2(f;\delta) \le C\omega_2(f;\sqrt{\delta}), \tag{3.3}$$

where the second order modulus of smoothness of $f \in C[0,1]$ is denoted by

$$\omega_2(f;\sqrt{\delta}) = \sup_{0<h\le\sqrt{\delta}} \sup_{x,x+2h\in[0,1]} |f(x+2h) - 2f(x+h) + f(x)|.$$

The usual modulus of continuity of $f \in C[0,1]$ is defined by

$$\omega(f;\sqrt{\delta}) = \sup_{0<h\le\sqrt{\delta}} \sup_{x,x+h\in[0,1]} |f(x+h) - f(x)|.$$

Now consider the following operator

$$L_n(f;q;x) = B_n^{\alpha}(f;q;x) - f\left(\frac{2q}{1+q}\frac{[n]_q}{[n+1]_q}x + \frac{1}{1+q}\frac{1}{[n+1]_q}\right) + f(x) \tag{3.4}$$

for $f \in C[0,1]$.

Lemma 3.4. *Let $g \in W^2$. Then we have*

$$\begin{aligned} |L_n(g;q;x) - g(x)| \le & \Bigg\{ \left(\frac{2+\alpha}{1+\alpha}\frac{4q^2}{(1+q)^2}\frac{[n]_q^2}{[n+1]_q^2} - \frac{8q}{1+q}\frac{[n]_q}{[n+1]_q} + 2\right)x^2 \\ & + \left(\frac{\alpha}{1+\alpha}\frac{[n]_q^2}{[n+1]_q^2} + \frac{1}{1+\alpha}\frac{1}{[n+1]_q}\right)x \\ & + \left.\frac{2q^2+3q+2}{(1+q+q^2)(1+q)^2}\frac{1}{[n+1]_q^2}\right]\Bigg\} \|g''\|. \end{aligned} \tag{3.5}$$

Proof. From (3.4), (2.1) and (2.2) it is immediately seen that

$$\begin{aligned} L_n(t-x;q;x) =& B_n^{\alpha}(t-x;q;x) - \left(\frac{2q}{1+q}\frac{[n]_q}{[n+1]_q}x + \frac{1}{1+q}\frac{1}{[n+1]_q} - x\right) \\ =& B_n^{\alpha}(t;q;x) - xB_n^{\alpha}(1;q;x) \\ & - \left(\frac{2q}{1+q}\frac{[n]_q}{[n+1]_q}x + \frac{1}{1+q}\frac{1}{[n+1]_q} - x\right) \\ =& 0. \end{aligned} \tag{3.6}$$

For $x \in [0,1]$ and $g \in W^2$, using the Taylor formula

$$g(t) - g(x) = (t-x)g'(x) + \int_x^t (t-u)g''(u)du$$

and (3.6) we have

$$\begin{aligned} & L_n(g;q;x)-g(x) \\ =& g'(x)L_n(t-x;q;x)+L_n\left(\int_x^t (t-u)g''(u)du;q;x\right) \\ =& L_n\left(\int_x^t (t-u)g''(u)du;q;x\right) \\ =& B_n^\alpha\left(\int_x^t (t-u)g''(u)du;q;x\right) \\ & -\int_x^{\frac{2q}{1+q}\frac{[n]_q}{[n+1]_q}x+\frac{1}{1+q}\frac{1}{[n+1]_q}}\left(\frac{2q}{1+q}\frac{[n]_q}{[n+1]_q}x+\frac{1}{1+q}\frac{1}{[n+1]_q}-u\right)g''(u)du. \end{aligned}$$

By means of the monotonicity of B_n^α this gives

$$\begin{aligned} & |L_n(g;q;x)-g(x)| \\ \le & B_n^\alpha\left(\left|\int_x^t (t-u)g''(u)du\right|;q;x\right) \\ + & \left|\int_x^{\frac{2q}{1+q}\frac{[n]_q}{[n+1]_q}x+\frac{1}{1+q}\frac{1}{[n+1]_q}}\left(\frac{2q}{1+q}\frac{[n]_q}{[n+1]_q}x+\frac{1}{1+q}\frac{1}{[n+1]_q}-u\right)g''(u)du\right|. \end{aligned} \tag{3.7}$$

On the other hand, it is clear that

$$\left|\int_x^t (t-u)g''(u)du\right| \le (t-x)^2\left\|g''\right\|. \tag{3.8}$$

Now let

$$I:=\int_x^{\frac{2q}{1+q}\frac{[n]_q}{[n+1]_q}x+\frac{1}{1+q}\frac{1}{[n+1]_q}}\left(\frac{2q}{1+q}\frac{[n]_q}{[n+1]_q}x+\frac{1}{1+q}\frac{1}{[n+1]_q}-u\right)g''(u)du.$$

Then we may write

$$\begin{aligned} I \le & \left[\left(\frac{2q}{1+q}\frac{[n]_q}{[n+1]_q}-1\right)x+\frac{1}{1+q}\frac{1}{[n+1]_q}\right]^2\left\|g''\right\| \\ =&\left\{\left(\frac{2q}{1+q}\frac{[n]_q}{[n+1]_q}-1\right)^2x^2+\left(\frac{4q}{(1+q)^2}\frac{[n]_q}{[n+1]_q^2}-\frac{2}{1+q}\frac{1}{[n+1]_q}\right)x\right. \\ & \left.+\frac{1}{(1+q)^2}\frac{1}{[n+1]_q^2}\right\}\left\|g''\right\|. \end{aligned}$$

Use of the facts $\frac{[n]_q}{[n+1]_q^2} < \frac{1}{[n+1]_q}$ and for $0 < q < 1$, $\frac{4q}{(1+q)^2} - \frac{2}{1+q} = \frac{2(q-1)}{(1+q)^2} < 0$ yields

$$\begin{aligned} I \leq & \left\{ \left(\frac{2q}{1+q} \frac{[n]_q}{[n+1]_q} - 1 \right)^2 x^2 + \left(\frac{4q}{(1+q)^2} - \frac{2}{1+q} \right) \frac{1}{[n+1]_q} x \right. \\ & \left. + \frac{1}{(1+q)^2} \frac{1}{[n+1]_q^2} \right\} \|g''\| \\ \leq & \left\{ \left(\frac{2q}{1+q} \frac{[n]_q}{[n+1]_q} - 1 \right)^2 x^2 + \frac{1}{(1+q)^2} \frac{1}{[n+1]_q^2} \right\} \|g''\| \\ = & \left\{ \left(\frac{4q^2}{(1+q)^2} \frac{[n]_q^2}{[n+1]_q^2} - \frac{4q}{1+q} \frac{[n]_q}{[n+1]_q} + 1 \right) x^2 + \frac{1}{(1+q)^2} \frac{1}{[n+1]_q^2} \right\} \|g''\| . \end{aligned} \tag{3.9}$$

Substituting (3.8) and (3.9) into (3.7), we have

$$\begin{aligned} & |L_n(g; q; x) - g(x)| \\ \leq & \left\{ B_n^\alpha \left((t-x)^2; q; x \right) + \left(\frac{4q^2}{(1+q)^2} \frac{[n]_q^2}{[n+1]_q^2} - \frac{4q}{1+q} \frac{[n]_q}{[n+1]_q} + 1 \right) x^2 \right. \\ & \left. + \frac{1}{(1+q)^2} \frac{1}{[n+1]_q^2} \right\} \|g''\| . \end{aligned} \tag{3.10}$$

Using (2.4), from (3.10) it follows that

$$\begin{aligned} |L_n(g; q; x) - g(x)| \leq & \left\{ \left(\frac{2+\alpha}{1+\alpha} \frac{4q^2}{(1+q)^2} \frac{[n]_q^2}{[n+1]_q^2} - \frac{8q}{1+q} \frac{[n]_q}{[n+1]_q} + 2 \right) x^2 \right. \\ & + \left(\frac{\alpha}{1+\alpha} \frac{[n]_q^2}{[n+1]_q^2} + \frac{1}{1+\alpha} \frac{1}{[n+1]_q} \right) x \\ & \left. + \frac{2q^2+3q+2}{(1+q+q^2)(1+q)^2} \frac{1}{[n+1]_q^2} \right] \Bigg\} \|g''\| . \end{aligned}$$

This completes the proof. □

Theorem 3.5. *Let $f \in C[0,1]$. Then for each $x \in [0,1]$ we have*

$$\begin{aligned} |B_n^\alpha(f; q; x) - f(x)| \leq & C\omega_2(f; \sqrt{\delta_n(x)}) \\ & + \omega \left(f; \left| \left(\frac{2q}{1+q} \frac{[n]_q}{[n+1]_q} - 1 \right) x + \frac{1}{1+q} \frac{1}{[n+1]_q} \right| \right), \end{aligned}$$

where

$$\delta_n(x) = \left(\frac{2+\alpha}{1+\alpha} \frac{4q^2}{(1+q)^2} \frac{[n]_q^2}{[n+1]_q^2} - \frac{8q}{1+q} \frac{[n]_q}{[n+1]_q} + 2 \right) x^2 + \left(\frac{\alpha}{1+\alpha} \frac{[n]_q^2}{[n+1]_q^2} + \frac{1}{1+\alpha} \frac{1}{[n+1]_q} \right) x + \frac{2+3q+2q^2}{(1+q+q^2)(1+q)^2} \frac{1}{[n+1]_q^2}$$

and C is a positive constant.

Proof. From (3.4), we have

$$|L_n(f;q;x)| \leq |B_n^\alpha(f;q;x)| + 2\|f\| \leq \|f\| B_n^\alpha(1;q;x) + 2\|f\| = 3\|f\| \tag{3.11}$$

and

$$B_n^\alpha(f;q;x) - f(x) = L_n(f-g;q;x) - (f-g)(x) + L_n(g;q;x) - g(x) + f\left(\frac{2q}{1+q} \frac{[n]_q}{[n+1]_q} x + \frac{1}{1+q} \frac{1}{[n+1]_q} \right) - f(x).$$

In the light of (3.5) and (3.11), this equality implies that

$$\begin{aligned}
&|B_n^\alpha(f;q;x) - f(x)| \\
\leq &|L_n(f-g;q;x)| + |(f-g)(x)| + |L_n(g;q;x) - g(x)| \\
&+ \left| f\left(\frac{2q}{1+q} \frac{[n]_q}{[n+1]_q} x + \frac{1}{1+q} \frac{1}{[n+1]_q} \right) - f(x) \right| \\
\leq &4\|f-g\| + |L_n(g;q;x) - g(x)| \\
&+ \omega\left(f; \left| \left(\frac{2q}{1+q} \frac{[n]_q}{[n+1]_q} - 1 \right) x + \frac{1}{1+q} \frac{1}{[n+1]_q} \right| \right) \\
\leq &4\|f-g\| + \left\{ \left(\frac{2+\alpha}{1+\alpha} \frac{4q^2}{(1+q)^2} \frac{[n]_q^2}{[n+1]_q^2} - \frac{8q}{1+q} \frac{[n]_q}{[n+1]_q} + 2 \right) x^2 \right. \\
&+ \left(\frac{\alpha}{1+\alpha} \frac{[n]_q^2}{[n+1]_q^2} + \frac{1}{1+\alpha} \frac{1}{[n+1]_q} \right) x \\
&\left. + \frac{2q^2+3q+2}{(1+q+q^2)(1+q)^2} \frac{1}{[n+1]_q^2} \right] \Bigg\} \|g''\| \\
&+ \omega\left(f; \left| \left(\frac{2q}{1+q} \frac{[n]_q}{[n+1]_q} - 1 \right) x + \frac{1}{1+q} \frac{1}{[n+1]_q} \right| \right) \\
= &4\|f-g\| + \delta_n(x)\|g''\| + \omega\left(f; \left| \left(\frac{2q}{1+q} \frac{[n]_q}{[n+1]_q} - 1 \right) x + \frac{1}{1+q} \frac{1}{[n+1]_q} \right| \right).
\end{aligned}$$

Hence taking infimum on the right-hand side over all $g \in W^2$ and considering (3.3), we get

$$\begin{aligned}
&|B_n^{\alpha}(f;q;x) - f(x)| \\
\leq & 4K_n(f;\delta_n(x)) + \omega\left(f;\left|\left(\frac{2q}{1+q}\frac{[n]_q}{[n+1]_q} - 1\right)x + \frac{1}{1+q}\frac{1}{[n+1]_q}\right|\right) \\
\leq & C\omega_2(f;\sqrt{\delta_n(x)}) + \omega\left(f;\left|\left(\frac{2q}{1+q}\frac{[n]_q}{[n+1]_q} - 1\right)x + \frac{1}{1+q}\frac{1}{[n+1]_q}\right|\right)
\end{aligned}$$

which is the desired result. □

References

[1] Agratini, O., *On a q-analogue of Stancu operators*, Cent. Eur. J. Math., **8**(2010), no. 1, 191-198.

[2] Andrews, G.E., Askey, R., Roy, R., *Special Functions*, Cambridge University Press, 1999.

[3] Dalmanoğlu, Ö., *Approximation by Kantorovich type q-Bernstein operators*, in: Proceedings of the 12th WSEAS International Conference on Applied Mathematics, Cairo, Egypt, (2007), 113-117, ISSN:1790-2769.

[4] Dalmanoğlu, Ö., Doğru, O., *On statistical approximation properties of Kantorovich type q-Bernstein operators*, Math. Comput. Model., **52**(2010), no. 5-6, 760-771.

[5] DeVore, R.A., Lorentz, G.G., *Constructive Approximation*, Springer-Verlag, Berlin, 1993.

[6] Finta, Z., Gupta, V., *Approximation by q-Durrmeyer operators*, J. Appl. Math. Comput., **29**(2009), no. 1-2, 401-415.

[7] Gupta, V., *Some approximation properties of of q-Durrmeyer operators*, Appl. Math. Comput., **197**(2008), no. 1, 172-178.

[8] Gupta, V., Heping, W., *The rate of convergence of q-Durrmeyer operators for* $0 < q < 1$, Math. Methods Appl. Sci., **31**(2008), no. 16, 1946-1955.

[9] Gupta,V., Finta, Z., *On certain q-Durrmeyer type operators*, Appl. Math. Comput., **209**(2009), no. 2, 415-420.

[10] Jiang, Yanjie; Li, Junming, *The rate of convergence of q-Berntein-Stancu polynomials*, Int. J. Wavelets, Multiresolut. Inf. Process., **7**(2009), no. 6, 773-779.

[11] Kac, V.G., Cheung, P., *Quantum Calculus*, Universitext, Springer-Verlag, New York, 2002.

[12] Marinković, S., Rajković, P., Stanković, M., *The inequalities for some types of q-integrals*, Comput.Math. Appl., **56**(2008), no. 10, 2490-2498.

[13] Nowak, G., *Approximation properties for generalized q-Bernstein polynomials*, J. Math. Anal. Appl., **350**(2009), no. 1, 50-55.

[14] Peetre, J., *Theory of interpolation of normed spaces*, Notas de Matematica, Rio de Janeiro, **39**(1963), 1-86.

[15] Phillips, G.M., *Bernstein polynomials based on the q-integers*, Ann. Numer. Math., **4**(1997), no. 1-4, 511-518.

[16] Stancu, D.D., *Approximation of functions by a new class of linear polynomial operators*, Rev. Roumaine Math. Pures Appl., **13**(1968), 1173-1194.

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