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On the Conjecture of Cao, Gonska and Kacsó

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Abstract. We consider the question if lower estimates in terms of the second order Ditzian-Totik modulus are possible, when we measure the pointwise approximation of continuous function by Bernstein operator. In this case we confirm the conjecture made by Cao, Gonska and Kacsó. To prove this we first establish sharp upper and lower bounds for pointwise approximation of the function $g(x) = x\ln(x) + (1-x)\ln(1-x)$, $x \in [0,1]$ by Bernstein operator.

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1. Introduction

In [6] Cao, Gonska and Kacsó formulated the following

Conjecture 1.1. *Let $T_n : C[a,b] \to C[a,b]$ be a sequence of linear operators and $\varepsilon_n > 0$, $\lim\limits_{n\to\infty} \varepsilon_n = 0$, $\varphi(x) = \varphi(x)_{[a,b]} = \sqrt{(x-a)(b-x)}$, and $0 \le \beta < \lambda \le 1$ fixed. If for every $f \in C[a,b]$ one has*

$$|T_n(f,x) - f(x)| \le C(f)\omega_2^{\varphi^\lambda}\left(f; \varepsilon_n \varphi^{1-\lambda}(x)\right), \tag{1.1}$$

then lower pointwise estimates

$$c(f)\omega_2^{\varphi^\beta}\left(f; \varepsilon_n \varphi^{1-\lambda}(x)\right) \le |T_n(f,x) - f(x)|,\ f \in C[a,b], \tag{1.2}$$

do not hold in general.

The case $\beta = 0$ was already solved by the same authors in Theorem 3.1 in [5]. The aim of this note is to confirm conjecture above for the case when T_n is replaced by the Bernstein operator B_n. Instead of T_n we consider further only the classical Bernstein operator B_n applied to a continuous on $[0,1]$ function $f(x)$ and defined by

$$B_n(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \cdot \binom{n}{k} x^k (1-x)^{n-k},\ x \in [0,1]. \tag{1.3}$$

By usual translation all considerations over the interval $[0,1]$ could be transformed into the interval $[a,b]$. Let us define the function

$$g(x) = x \ln x + (1-x)\ln(1-x),\ x \in (0,1) \tag{1.4}$$

and $g(0) = g(1) = 0$. This function was studied and used by many authors in different problems in approximation theory - see [1,2,5,6,9,10,11,12,13,14]. For example the function $g(x)$ was used to establish Theorem 3.1 in [5]. Also g was studied to obtain direct pointwise estimates for approximation of a continuous function by linear positive operator L in [13-Lemma 3.2]. V.Maier considered the function g to establish the saturation order of Kantorovich operator (see [11,12] and Ch. 10 in [3]). The first uniform estimate for approximation of $g(x)$ by B_n was given by Berens and Lorentz in [1]:

$$B_n(g,x) - g(x) \leq \frac{7}{n}, \text{ for all } x \in [0,1].$$

Different problems in approximation and learning theory, connected with approximation of g by B_n are also studied in [2]. The problem to evaluate in a pointwise form the remainder term

$$R_n(g,x) := B_n(g,x) - g(x),\ x \in [0,1] \tag{1.5}$$

was formulated by the author in [14] as open problem during the fifth Romanian-German seminar on approximation theory, held in Sibiu, Romania in 2002. More precisely, we propose to find (best) bounds of the type

$$k_1 \cdot \frac{x^{\alpha_1}(1-x)^{\alpha_2}}{n^{\beta}} \leq R_n(g,x) \leq K_2 \cdot \frac{x^{a_1}(1-x)^{a_2}}{n^b}, \tag{1.6}$$

for every $x \in [0,1]$, where k_1, K_1 are positive numbers, independent of x and n. Some days after the conference prof. A.Lupaş sent to me the proof of inequality (1.6) with $\alpha_1 = \alpha_2 = \beta = 1$, $k_1 = \frac{1}{2}$ and $a_1 = a_2 = b = \frac{1}{2}$, $K_2 = \sqrt{2}$, i.e.

Theorem 1.2. (see [10]) *For all $x \in [0,1]$ the following holds true*

$$\frac{x(1-x)}{2n} \leq R_n(g,x) \leq \sqrt{2} \cdot \sqrt{\frac{x(1-x)}{n}}. \tag{1.7}$$

Our first statement is motivated by the result of Lupaş and considerations, made in [5,6,13]. We prove that the values of $\alpha_1 = \alpha_2 = 1$ and $a_1 = a_2 = \frac{1}{2}$ in (1.7) are optimal , namely

Theorem 1.3. *It is not possible to find $a_1 > \frac{1}{2}$, or $a_2 > \frac{1}{2}$, or $\alpha_1 < 1$, or $\alpha_2 < 1$, such that*

$$k_1 \cdot \frac{x^{\alpha_1}(1-x)^{\alpha_2}}{n} \leq R_n(g,x) \leq K_2 \cdot \frac{x^{a_1}(1-x)^{a_2}}{\sqrt{n}}, \tag{1.8}$$

holds true for all $x \in [0,1]$ with some positive numbers k_1, K_2, independent of x and n.

Our next result confirms the conjecture of Cao, Gonska, Kacsó in [6] and states the following

Theorem 1.4. *Let* $\varphi(x) = \sqrt{x(1-x)}$, $x \in [0,1]$ *and* $0 \le \beta < \lambda \le 1$ *be fixed. For the function* $g(x)$, *defined in* (1.4) *one has*

$$|B_n(g,x) - g(x)| \le C(g)\omega_2^{\varphi^\lambda}\left(g; \frac{1}{\sqrt{n}}\varphi^{1-\lambda}(x)\right), \tag{1.9}$$

but the lower pointwise estimate

$$c(g)\omega_2^{\varphi^\beta}\left(g; \frac{1}{\sqrt{n}}\varphi^{1-\lambda}(x)\right) \le |B_n(g,x) - g(x)|, \tag{1.10}$$

is not valid.

In Section 2 we give the proof of Theorem 1.3. In Section 3 we establish the proof of Theorem 1.4.

2. Proof of Theorem 1.3

Proof. Due to symmetry it is enough to consider in (1.8) only $x \in [0, \frac{1}{2}]$ and to study the possible values of the parameters α_1 and a_1. It is easy to compute that

$$g''(x) = \frac{1}{x(1-x)},\ x \in (0,1), \tag{2.1}$$

i.e. g is a convex function on $[0,1]$. Therefore

$$B_n(g,x) \ge g(x), \text{for all}\, x \in [0,1]. \tag{2.2}$$

If $S_n(g,x)$ is the piecewise linear interpolant for g at the points $0, \frac{1}{n}, \ldots, 1$, then

$$\begin{aligned} B_n(S_ng,x) &= B_n(g,x), \\ B_n(S_ng,x) &\ge S_n(g,x), \end{aligned} \tag{2.3}$$

due to the fact that S_ng is also convex function. Consequently from (2.2)-(2.3) we get

$$B_n(g,x) - g(x) \ge S_n(g,x) - g(x). \tag{2.4}$$

First let us consider the r.h.s. of (1.8). We suppose that (1.8) holds with $a_1 > \frac{1}{2}$. Then from (2.4) it follows that

$$S_n(g,x) - g(x) \le K_2 \cdot \frac{x^{a_1}(1-x)^{a_2}}{\sqrt{n}},\ x \in [0, \frac{1}{2}]. \tag{2.5}$$

We compute for $0 \le x \le \frac{1}{n}$ that

$$\begin{aligned} S_n(g,x) &= nx \cdot g(\tfrac{1}{n}) = nx\left[\tfrac{1}{n}\ln(\tfrac{1}{n}) + (1-\tfrac{1}{n})\ln(1-\tfrac{1}{n})\right] \\ &= x\left[\ln(\tfrac{1}{n}) + (n-1)\ln(1-\tfrac{1}{n})\right]. \end{aligned} \tag{2.6}$$

Also we verify that for $x \in [0, \frac{1}{2}]$,

$$g(x) = x\ln x + (1-x)\ln(1-x) \le x\ln x. \tag{2.7}$$

Consequently (2.5) and (2.7) yield

$$x\left[-\ln n + (n-1)\ln(1-\frac{1}{n})\right] - x\ln x \le K_2 \cdot \frac{x^{a_1}(1-x)^{a_2}}{\sqrt{n}} \tag{2.8}$$

for $0 < x \leq \frac{1}{n}$. Therefore

$$-x\ln(nx) + x\left[(n-1)\ln(1-\frac{1}{n})\right] \leq K_2 \cdot \frac{x^{a_1}(1-x)^{a_2}}{\sqrt{n}} \tag{2.9}$$

Hence we get

$$x\ln\left[(1-\frac{1}{n})^{n-1}\cdot\frac{1}{nx}\right] \leq K_2 \cdot \frac{x^{a_1}(1-x)^{a_2}}{\sqrt{n}}.$$

Consequently

$$x^{1-a_1}\cdot\sqrt{n} \leq \frac{K_2\cdot(1-x)^{a_2}}{\ln\left[(1-\frac{1}{n})^{n-1}\cdot\frac{1}{nx}\right]}. \tag{2.10}$$

We set $x = \frac{1}{2en}$ in (2.10) and take $n \to \infty$. Then we arrive at

$$+\infty = \lim_{n\to\infty} n^{a_1-1+\frac{1}{2}} \leq \frac{K_2}{\ln 2}, \tag{2.11}$$

when $a_1 > \frac{1}{2}$, which is a contradiction.

To study the best possible value of α_1 in (1.8) we may use the following estimate, proved firstly by Cao in 1964 for all continuous functions, and in particular for $g(x)$-see [4]:

$$|B_n(g,x) - g(x)| \leq C\omega_2(g, \sqrt{\frac{x(1-x)}{n}}). \tag{2.12}$$

This nice estimate can not help us to establish the impossibility of the first inequality in (1.8). We suppose that (1.8) holds with $\alpha_1 < 1$. It is easy to observe that

$$R_n(g,x) \leq |g(x)|. \tag{2.13}$$

Then we would have

$$k_1\frac{(1-x)^{\alpha_2}}{n} \leq x^{-\alpha_1}|g(x)|,$$

which for $x \to 0$ gives

$$\frac{k_1}{n} \leq 0$$

a contradiction. The proof of Theorem 1.3 is completed. □

3. Proof of Theorem 1.4

Proof. We recall the definition of the moduli $\omega_2^{\varphi^\lambda}$, $0 \leq \lambda \leq 1$, which is in complete analogy to those of $\omega_2(f,\cdot)$, $(\lambda = 0)$ and $\omega_2^{\varphi}(f,\cdot)$, $(\lambda = 1)$, (see [8], Chap.2):

$$\omega_2^{\varphi^\lambda}(f,t) = \sup_{0\leq h\leq t}\|\Delta^2_{h\varphi^\lambda} f\|_\infty, \tag{3.1}$$

where

$$\Delta^2_{h\varphi^\lambda} f(x) := \begin{cases} f(x-h\varphi^\lambda(x)) - 2f(x) + f(x+h\varphi^\lambda(x)), \\ \text{if } [x-h\varphi^\lambda(x), x+h\varphi^\lambda(x)] \subseteq [0,1]; \\ 0, \text{otherwise.} \end{cases} \tag{3.2}$$

The direct pointwise estimate (1.9) was proved by Ditzian in [7] for all continuous functions, defined in $[0,1]$ and in particular it holds for $g(x)$ too. We suppose that (1.10) holds true. Setting $x=\frac{1}{2}$ in (3.2) we obtain

$$\Delta^2_{h\varphi^\beta} g\left(\frac{1}{2}\right) = h^2\cdot\varphi^{2\beta}(\frac{1}{2})\cdot g''(\xi) \geq h^2\cdot(\frac{1}{2})^{2\beta}\cdot\frac{1}{\frac{1}{2}(1-\frac{1}{2})} = h^2\cdot 2^{2(1-\beta)}.$$

Hence by

$$t := \frac{1}{\sqrt{n}}\varphi^{1-\lambda}(x),\ x\in[0,1] - \text{fixed}$$

it follows

$$\omega_2^{\varphi^\beta}(g,t) \geq t^2\cdot 2^{2(1-\beta)} = \frac{1}{n}(x(1-x))^{1-\lambda}\cdot 2^{2(1-\beta)}. \tag{3.3}$$

From our supposition and (3.3) we get

$$c(g)\cdot 2^{2(1-\beta)}\cdot\frac{x^{1-\lambda}(1-x)^{1-\lambda}}{n} \leq |B_n(g,x)-g(x)| \tag{3.4}$$

for $0\leq\beta<\lambda\leq 1$. It is clear that for $\lambda=1$ (3.4) is not possible, because due to (1.7) it would lead to

$$c(g)\cdot 2^{2(1-\beta)} \leq \sqrt{2}\cdot\sqrt{\frac{x(1-x)}{n}},\ \text{for all}\, x\in[0,1], \tag{3.5}$$

which is a contradiction.

Consequently for $0\leq\beta<\lambda<1$ (3.4) would imply, that (1.8) is valid with $\alpha_1 = 1-\lambda<1$, which contradicts the statement of Theorem 1.3. Thus the proof of Theorem 1.4 is completed. □

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