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Non-isomorphic contact structures on the torus T^3

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Abstract. In this paper, we prove the existence of infinitely many number non-isomorphic contact structures on the torus T^3. Moreover, this structures are explicitly given by $\omega_n = \cos n\theta_3 d\theta_1 + \sin n\theta_3 d\theta_2, (n \in \mathbb{N})$.

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1. Introduction

In the acts of Colloquium of Brussels in 1958, P. Libermann [3] addressed the study of the automorphisms of the contact structures on a differentiable manifold M. She has proved that these automorphisms correspond bijectively to functions on this manifold. This allows to transport the Lie algebra structure on the vector space $F(M)$ of the functions on M. We obtain, for two given functions $f, g \in F(M)$, a Poisson bracket $[f, g]$ that depends of the contact form ω. The study of the infinite dimensional Lie algebras obtained is far from being advanced. Thus, in 1973 A. Lichnerowicz [4] who hoped to distinguish the contact structures by their Lie algebras, has given a series of results that are all however of general caracter. Some works that have appeared after have emphasis on the similarities of these algebras. In 1979, R. Lutz [7] has proved the existence of infinitely many non-isomorphic contact structures on the sphere S^3. In 1989, as reported by R. Lutz [7] himself, I have opened in my thesis [1] new perspectives in the other direction by studying the sub-algebras of finite dimension of these algebras. We know that if two contact structures $[\omega_1]$ and $[\omega_2]$ are isomorphic then their Lie algebras (of infinite dimension of course) $A([\omega_1])$ and $A([\omega_2])$ are also isomorphic.

Given an n-dimensional smooth manifold M, and a point $p \in M$, a contact element of M with contact point p is an $(n-1)$-dimensional linear subspace of the tangent space to M at p. A contact contact element can be given by the zeros of a 1-form on the tangent space to M at p. However, if a contact element is given by the zeros of a1-form ω, then it will also be given by the zeros of $\lambda\omega$ where $\lambda \neq 0$. thus

$\{\lambda\omega : \lambda \neq 0\}$ all give the same contact element. It follows that the space of all contact elemnts of M can be identified with a quotient of the cotangent bundle PT^*M, where $PT^*M = T^*M/\mathcal{R}$, where, for $\omega_i \in T_p^*M$, $\omega_1 \mathcal{R} \omega_2$ iff there exists $\lambda \neq 0 : \omega_1 = \lambda\omega_2$.

A contact structure on an odd dimensional manifold M, of dimension $2k+1$, is a smooth distribution of contact elements, denoted by ξ, which is generic at each point. The genericity condition is that ξ is non-integrable.

Assume that we have a smooth distribution of contact elements ξ given locally by a differential 1-form α; i.e. a smooth section of the cotangent bundle. The non-integrability condition can be given explicitly as $\alpha \wedge (d\alpha)^k \neq 0$.

Notice that if ξ is given by the differential 1-form α, then the same distribution is given locally by $\beta = f\alpha$, where f is a non-zero smooth function. If ξ is co-orientable then α is defined globally.

If α is a contact form for a given contact structure, the Reeb vector field R can be defined as the unique element of the kernel of $d\alpha$ such that $\alpha(R) = 1$.

For more details, we can consult the references $[5, 6, 8]$.

2. The main result

The main result is contained in the following theorem:

Theorem 2.1. *On the torus T^3 the contact structures defined by the contact forms $\omega_n = \cos n\theta_3 d\theta_1 + \sin n\theta_3 d\theta_2$, $(n \in \mathbb{N})$ are non-isomorphic.*

To establish this result, we need the following lemma.

Lemma 2.2. *Let f a $C^\infty-$function on the torus T^3 and R_n the Reeb field of ω_n defined by*

$$R_n = \cos n\theta_3 \frac{\partial}{\partial\theta_1} + \sin n\theta_3 \frac{\partial}{\partial\theta_2}.$$

If $R_n(f) = 0$, then f depends only on θ_3.

Proof. $R_n(f) = 0$ means that f is constant along the integral curves of R_n whose equations are:

$$\begin{aligned} \frac{d\theta_1}{dt} &= \cos n\theta_3, \\ \frac{d\theta_2}{dt} &= \sin n\theta_3, \\ \frac{d\theta_3}{dt} &= 0. \end{aligned}$$

So, we have

$$\begin{aligned} \theta_1 &= t\cos nk_3 + k_1, \\ \theta_2 &= \sin nk_3 + k_2, \\ \theta_3 &= k_3, \end{aligned}$$

where k_1, k_2 and k_3 are real constants.

When $\tan k_3$ is irrational, the trajectories are dense on a torus T^2, so by continuity f is constant on this torus. Hence, we get $\frac{\partial f}{\partial \theta_1} = \frac{\partial f}{\partial \theta_2} = 0$ for θ_1, θ_2 arbitrary and θ_3 in a dense subset of the circle. It follow that f is constant with respect to θ_1 and θ_2. This completes the proof of the lemma. □

Proof of the theorem. It suffices to prove that the structures $[\omega_1]$ and $[\omega_2]$ are non-isomorphic.

From [1] we recall that the Poisson brackets associated to $[\omega_1]$ and $[\omega_2]$ are given respectively by:

$$\begin{aligned}
[f,g]_1 &= \left(f\frac{\partial g}{\partial \theta_1} - g\frac{\partial f}{\partial \theta_1} + \frac{\partial f}{\partial \theta_3}\frac{\partial g}{\partial \theta_2} - \frac{\partial f}{\partial \theta_2}\frac{\partial g}{\partial \theta_3} \right)\cos\theta_3 \\
&+ \left(f\frac{\partial g}{\partial \theta_2} - g\frac{\partial f}{\partial \theta_2} + \frac{\partial f}{\partial \theta_1}\frac{\partial g}{\partial \theta_3} - \frac{\partial f}{\partial \theta_3}\frac{\partial g}{\partial \theta_1} \right)\sin\theta_3, \\
[f,g]_2 &= \left(f\frac{\partial g}{\partial \theta_1} - g\frac{\partial f}{\partial \theta_1} + \frac{1}{2}\frac{\partial f}{\partial \theta_3}\frac{\partial g}{\partial \theta_2} - \frac{1}{2}\frac{\partial f}{\partial \theta_2}\frac{\partial g}{\partial \theta_3} \right)\cos 2\theta_3 \\
&+ \left(f\frac{\partial g}{\partial \theta_2} - g\frac{\partial f}{\partial \theta_2} + \frac{1}{2}\frac{\partial f}{\partial \theta_1}\frac{\partial g}{\partial \theta_3} - \frac{1}{2}\frac{\partial f}{\partial \theta_3}\frac{\partial g}{\partial \theta_1} \right)\sin 2\theta_3.
\end{aligned}$$

Suppose that $[\omega_1]$ and $[\omega_2]$ are isomorphic that is $F^*\omega_1 = \lambda\omega_2$, where λ is a function on T^3 without zeros and F be this diffeomorphism defined from T^3 into T^3 by:

$$F(\theta_1,\theta_2,\theta_3) = (u(\theta_1,\theta_2,\theta_3), v(\theta_1,\theta_2,\theta_3), w(\theta_1,\theta_2,\theta_3)).$$

We obtain the two equations

$$\frac{\partial u}{\partial \theta_1}\cos w + \frac{\partial v}{\partial \theta_1}\sin w = \lambda\cos 2\theta_3. \tag{2.1}$$

$$\frac{\partial u}{\partial \theta_2}\cos w + \frac{\partial v}{\partial \theta_2}\sin w = \lambda\sin 2\theta_3. \tag{2.2}$$

Let $\Phi(\theta_1,\theta_2,\theta_3) = \cos\theta_3$, $\Psi(\theta_1,\theta_2,\theta_3) = \cos\theta_1$ and $\Omega(\theta_1,\theta_2,\theta_3) = -\sin\theta_1$. Thus we have $[\Phi,\Psi]_1 = \Omega$, $[\Psi,\Omega]_1 = \Phi$ and $[\Omega,\Phi]_1 = -\Psi$.

Then Φ, Ψ and Ω generate a three dimensiononal sub-algebra of $A\ [\omega_1]$ isomorphic to $SL_2(\mathbb{R})$ and consequently, we deduce that the functions $\Phi\circ F$, $\Psi\circ F$ and $\Omega\circ F$ generate a three dimensional sub-algebra of $A\ [\omega_2]$ isomorphic to $SL_2(\mathbb{R})$.
Thus, we have by analogy

$$\begin{aligned}
[\cos w, \cos u]_2 &= -\sin u, \\
[\cos u, -\sin u]_2 &= \cos w, \\
[-\sin u, \cos w]_2 &= -\cos u.
\end{aligned}$$

From this equations, it follows that

$$\frac{\partial u}{\partial \theta_1}\cos 2\theta_3 + \frac{\partial u}{\partial \theta_2}\sin 2\theta_3 = -\cos w. \tag{2.3}$$

If $\Phi(\theta_1,\theta_2,\theta_3) = \sin\theta_3$, $\Psi(\theta_1,\theta_2,\theta_3) = \cos\theta_2$ and $\Omega(\theta_1,\theta_2,\theta_3) = -\sin\theta_2$.

We obtain similarly

$$\frac{\partial v}{\partial \theta_1}\cos 2\theta_3 + \frac{\partial v}{\partial \theta_2}\sin 2\theta_3 = -\sin w. \tag{2.4}$$

We take now

$$\Phi(\theta_1,\theta_2,\theta_3) = 1 \text{ and } \Psi(\theta_1,\theta_2,\theta_3) = -\cos\theta_3,$$

we get

$$\frac{\partial\left(\cos w\right)}{\partial \theta_1}\cos 2\theta_3 + \frac{\partial\left(\cos w\right)}{\partial \theta_2}\sin 2\theta_3 = 0. \tag{2.5}$$

From (5) and lemma 2, it follows that the function $\cos w$ and consequently the function w depend only on θ_3.
Differentiating (3) and (4) with respect to θ_1 and θ_2, we get after taking into account the form of Reeb field R_n the four equations

$$R_2\left(\frac{\partial u}{\partial \theta_1}\right) = R_2\left(\frac{\partial u}{\partial \theta_2}\right) = R_2\left(\frac{\partial v}{\partial \theta_1}\right) = R_2\left(\frac{\partial v}{\partial \theta_2}\right) = 0,$$

from those, we deduce that the functions $\frac{\partial u}{\partial \theta_1}, \frac{\partial u}{\partial \theta_2}, \frac{\partial v}{\partial \theta_1}$ and $\frac{\partial v}{\partial \theta_2}$ depend only on θ_3.
The diffeomorphism F can now be completly caracterized in the following way :

$$u\left(\theta_1,\theta_2,\theta_3\right) = \theta_1\alpha_1(\theta_3) + \theta_2\beta_1(\theta_3) + \gamma_1(\theta_3),$$

$$v\left(\theta_1,\theta_2,\theta_3\right) = \theta_1\alpha_2(\theta_3) + \theta_2\beta_2(\theta_3) + \gamma_2(\theta_3),$$

$$w\left(\theta_1,\theta_2,\theta_3\right) = \gamma_3(\theta_3),$$

where the functions $\alpha_i, \beta_i, \gamma_j$, $i = 1, 2$ and $j = 1, 2, 3$ are defined on the torus T^3.
So F is a diffeomorphism iff the functions α_i and β_i take only integer values and subject to the condition

$$\alpha_1\beta_2 - \alpha_2\beta_1 = \pm 1.$$

We return now to the equations (1) and (2),we obtain

$$\left(\alpha_1 - \beta_2\right)\sin\left(w + 2\theta_3\right) - \left(\alpha_1 + \beta_2\right)\sin\left(w - 2\theta_3\right)$$

$$+\left(\alpha_2 - \beta_1\right)\cos\left(w - 2\theta_3\right) - \left(\alpha_2 + \beta_1\right)\cos\left(w + 2\theta_3\right) = 0.$$

Thus if $w = \pm 2\theta_3$, F is not invertible. In the contrary case, the quantities $\sin\left(w + 2\theta_3\right), \sin\left(w - 2\theta_3\right), \cos\left(w - 2\theta_3\right)$ and $\cos\left(w + 2\theta_3\right)$ are linearly independant, so $\alpha_i = \beta_i = 0$.
In all cases this diffeomorphism do not exist and the contact structures $[\omega_1]$ and $[\omega_2]$ are not isomorphic.
Consequently, there are infinitely many non-isomorphic contact structues $[\omega_n]$ on the torus T^3 given by

$$\omega_n = \cos n\theta_3 d\theta_1 + \sin n\theta_3 d\theta_2, \left(n \in \mathbb{N}\right).$$

This completes the proof of the theorem. □

3. Conclusion

The technics used in this work to find non-isomorphic contact structures can be extended to the sphere S^3 in a first steep and may be to other manifolds suitably choosen. It is also interesting to find the group of diffeomorphisms that leaves the contact structure invariante.

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References

[1] Aggoun, S., *Thesis*, Mulhouse (1989).

[2] Aggoun, S., *Invariant contact form*, Far East. J. Math. Sc. (FJMS), **21**(2006), no. 2, 163-180.

[3] Libermann, P., *Sur les automorphismes infinitésimaux des structures symplectiques et des structures de contact*, Colloque de géométrie différentielle globale, Bruxelles, 1958.

[4] Lichnerowicz, A., *Les variétés de Poisson et leurs algèbres de Lie associées*, J. of Diff. Geometry, **12**(1977), no. 2, 253-300.

[5] Lichnerowicz, A., *Les variétés de Jacobi et leurs algèbres de Lie associées,* J. Math. Pures Appl., **57**(1978), no. 9, 453-488.

[6] Lichnerowicz, A., *Déformations de l'algèbre de Lie des automorphismes infinitésimaux d'une structure de contact,* C. R. Acad. Sci. Paris Sér. A-B, **290**(1980), A241-A245.

[7] Lutz, R., *Sur la géométrie des structures de contact invariantes*, Ann. Inst. Fourier, Grenoble, **29**(1979), no. 1, 283-306.

[8] Lutz, R., *Quelques remarques historiques et prospectives sur la géométrie de contact*, Rend. Sem. Fac. Sci. Univ. Cagliari, **58**(1988), 361-393.

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