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A nonsmooth sublinear elliptic problem in $\mathbb{R}^N$ with perturbations

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Abstract. We study a differential inclusion problem in $\mathbb{R}^{\mathbb{N}}$ involving the p-Laplace operator and a $(p-1)$-sublinear term, $p > N > 1$. Our main result is a multiplicity theorem; we also show the non-sensitivity of our problem with respect to small perturbations.

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1. Introduction

Very recently, Kristály, Marzantowicz and Varga (see [5]) studied a quasilinear differential inclusion problem in $\mathbb{R}^{\mathbb{N}}$ involving a suitable sublinear term. The aim of the present paper is to show that under the same assumptions, a more precise conclusion can be concluded by exploiting a recent result of Iannizzotto (see [3]). To be more precise, we recall the assumptions and the relevant results from [5].

Let $p > 2$ and $F : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz function such that

($\tilde{\mathbf{F}}1$) $\displaystyle\lim_{t\to 0} \frac{\max\{|\xi| : \xi \in \partial F(t)\}}{|t|^{p-1}} = 0;$

($\tilde{\mathbf{F}}2$) $\displaystyle\limsup_{|t|\to+\infty} \frac{F(t)}{|t|^p} \leq 0;$

($\tilde{\mathbf{F}}3$) There exists $\tilde{t} \in \mathbb{R}$ such that $F(\tilde{t}) > 0$, and $F(0) = 0$.

Here and in the sequel, ∂ stands for the generalized gradient of a locally Lipschitz function; see for details Section 2. We consider the differential inclusion problem

$$(\tilde{P}_{\lambda,\mu}) \qquad \begin{cases} -\triangle_p u + |u|^{p-2}u \in \lambda\alpha(x)\partial F(u(x)) + \mu\beta(x)\partial G(u(x)) & \text{on } \mathbb{R}^N, \\ u(x) \to 0 \text{ as } |x| \to \infty, \end{cases}$$

where $p > N \geq 2$, the numbers λ, μ are positive, and $G : \mathbb{R} \to \mathbb{R}$ is any locally Lipschitz function. Furthermore, we assume that $\beta \in L^1(\mathbb{R}^N)$ is any function, and

($\tilde{\alpha}$) $\alpha \in L^1(\mathbb{R}^N) \cap L^\infty_{\text{loc}}(\mathbb{R}^N)$, $\alpha \geq 0$, and $\sup_{R>0} \text{essinf}_{|x|\leq R}\alpha(x) > 0$.

The functional space where the solutions of $(\tilde{P}_{\lambda,\mu})$ are sought is the usual Sobolev space $W^{1,p}(\mathbb{R}^N)$, endowed with its standard norm

$$\|u\| = \left(\int_{\mathbb{R}^N} |\nabla u(x)|^p + \int_{\mathbb{R}^N} |u(x)|^p\right)^{1/p}.$$

The main application in Kristály, Marzantowicz and Varga [5] is as follows.

Theorem A. *Assume that $p > N \geq 2$. Let $\alpha, \beta \in L^1(\mathbb{R}^N)$ be two radial functions, α fulfilling $(\tilde{\alpha})$, and let $F, G : \mathbb{R} \to \mathbb{R}$ be two locally Lipschitz functions, F satisfying the conditions $(\tilde{\mathbf{F}}1)$-$(\tilde{\mathbf{F}}3)$. Then there exists a non-degenerate compact interval $[a, b] \subset]0, +\infty[$ and a number $\tilde{r} > 0$, such that for every $\lambda \in [a, b]$ there exists $\mu_0 \in]0, \lambda + 1]$ such that for each $\mu \in [0, \mu_0]$, the problem $(\tilde{P}_{\lambda,\mu})$ has at least three distinct, radially symmetric solutions with L^∞-norms less than $\tilde{r}$.*

To be more precise, (weak) solutions for $(\tilde{P}_{\lambda,\mu})$ are in the following sense: We say that $u \in W^{1,p}(\mathbb{R}^N)$ is a solution of problem $(\tilde{P}_{\lambda,\mu})$, if there exist $\xi_F(x) \in \partial F(u(x))$ and $\xi_G(x) \in \partial G(u(x))$ for a. e. $x \in \mathbb{R}^N$ such that for all $v \in W^{1,p}(\mathbb{R}^N)$ we have

$$\int_{\mathbb{R}^N} (|\nabla u|^{p-2}\nabla u \nabla v + |u|^{p-2}uv)dx = \lambda \int_{\mathbb{R}^N} \alpha(x)\xi_F v dx + \mu \int_{\mathbb{R}^N} \beta(x)\xi_G v dx. \tag{1.1}$$

Our main result reads as follows:

Theorem 1.1. *Assume that $p > N \geq 2$. Let $\alpha \in L^1(\mathbb{R}^N)$ be a radial function fulfilling $(\tilde{\alpha})$, and let $F : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz function satisfying the conditions $(\tilde{\mathbf{F}}1)$-$(\tilde{\mathbf{F}}3)$. Then there exists $\lambda_0 > 0$ such that for each non-degenerate compact interval $[a, b] \subset]\lambda_0, +\infty[$ there exists a number $r > 0$ with the following property: for every $\lambda \in [a, b]$, every radially symmetric function $\beta \in L^1(\mathbb{R}^N)$ and every locally Lipschitz function $G : \mathbb{R} \to \mathbb{R}$, there exists $\delta > 0$ such that for each $\mu \in [0, \delta]$, the problem $(\tilde{P}_{\lambda,\mu})$ has at least three distinct, radially symmetric solutions with L^∞-norms less than r.*

Remark 1.2. (a) Note that since $p > N$, any element $u \in W^{1,p}(\mathbb{R}^N)$ is homoclinic, i.e., $u(x) \to 0$ as $|x| \to \infty$. This is a consequence of Morrey's embedding theorem.

(b) The terms in the right hand side of (1.1) are well-defined. Indeed, due to Morrey's embedding theorem, i.e., $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$ is continuous ($p > N$), we have $u \in L^\infty(\mathbb{R}^N)$. Thus, there exists a compact interval $I_u \subset \mathbb{R}$ such that $u(x) \in I_u$ for a.e. $x \in \mathbb{R}^N$. Since the set-valued mapping ∂F is upper-semicontinuous, the set $\partial F(I_u) \subset \mathbb{R}$ is bounded; let $C_F = \sup |\partial F(I_u)|$. Therefore,

$$|\int_{\mathbb{R}^N} \alpha(x)\xi_F v dx| \leq C_F \|\alpha\|_{L^1} \|v\|_\infty < \infty.$$

Similar argument holds for the function G.

(c) Note that no hypothesis on the growth of G is assumed; therefore, the last term in $(\tilde{P}_{\lambda,\mu})$ may have an arbitrary growth.

The paper is organized as follows. In Section 2 we recall some basic elements from the theory of locally Lipschitz functions, a recent non-smooth three critical points result of Ricceri-type proved by Iannizzotto [3], and a compactness embedding theorem. In Section 3 we prove Theorem 1.1.

2. Preliminaries

2.1. Locally Lipschitz functions

Let $(X, \|\cdot\|)$ be a real Banach space and X^* its dual. A function $h : X \to \mathbb{R}$ is called locally Lipschitz if each point $u \in X$ possesses a neighborhood U_u of u such that

$$|h(u_1) - h(u_2)| \leq L\|u_1 - u_2\|, \ \forall u_1, u_2 \in U_u,$$

for a constant $L > 0$ depending on U_u. The generalized gradient of h at $u \in X$ is defined as being the subset of X^*

$$\partial h(u) = \{x^* \in X^* : \langle x^*, z\rangle \leq h^0(u; z) \ \text{ for all } \ z \in X\},$$

which is nonempty, convex and w^*-compact, where $\langle\cdot,\cdot\rangle$ is the duality pairing between X^* and X, $h^0(u; z)$ being the generalized directional derivative of h at the point $u \in X$ along the direction $z \in X$, namely

$$h^0(u; z) = \limsup_{\substack{w \to u \\ t \to 0^+}} \frac{h(w + tz) - h(w)}{t},$$

see [2]. Moreover, $h^0(u; z) = \max\{\langle x^*, z\rangle : x^* \in \partial h(u)\}$, $\forall z \in X$. It is easy to verify that $(-h)^0(u; z) = h^0(u; -z)$, and for locally Lipschitz functions $h_1, h_2 : X \to \mathbb{R}$ one has

$$(h_1 + h_2)^0(u; z) \leq h_1^0(u; z) + h_2^0(u; z), \ \forall u, z \in X,$$

and

$$\partial(h_1 + h_2)(u) \subseteq \partial h_1(u) + \partial h_2(u).$$

The Lebourg's mean value theorem says that for every $u, v \in X$ there exist $\theta \in]0, 1[$ and $x_\theta^* \in \partial h(\theta u + (1 - \theta)v)$ such that $h(u) - h(v) = \langle x_\theta^*, u - v\rangle$. If h_2 is continuously Gâteaux differentiable, then $\partial h_2(u) = h_2'(u)$; $h_2^0(u; z)$ coincides with the directional derivative $h_2'(u; z)$ and the above inequality reduces to $(h_1 + h_2)^0(u; z) = h_1^0(u; z) + h_2'(u; z)$, $\forall u, z \in X$.

A point $u \in X$ is a *critical point of* h if $0 \in \partial h(u)$, i.e. $h^0(u, w) \geq 0$, $\forall w \in X$, see [1]. We define $\lambda_h(u) = \inf\{\|x^*\| : x^* \in \partial h(u)\}$. Of course, this infimum is attained, since $\partial h(u)$ is w^*-compact.

2.2. A nonsmooth Ricceri-type critical point theorem

We recall a non-smooth version of a Ricceri-type (see [7]) three critical point theorem proved by Iannizzotto [3]. Before to do that, we need a notion: let X be a Banach space; a functional $I_1 : X \to \mathbb{R}$ is of type (N) if $I_1(u) = \varphi(\|u\|)$ for every $u \in X$, where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous differentiable, convex, increasing mapping with $\varphi(0) = \varphi'(0) = 0$.

Theorem 2.1. [3, Corollary 7] *Let X be a separable and reflexive real Banach space with uniformly convex topological dual X^*, let $I_1 : X \to \mathbb{R}$ be functional of type (N), $I_2 : X \to \mathbb{R}$ be a locally Lipschitz functional with compact derivative such that $I_2(u_0) = 0$. Setting the numbers*

$$\tau = \max\left\{0, \limsup_{\|u\|\to\infty} \frac{I_2(u)}{I_1(u)}, \limsup_{u\to 0} \frac{I_2(u)}{I_1(u)}\right\}, \tag{2.1}$$

$$\chi = \sup_{I_1(u)>0} \frac{I_2(u)}{I_1(u)}, \tag{2.2}$$

assume that $\tau < \chi$.

Then, for each compact interval $[a, b] \subset (1/\chi, 1/\tau)$ (with the conventions $1/0 = \infty$ and $1/\infty = 0$) there exists $\kappa > 0$ with the following property: for every $\lambda \in [a, b]$ and every locally Lipschitz functional $I_3 : X \to \mathbb{R}$ with compact derivative, there exists $\delta > 0$ such that for each $\mu \in [0, \delta]$, the inclusion

$$0 \in \partial I_1(u) - \lambda\partial I_2(u) - \mu\partial I_3(u)$$

admits at least three solutions in X having norm less than κ.

2.3. Embeddings

The embedding $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$ is continuous (due to Morrey's theorem $(p > N)$) but it is not compact. As usual, we may overcome this gap by introducing the subspace of radially symmetric functions of $W^{1,p}(\mathbb{R}^N)$. The action of the orthogonal group $O(N)$ on $W^{1,p}(\mathbb{R}^N)$ can be defined by

$$(gu)(x) = u(g^{-1}x),$$

for every $g \in O(N)$, $u \in W^{1,p}(\mathbb{R}^N)$, $x \in \mathbb{R}^N$. It is clear that this compact group acts linearly and isometrically; in particular $\|gu\| = \|u\|$ for every $g \in O(N)$ and $u \in W^{1,p}(\mathbb{R}^N)$. The subspace of radially symmetric functions of $W^{1,p}(\mathbb{R}^N)$ is defined by

$$W^{1,p}_{\text{rad}}(\mathbb{R}^N) = \{u \in W^{1,p}(\mathbb{R}^N) : gu = u \text{ for all } g \in O(N)\}.$$

Proposition 2.2. [6] *The embedding $W^{1,p}_{\text{rad}}(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$ is compact whenever $2 \le N < p < \infty$.*

3. Proof of Theorem 1.1

Let $I_1 : W^{1,p}(\mathbb{R}^N) \to \mathbb{R}$ be defined by

$$I_1(u) = \frac{1}{p}\|u\|^p,$$

and let $I_2, I_3 : L^\infty(\mathbb{R}^N) \to \mathbb{R}$ be

$$I_2(u) = \int_{\mathbb{R}^N} \alpha(x)F(u(x))dx \quad \text{and} \quad I_3(u) = \int_{\mathbb{R}^N} \beta(x)G(u(x))dx.$$

Since $\alpha, \beta \in L^1(\mathbb{R}^N)$, the functionals I_2, I_3 are well-defined and locally Lipschitz, see Clarke [2, p. 79-81]. Moreover, we have

$$\partial I_1(u) \subseteq \int_{\mathbb{R}^N} \alpha(x)\partial F(u(x))dx, \quad \partial I_2(u) \subseteq \int_{\mathbb{R}^N} \beta(x)\partial G(u(x))dx.$$

The energy functional $\mathcal{E}_{\lambda,\mu} : W^{1,p}(\mathbb{R}^N) \to \mathbb{R}$ associated to problem $(\tilde{P}_{\lambda,\mu})$, is given by

$$\mathcal{E}_{\lambda,\mu}(u) = I_1(u) - \lambda I_2(u) - \mu I_3(u), \quad u \in W^{1,p}(\mathbb{R}^N).$$

It is clear that the critical points of the functional $\mathcal{E}_{\lambda,\mu}$ are solutions of the problem $(\tilde{P}_{\lambda,\mu})$ in the sense of relation (1.1).

Since α, β are radially symmetric, then $\mathcal{E}_{\lambda,\mu}$ is $O(N)$-invariant, i.e. $\mathcal{E}_{\lambda,\mu}(gu) = \mathcal{E}_{\lambda,\mu}(u)$ for every $g \in O(N)$ and $u \in W^{1,p}(\mathbb{R}^N)$. Therefore, we may apply a non-smooth version of the *principle of symmetric criticality,* proved by Krawcewicz-Marzantowicz [4], whose form in our setting is as follows.

Proposition 3.1. *Any critical point of* $\mathcal{E}^{\text{rad}}_{\lambda,\mu} = \mathcal{E}_{\lambda,\mu}|_{W^{1,p}_{\text{rad}}(\mathbb{R}^N)}$ *will be also a critical point of* $\mathcal{E}_{\lambda,\mu}$.

Therefore, it remains to find critical point for the functional $\mathcal{E}^{\text{rad}}_{\lambda,\mu}$; here, we will check the assumptions of Theorem 2.1 with the choice $X = W^{1,p}_{\text{rad}}(\mathbb{R}^N)$.

It is standard that X is a reflexive, separable Banach space with uniformly convex topological dual X^*. The functional I_1 is of type (N) on X since $I_1(u) = \varphi(\|u\|)$ where $\varphi(s) = \frac{s^p}{p}$, $s \geq 0$.

Proposition 3.2. ∂I_2 *is compact on* $X = W^{1,p}_{\text{rad}}(\mathbb{R}^N)$.

Proof. Let $\{u_n\}$ be a bounded sequence in X and let $u_n^* \in \partial I_2(u_n)$. It is clear that u_n^* is also bounded in X^* by exploiting Remark 1.2 (b) and hypothesis $(\tilde{\alpha})$. Thus, up to a subsequence, we may assume that $u_n^* \to u^*$ weakly in X^* for some $u^* \in X^*$. By contradiction, let us assume that $\|u_n^* - u^*\|_* > M, \ \ \forall n \in \mathbb{N}$, for some $M > 0$. In particular, there exists $v_n \in X$ with $\|v_n\| \leq 1$ such that

$$(u_n^* - u^*)(v_n) > M.$$

Once again, up to a subsequence, we may suppose that $v_n \to v$ weakly in X for some $v \in X$. Now, applying Proposition 2.2, we may also assume that

$$\|v_n - v\|_{L^\infty} \to 0.$$

Combining the above facts, we obtain that

$$\begin{aligned} M < (u_n^* - u^*)(v_n) &= (u_n^* - u^*)(v) + u_n^*(v_n - v) + u^*(v - v_n) \\ &\leq (u_n^* - u^*)(v) + C\|v_n - v\|_{L^\infty} + u^*(v - v_n) \end{aligned}$$

for some $C > 0$. Since all the terms from the right hand side tend to 0, we get a contradiction. □

Proposition 3.3. $\lim\limits_{u\to 0} \frac{I_2(u)}{I_1(u)} = 0$.

Proof. Due to ($\tilde{\mathbf{F}}1$), for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that

$$|\xi| \leq \varepsilon |t|^{p-1}, \quad \forall t \in [-\delta(\varepsilon), \delta(\varepsilon)], \ \forall \xi \in \partial F(t). \tag{3.1}$$

For any $0 < t \leq \frac{1}{p}\left(\frac{\delta(\varepsilon)}{c_\infty}\right)^p$ define the set

$$S_t = \{\ u \in W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N) : \|u\|^p < pt\},$$

where $c_\infty > 0$ denotes the best constant in the embedding $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$.

Note that $u \in S_t$ implies that $\|u\|_\infty \leq \delta(\varepsilon)$; indeed, we have $\|u\|_\infty \leq c_\infty \|u\| < c_\infty (pt)^{1/p} \leq \delta(\varepsilon)$. Fix $u \in S_t$; for a.e. $x \in \mathbb{R}^N$, Lebourg's mean value theorem and (3.1) imply the existence of $\xi_x \in \partial F(\theta_x u(x))$ for some $0 < \theta_x < 1$ such that

$$|F(u(x))| = |F(u(x)) - F(0)| = |\xi_x u(x)| \leq \varepsilon |u(x)|^p.$$

Consequently, for every $u \in S_t$ we have

$$\begin{aligned} |I_2(u)| &= |\int_{\mathbb{R}^N} \alpha(x) F(u(x)) dx| \leq \varepsilon \int_{\mathbb{R}^N} \alpha(x) |u(x)|^p dx \\ &\leq \varepsilon \|\alpha\|_{L^1} \|u\|_\infty^p \leq \varepsilon \|\alpha\|_{L^1} c_\infty^p \|u\|^p. \end{aligned}$$

Therefore, for every $u \in S_t \setminus \{0\}$ with $0 < t \leq \frac{1}{p}\left(\frac{\delta(\varepsilon)}{c_\infty}\right)^p$ we have

$$0 \leq \frac{|I_2(u)|}{I_1(u)} \leq \varepsilon \|\alpha\|_{L^1} c_\infty^p p.$$

Since $\varepsilon > 0$ is arbitrary, we obtain the required limit. □

Proposition 3.4. $\limsup_{\|u\| \to \infty} \frac{I_2(u)}{I_1(u)} \leq 0.$

Proof. By ($\tilde{\mathbf{F}}2$), for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that

$$F(t) \leq \varepsilon |t|^p, \quad \forall |t| \in [\delta(\varepsilon), \infty[. \tag{3.2}$$

Consequently, for every $u \in W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ we have

$$\begin{aligned} I_2(u) &= \int_{\mathbb{R}^N} \alpha(x) F(u(x)) dx \\ &= \int_{\{x \in \mathbb{R}^N : |u(x)| > \delta(\varepsilon)\}} \alpha(x) F(u(x)) dx + \int_{\{x \in \mathbb{R}^N : |u(x)| \leq \delta(\varepsilon)\}} \alpha(x) F(u(x)) dx \\ &\leq \varepsilon \int_{\{x \in \mathbb{R}^N : |u(x)| > \delta(\varepsilon)\}} \alpha(x) |u(x)|^p dx + \max_{|t| \leq \delta(\varepsilon)} |F(t)| \int_{\{x \in \mathbb{R}^N : |u(x)| \leq \delta(\varepsilon)\}} \alpha(x) dx \\ &\leq \varepsilon \|\alpha\|_{L^1} c_\infty^p \|u\|^p + \|\alpha\|_{L^1} \max_{|t| \leq \delta(\varepsilon)} |F(t)|. \end{aligned}$$

Therefore, for every $u \in W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N) \setminus \{0\}$, we have

$$\frac{I_2(u)}{I_1(u)} \leq \varepsilon p \|\alpha\|_{L^1} c_\infty^p + p \|\alpha\|_{L^1} \max_{|t| \leq \delta(\varepsilon)} |F(t)| \|u\|^{-p}.$$

Once $\|u\| \to \infty$, the claim is proved, taking into account that $\varepsilon > 0$ is arbitrary. □

Due to hypothesis ($\tilde{\alpha}$), one can fix $R > 0$ such that $\alpha_R = \text{essinf}_{|x|\leq R}\alpha(x) > 0$. For $\sigma \in]0,1[$ define the function

$$w_\sigma(x) = \begin{cases} 0, & \text{if} \quad x \in \mathbb{R}^N \setminus B_N(0,R); \\ \tilde{t}, & \text{if} \quad x \in B_N(0,\sigma R); \\ \frac{\tilde{t}}{R(1-\sigma)}(R - |x|), & \text{if} \quad x \in B_N(0,R) \setminus B_N(0,\sigma R), \end{cases}$$

where $B_N(0,r)$ denotes the $N-$dimensional open ball with center 0 and radius $r > 0$, and $\tilde{t}$ comes from ($\tilde{\mathbf{F}}$3). Since $\alpha \in L^\infty_{\text{loc}}(\mathbb{R}^N)$, then $M(\alpha, R) = \sup_{x \in B_N(0,R)} \alpha(x) < \infty$. A simple estimate shows that

$$I_2(w_\sigma) \geq \omega_N R^N [\alpha_R F(\tilde{t})\sigma^N - M(\alpha, R) \max_{|t|\leq|\tilde{t}|} |F(t)|(1 - \sigma^N)].$$

When $\sigma \to 1$, the right hand side is strictly positive; choosing σ_0 close enough to 1, for $u_0 = w_{\sigma_0}$ we have $I_2(u_0) > 0$.

Proof of Theorem 1.1. It remains to combine Theorem 2.1 with Propositions 3.1-3.4. The definitions of the number τ and χ, see relations (2.2)-(2.1), show that $\tau = 0$ and

$$\lambda_0 := \chi^{-1} = \inf_{I_2(u)>0} \frac{I_1(u)}{I_2(u)}$$

is well-defined, positive which is the number appearing in the statement of Theorem 1.1. □

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References

[1] Chang, K.-C., *Variational methods for non-differentiable functionals and their applications to partial differential equations,* J. Math. Anal. Appl., **80**(1981), 102-129.

[2] Clarke, F.H., *Nonsmooth analysis and Optimization,* Wiley, New York, 1983.

[3] Iannizzotto, A., *Three solutions for a partial differential inclusion via nonsmooth critical point theory,* Set-Valued and Variational Analysis, in press. DOI: 10.1007/s11228-010-0145-9

[4] Krawcewicz, W., Marzantowicz, W., *Some remarks on the Lusternik-Schnirelman method for non-differentiable functionals invariant with respect to a finite group action,* Rocky Mountain J. Math., **20**(1990), 1041-1049.

[5] Kristály, A., Marzantowicz, W., Varga, Cs., *A non-smooth three critical points theorem with applications in differential inclusions,* J. Global Optim., **46** (2010), no. 1, 4962.

[6] Kristály, A., Varga, Cs., *On a class of quasilinear eigenvalue problems in* $\mathbf{R}^N$, Math. Nachr., **278**(2005), no. 15, 17561765.

[7] Ricceri, B., *A further three critical points theorem,* Nonlinear Analysis, **71**(2009), 4151-4157.

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