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On analytic functions with generalized bounded variation

Ningegowda Ravikumar and Satyanarayana Latha

Abstract. In this paper we study a class introduced by Bhargava and Nanjunda Rao which unifies a number of classes studied previously by Mocanu and others. This class includes several known classes of analytic functions such as convex and starlike functions of order β, α-convex functions, functions with bounded boundary rotation, bounded radius rotation and bounded Mocanu variation. Several interesting properties like inclusion results, arclength problem, coefficient bounds and distortion results of this class are discussed.

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1. First section (Introduction)

Let $\mathcal{A}$ denote the class of analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

in the unit disc $\mathcal{E} = \{z; |z| < 1\}$. Let P designate the class of functions p which are analytic, have positive real part in $\mathcal{E}$ and satisfy $p(0) = 1$. Let M_k denote the class of real-valued functions $\mu(t)$ of bounded variation on $[0, 2\pi]$ which satisfy the conditions,

$$\int_0^{2\pi} d\mu(t) = 2, and \quad \int_0^{2\pi} |d\mu(t)| \leq k. \tag{1.2}$$

M_2 is clearly the class of nondecreasing functions on $[0, 2\pi]$ satisfying

$$\int_0^{2\pi} d\mu(t) = 2.$$

If $\mu(t) \in M_k$ with $k > 2$ we can write $\mu(t) = \alpha(t) - \beta(t)$ where $\alpha(t)$ and $\beta(t)$ are both nondecreasing functions on $[0, 2\pi]$ and satisfy

$$\int_0^{2\pi} d\alpha(t) \leq \frac{k}{2} + 1, and \quad \int_0^{2\pi} d\beta(t) \leq \frac{k}{2} - 1. \tag{1.3}$$

Let P_k denote the class of functions p analytic in $\mathcal{E}$ such that $p(0) = 1$, $z = re^{i\theta}$. and having the representation

$$p(z) = \frac{1}{2} \int_0^{2\pi} \frac{1 + ze^{-it}}{1 - ze^{-it}} d\mu(t), \tag{1.4}$$

where $\mu(t) \in M_k$. This class has been studied by Pinchuk [5] .

Clearly $P_2 = P$. We can write for $p(z) \in P_k$ as

$$p(z) = \frac{1}{2}\left(\frac{k}{2} + 1\right) P_1(z) - \frac{1}{2}\left(\frac{k}{2} - 1\right) P_2(z)$$

where $P_1, P_2 \in P$.

Definition 1.1. *Let $f \in \mathcal{A}$ with $\frac{f(z)f'(z)}{z} \neq 0$ in $\mathcal{E}$, and let*

$$J_f = J_f(\alpha, b, c) = (1 - \alpha)\left[1 - \frac{1}{c} + \frac{z}{c}\frac{f'(z)}{f(z)}\right] + \alpha\left[1 + \frac{z}{b}\frac{f''(z)}{f'(z)}\right]$$

where α, $b \neq 0 \, and \, c \neq 0$ are complex numbers.

Let $B_k(\alpha, b, c)$ be the class of all fuctions f in $\mathcal{E}$, such that if $J_f \in P_k$ for $z \in \mathcal{E}, k \geq 2$.

This class is a particular case of the class studied earlier by Bhargava and Nanjunda Rao [1] which unifies and generalizes various classes studied earlier by Robertson [6], Moulis [3], Pinchuk [5], Padmanabhan and Parvatham [4], and Khalida Inayat Noor and Ali Muhammad [2].

For $f, g \in \mathcal{A}$ given by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ the Hardmard product is given by $(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n$.

Let Γ denote the Gamma function of Euler and $G(l, m, n; z)$ be the analytic function for z in $\mathcal{E}$ defined by

$$G(l, m, n; z) = \frac{\Gamma(n)}{\Gamma(l)\Gamma(n - l)} \int_0^1 u^{l-1}(1 - u)^{n-l-1}(1 - zu)^{-n} du, \tag{1.5}$$

where $\Re\{l\} > 0$, $and, \Re\{l - n\} > 0$. Also we define

$$N(\alpha, b, c, r) = r\left[G\left(\frac{2b}{\alpha c}, M, M + 1, r\right)\right]^{\frac{1}{M}}. \tag{1.6}$$

and

$$f_\theta(\alpha, b, c, z) = \left[M \int_0^z t^{M-1}(1 - e^{i\theta}t)^{\frac{-2b}{\alpha}} dt\right]^{\frac{1}{M}} \tag{1.7}$$

where $M = 1 + \frac{(1-\alpha)b}{\alpha c}$, $\alpha \neq 0$, $0 \leq \theta \leq 2\pi$.

2. Second section

We use the following lemmas to prove the main results.

Lemma 2.1. *Let p be analytic in $\mathcal{E}$ and $p(0) = 1$, then $\alpha \geq 0, z \in \mathcal{E}$, $\left(p + \frac{\alpha z p'}{p}\right) \in P_k$ implies $p \in P_k$.*

Lemma 2.2. [7] *Let $f \in \mathcal{A}$ with $\frac{f(z)f'(z)}{z} \neq 0$ in $\mathcal{E}$, then f is univalent in $\mathcal{E}$ if and only if for $0 \leq \theta_1 < \theta_2 \leq 2\pi$ and $0 < r < 1$, we have*

$$\int_{\theta_1}^{\theta_2} \left[\Re\left\{1 + z\frac{f''(z)}{f'(z)} + (\beta - 1)z\frac{f'(z)}{f(z)}\right\} - \alpha\Im z\frac{f'(z)}{f(z)}\right] d\theta > -\pi$$

with $z = re^{i\theta}$, $\beta > 0$ and α real.

Theorem 2.3. *$f \in B_k(\alpha, b, c)$, $\alpha \neq 0, b \neq 0, c \neq 0$, if and only if there is a function $g \in B_k(0, b, 1) = R_k$ such that*

$$f(z) = \left[M\int_0^z t^{M-1}\left(\frac{g(t)}{t}\right)^{\frac{b}{\alpha}} dt\right]^{\frac{1}{M}}, \tag{2.1}$$

where $M = 1 + \frac{(1-\alpha)b}{\alpha c}$.

Proof. Using (2.1) we get,

$$(1-\alpha)\frac{z}{c}\frac{f'(z)}{f(z)} + \frac{\alpha}{b}z\frac{f''(z)}{f'(z)} = \frac{1-\alpha}{c} + z\frac{g'(z)}{g(z)} - 1$$

$$(1-\alpha)\left[1 - \frac{1}{c} + \frac{z}{c}\frac{f'(z)}{f(z)}\right] + \alpha\left[1 + \frac{z}{b}\frac{f''(z)}{f'(z)}\right] = z\frac{g'(z)}{g(z)}$$

If J_f belongs to P_k, so does the left hand side and conversely. □

Putting $c = 1$ and $b = 1 - \beta$ in above Theorem we get the following corollary.

Corollary 2.4. [2] *$f \in B_k(\alpha, \beta)$, $\alpha \neq 0$, if and only if there is a function $g \in B_k(0, \beta) = R_k$ such that*

$$f(z) = \left[M\int_0^z t^{M-1}\left(\frac{g(t)}{t}\right)^{\frac{1-\beta}{\alpha}} dt\right]^{\frac{1}{M}},$$

where $M = 1 + \frac{(1-\alpha)(1-\beta)}{\alpha}$.

Theorem 2.5. *Let $f \in B_k(\alpha, b, c)$ then the function*

$$g(z) = z\left(\frac{f(z)}{z}\right)^{\frac{1-\alpha}{c}} (f'(z))^{\frac{\alpha}{b}} \tag{2.2}$$

belongs to R_k for all $z \in \mathcal{E}$.

Proof. Logarithmic differentiation of (2.2) yields

$$z\frac{g'(z)}{g(z)} = 1 + \frac{(1-\alpha)}{c} z\frac{f'(z)}{f(z)} - \frac{(1-\alpha)}{c} + \frac{\alpha}{b} z\frac{f''(z)}{f'(z)}$$
$$= (1-\alpha)\left[1 - \frac{1}{c} + \frac{z}{c}\frac{f'(z)}{f(z)}\right] + \alpha\left[1 + \frac{z}{b}\frac{f''(z)}{f'(z)}\right]$$

Since $f \in B_k(\alpha, b, c)$ the result follows. □

For the parametric values $c = 1$ and $b = 1 - \beta$ we get the following result.

Corollary 2.6. [2] *Let $f \in B_k(\alpha, \beta)$ then the function*

$$g(z) = z\left(\frac{f(z)}{z}\right)^{1-\alpha} (f'(z))^{\frac{\alpha}{1-\beta}}$$

belongs to R_k for all $z \in \mathcal{E}$.

Remark 2.7. The above Theorem can also be obtained as a particular case of Theorem 3.1 by Bhargava and Nanjunda Rao [1].

Theorem 2.8. *$B_k(\alpha, b, c) \subset R_k$, for $\alpha > 0, b \neq 0$.*

Proof. Let $\frac{zf'(z)}{f(z)} = p(z)$, p analytic in $\mathcal{E}$, with $p(0) = 1$. Now

$$\frac{1}{b}\left\{(1-\alpha)b\left[1 - \frac{1}{c} + \frac{z}{c}\frac{f'(z)}{f(z)}\right] + \alpha\left[b + \frac{zf''(z)}{f'(z)}\right]\right\}$$
$$= \frac{\alpha}{b}\left\{\frac{(1-\alpha)}{\alpha c} b[(c-1) + p(z)] + \alpha\left[b + \frac{zf''(z)}{f'(z)}\right]\right\}$$
$$= \frac{\alpha}{b}\left\{(M-1)[(c-1) + p(z)] + \left[b + \frac{zp'(z)}{p(z)} + p(z) - 1\right]\right\}$$
$$= \frac{\alpha}{b}\left[Mp(z) + \frac{zp'(z)}{p(z)} + (M-1)(c-1) + (b-1)\right]$$
$$= \frac{\alpha}{b}\left[M\left\{p(z) + \frac{1}{M}\frac{zp'(z)}{p(z)}\right\} + (M-1)(c-1) + (b-1)\right] \in P_k.$$

Therefore $\left\{p(z) + \frac{1}{M}\frac{zp'(z)}{p(z)}\right\} \in P_k$, and by using Lemma 2.1. it follows that $p \in P_k$, $z \in \mathcal{E}$. This proves that $f \in R_k$. □

Corollary 2.9. [2] *$B_k(\alpha, \beta) \subset R_k$, for $\alpha > 0, 0 \leq \beta < 1$.*

Theorem 2.10. i. *$B_k(\alpha, b, c) \subset B_{k_1}(\alpha_1, b, c)$, $0 < \alpha \leq \alpha_1$, and $k_1 = k\left(\frac{2\alpha_1 - \alpha}{\alpha}\right)$.*
ii. *$B_k(\alpha, b, c) \subset B_k(\alpha_1, b, c)$, $0 \leq \alpha_1 < \alpha$.*

Proof. (*i*) Let $f \in B_k(\alpha, b, c)$ then

$$\frac{1}{b}\left\{(1-\alpha_1)b\left[1 - \frac{1}{c} + \frac{z}{c}\frac{f'(z)}{f(z)}\right] + \alpha_1\left[b + \frac{zf''(z)}{f'(z)}\right]\right\}$$
$$= \frac{\alpha_1}{\alpha}\left\{(1-\alpha)b\left[1 - \frac{1}{c} + \frac{z}{c}\frac{f'(z)}{f(z)}\right] + \alpha\left[b + \frac{zf''(z)}{f'(z)}\right]\right\} - \frac{(\alpha_1 - \alpha)}{\alpha} b\left[1 - \frac{1}{c} + \frac{z}{c}\frac{f'(z)}{f(z)}\right]$$

$$= b\left[\frac{\alpha_1}{\alpha}h_1(z) - \frac{(\alpha_1-\alpha)}{\alpha}h_2(z)\right], \quad h_1, h_2 \in P_k. \tag{2.3}$$

by using Definition 1.1 and Theorem 2.8. From (2.3) it follows that

$$\int_0^{2\pi} |\Re J_f| d\theta \leq \left[\frac{\alpha_1}{\alpha} + \frac{(\alpha_1-\alpha)}{\alpha}\right] k\pi = \left(\frac{2\alpha_1-\alpha}{\alpha}\right) k\pi.$$

(ii) Let $f \in B_k(\alpha, b, c)$. Then

$$(1-\alpha_1)\left[1 - \frac{1}{c} + \frac{z}{c}\frac{f'(z)}{f(z)}\right] + \alpha_1\left[1 + \frac{z}{b}\frac{f''(z)}{f'(z)}\right]$$

$$= \left(1-\frac{\alpha_1}{\alpha}\right)\left[1-\frac{1}{c}+\frac{z}{c}\frac{f'(z)}{f(z)}\right] + \frac{\alpha_1}{\alpha}\left\{(1-\alpha)\left[1-\frac{1}{c}+\frac{z}{c}\frac{f'(z)}{f(z)}\right] + \alpha\left[1+\frac{z}{b}\frac{f''(z)}{f'(z)}\right]\right\}$$

$= \left(1-\frac{\alpha_1}{\alpha}\right)H_1(z) + \frac{\alpha_1}{\alpha}H_2(z), \quad H_1, H_2 \in P_k$, $z \in \mathcal{E}$, since P_k is a convex set. Therefore $f \in B_k(\alpha_1, b, c)$, for $z \in \mathcal{E}$. □

Corollary 2.11. [2]

i. $B_k(\alpha,\beta) \subset B_{k_1}(\alpha_1,\beta)$, $0 < \alpha \leq \alpha_1$, *and* $k_1 = k\left(\frac{2\alpha_1-\alpha}{\alpha}\right)$

ii. $B_k(\alpha,\beta) \subset B_k(\alpha_1,\beta)$, $0 \leq \alpha_1 < \alpha$.

Theorem 2.12. *Let $f \in B_k(\alpha, b, c)$. Then f is univalent in $\mathcal{E}$ for $k \leq \frac{2(3\alpha c - bc + 2b - 2b\alpha)}{bc}$.*

Proof. Since $f \in B_k(\alpha, b, c)$, also we have $z = re^{i\theta}$, $0 \leq r < 1$, $0 \leq \theta_1 < \theta_2 \leq 2\pi$

$$\int_{\theta_1}^{\theta_2} \Re\left\{\frac{(1-\alpha)}{\alpha c}\left[c - 1 + \frac{zf'(z)}{f(z)}\right] + \frac{1}{b}\left[1 + \frac{zf''(z)}{f'(z)}\right]\right\} d\theta \geq -\left(\frac{k}{2}-1\right)\frac{\pi}{\alpha} - \left(\frac{b-1}{b}\right)2\pi$$

$$\int_{\theta_1}^{\theta_2} \Re\left\{\left[1 + \frac{zf''(z)}{f'(z)}\right] + \left[\frac{b(1-\alpha)}{\alpha c} - 1\right]\frac{zf'(z)}{f(z)}\right\} d\theta$$

$$\geq -\left[\left(\frac{k}{2}-1\right)\frac{b}{\alpha} + 2(b-1) + \frac{2(1-\alpha)(c-1)b}{\alpha c}\right]\pi$$

by using Lemma 2.2, that f is univalent in $\mathcal{E}$ if $k \leq \frac{2(3\alpha c - bc + 2b - 2b\alpha)}{bc}$. □

Corollary 2.13. *Let $f \in B_k(\alpha, \beta)$. Then f is univalent in $\mathcal{E}$ for $k \leq \frac{2(\alpha + 2\alpha\beta - \beta + 1)}{(1-\beta)}$.*

Theorem 2.14. *Let $f \in B_k(\alpha, b, c)$, $\alpha > 0$ and $L_r(f)$ denote the length of the curve $C = f(re^{i\theta})$, $0 \leq \theta \leq 2\pi$ and $N(r) = \max_{0\leq\theta\leq 2\pi} |f(re^{i\theta})|$, then for $0 < r < 1$,*

$$L_r(f) \leq \frac{N(r)b}{\alpha}\left\{k + \frac{(\alpha-1)}{c}[(c-1)2 + k] + \frac{\alpha}{b}(1-b)2\right\}\pi, \quad \alpha > 0.$$

Proof. We have, $z = re^{i\theta}$

$$L_r(f) = \int_0^{2\pi} |zf'(z)| d\theta = \int_0^{2\pi} zf'(z)e^{-iarg(zf'(z))} d\theta.$$

On integration we get,

$$L_r(f) = \int_0^{2\pi} f(z)e^{-iarg(zf'(z))}\Re\left\{\frac{(zf'(z))'}{f'(z)}\right\} d\theta$$

$$\leq \frac{N(r)b}{\alpha}\int_0^{2\pi}\left|\Re J_f+(\alpha-1)\left[1-\frac{1}{c}+\frac{z}{c}\frac{f'(z)}{f(z)}\right]-\alpha+\frac{\alpha}{b}\right|d\theta$$
$$\leq \frac{N(r)b}{\alpha}\left\{k\pi+(\alpha-1)\left[\left(1-\frac{1}{c}\right)2\pi+\frac{\pi k}{c}\right]+\alpha(\frac{1}{b}-1)2\pi\right\}$$
$$\leq \frac{N(r)b}{\alpha}\left\{k+\frac{(\alpha-1)}{c}[(c-1)2+k]+\frac{\alpha}{b}(1-b)2\right\}\pi.$$

□

Corollary 2.15. [2] *Let $f\in B_k(\alpha,\beta)$, $\alpha>2$ and $L_r(f)$ denote the length of the curve $C=f(re^{i\theta})$, $0\leq\theta\leq 2\pi$ and $N(r)=\max\limits_{0\leq\theta\leq 2\pi}|f(re^{i\theta})|$, then for $0<r<1$,*

$$L_r(f)\leq(1-\beta)N(r)\left[k+\frac{2\beta}{1-\beta}\right]\pi,\quad \alpha>0.$$

Theorem 2.16. *Let f given by (1.1) belongs to $B_k(\alpha,b,c)$ for $\alpha\geq 0$. Then for $n\geq 2$, $na_n=O(1)N\left(\frac{n-1}{n}\right)$, where $O(1)$ is a constant depending on α,b,c,k only.*

Proof. We have,

$$na_n=\frac{1}{2\pi r^n}\int_0^{2\pi} zf'(z)e^{-in\theta}d\theta,\quad z=re^{i\theta}$$
$$na_n\leq\frac{1}{2\pi r^n}\int_0^{2\pi}|zf'(z)|d\theta=\frac{1}{2\pi r^n}L_r(f).$$

By using Theorem 2.14 and $r=\frac{n-1}{n}$, we get the required result. □

Corollary 2.17. [2] *Let f given by (1.1) belongs to $B_k(\alpha,\beta)$ for $\alpha\geq 0$. Then for $n\geq 2$, $na_n=O(1)N\left(\frac{n-1}{n}\right)$, where $O(1)$ is a constant depending on α,β,k only.*

Theorem 2.18. *Let $f\in B_2(\alpha,b,c)$, $\alpha\neq 0, b\neq 0, c\neq 0$ and $|z|=r$ $(0<r<1)$. Then*

$$(i)\ N(\alpha,b,c,-r)\leq|f(z)|\leq N(\alpha,b,c,r),\quad for\,\alpha>0.$$
$$(ii)\ N(\alpha,b,c,r)\leq|f(z)|\leq N(\alpha,b,c,-r),\quad for\,\alpha<0.$$

This result is sharp and equality occurs, for the function $f_\theta(\alpha,b,c,z)$ defined by (1.7), with suitably chosen θ.

Proof. We consider $\alpha>0$. From Theorem 2.3, certifies the existence of $f\in B_2(\alpha,b,c)$ if and only if there exists a $g\in R_2=S^*$ such that

$$f(z)=\left[M\int_0^z t^{M-1}\left(\frac{g(t)}{t}\right)^{\frac{b}{\alpha}}dt\right]^{\frac{1}{M}},\quad \text{where}\, M=1+\frac{(1-\alpha)b}{\alpha c}.\tag{2.4}$$

Taking $z=r$, $t=\rho e^{i\theta}$ and integrating , we get from(2.4),

$$f(r)=\left[Me^{i\theta M}\int_0^r\rho^{M-1}\left(\frac{g(\rho)}{\rho}\right)d\rho\right]^{\frac{1}{M}}.\tag{2.5}$$

Since g is starlike , we have

$$\frac{\rho}{(1+\rho)^2}\leq|g(t)|\leq\frac{\rho}{(1-\rho)^2}.\tag{2.6}$$

Using (2.6) in (2.5), we get

$$|f(r)|^M \leq M\int_0^r \rho^{M-1}(1-\rho)^{\frac{-2b}{\alpha}}d\rho = Mr^M\int_0^1 u^{M-1}(1-ru)^{\frac{-2b}{\alpha}}du. \tag{2.7}$$

Therefore $|f(r)| \leq N(\alpha, b, c, r), \quad \alpha > 0$.

It remains only to prove that the left-hand inequality. We consider the straight line Γ^* joining 0 to $f(z) = Re^{i\phi}$. Γ^* is the image of a Jordan arc γ in $\mathcal{E}$ connecting 0 and $z = re^{i\theta}$. If z_0 is a point on the circumference $|z| = r$ such that

$$|f(z_0)| = \min_{0\leq\theta\leq 2\pi} |f(re^{i\theta})|.$$

Using (2.5) and (2.6), we get

$$|f(z_0)|^M \geq M\int_0^r \rho^{M-1}(1+\rho)^{\frac{-2b}{\alpha}}d\rho = Mr^M\int_0^1 u^{M-1}(1+ru)^{\frac{-2b}{\alpha}}du.$$

$$|f(z)| \geq N(\alpha, b, c, -r), \quad \alpha > 0.$$

Proof of (ii) is analogous to proof of (i). □

Corollary 2.19. [2] *Let* $f \in B_2(\alpha, \beta), \quad \alpha \neq 0, 0 < \beta < 1$ *and* $|z| = r \ (0 < r < 1)$. *Then*

$$(i)\ N(\alpha, \beta, -r) \leq |f(z)| \leq N(\alpha, \beta, r), for\, \alpha > 0.$$

$$(ii)\ N(\alpha, \beta, r) \leq |f(z)| \leq N(\alpha, \beta, -r), for\, \alpha < 0.$$

This result is sharp and equality occurs, for the function $f_\theta(\alpha, b, c, z)$ *defined by (1.7), with suitably chosen* θ.

Remark 2.20. The above Theorem can be obtained as a particular case of Corollary 3.2 by Bhargava and Nanjunda Rao [1].

Theorem 2.21. *Let* $f \in B_2(\alpha, 1, c)$, $\alpha > 0$. *Then, for* $|z| = r \ (0 < r < 1)$, *we have*

$$\frac{r + |\alpha-1|(1+r)^2 N(\alpha, 1, c, -r)}{\alpha r(1+r)^2} \leq |f'(z)| \leq \frac{r + |\alpha-1|(1-r)^2 N(\alpha, 1, c, -r)}{\alpha r(1-r)^2}.$$

This result is sharp.

Theorem 2.22. *Let* $f \in B_2(\alpha, b, c)$, $\alpha \neq 0, \ b \neq 0$. *and be given by (1.1). Then*

$$|a_2| \leq \frac{2b}{|(1-\alpha)b + 2\alpha c|}.$$

Proof. By using Theorem 2.18, we have

$$N(\alpha, b, c, r) = r + \frac{2b}{(1-\alpha)b + 2\alpha c}r^2 + O(r^3),$$

and

$$|f(r)| = r + a_2 r^2 + O(r^3).$$

Therefore, we have

$$a_2 \leq \frac{2b}{(1-\alpha)b + 2\alpha c} \quad (\alpha > 0).$$ □

Corollary 2.23. [2] *Let* $f \in B_2(\alpha,\beta)$,$\alpha \neq 0,\ 0<\beta<1$. *and be given by* (1.1). *Then*

$$|a_2| \leq \frac{2(1-\beta)}{|(1-\alpha)(1-\beta+2\alpha|}.$$

Remark 2.24. The above Theorem can be obtained as a particular case of Corollary 3.1 by Bhargava and Nanjunda Rao [1].

Theorem 2.25. *Let* $f \in B_k(1,b,c)$. *Then, with* $|z|=r,\ r_1=\frac{1-r}{1+r}$ *we have*

$$\frac{2^{m-1}}{l}[G(l,m,n,-1)-r^l{}_1G(l,m,n,-r_1)] \leq |f(z)|$$
$$\leq \frac{2^{m-1}}{l}[G(l,m,n,-1)-r_1^{-l}G(l,m,n,-r_1^{-1})]$$

where $l=\left(\frac{k}{2}-1\right)b+1, \quad m=2(1-b), \quad n=\left(\frac{k}{2}-1\right)b+2.$

Proof. Since $f \in B_k(1,b,c)$. we have from (2.4)

$$f'(z)=\left(\frac{g(z)}{z}\right)^b, \quad g \in R_k.$$

Since $g \in R_k$

$$\frac{(1-|z|)^{\frac{k}{2}-1}}{(1+|z|)^{\frac{k}{2}+1}} \leq |g(z)| \leq \frac{(1+|z|)^{\frac{k}{2}-1}}{(1-|z|)^{\frac{k}{2}+1}}.$$

Therefore, we have

$$|f'(z)| \geq \frac{(1-|z|)^{\left(\frac{k}{2}-1\right)b}}{(1+|z|)^{\left(\frac{k}{2}+1\right)b}}.$$

Let d_r denote the radius of the largest Schlicht disc centered at the origin contained in the image $|z|<r$ under $f(z)$.

$$d_r=|f(z_0)|=\int_c |f'(z)||dz| \geq \int_c \frac{(1-|z|)^{\left(\frac{k}{2}-1\right)b}}{(1+|z|)^{\left(\frac{k}{2}+1\right)b}}|dz| \geq \int_0^{|z|} \frac{(1-s)^{\left(\frac{k}{2}-1\right)b}}{(1+s)^{\left(\frac{k}{2}+1\right)b}}ds$$

$$=\int_0^{|z|}\left[\frac{1-s}{1+s}\right]^{\left(\frac{k}{2}+1\right)b}\frac{ds}{(1+s)^{2b}}$$

Replacing $\frac{1-s}{1+s}=t$ we get

$$\geq \frac{-2}{4^b}\int_1^{\frac{1-|z|}{1+|z|}} t^{\left(\frac{k}{2}-1\right)b}(1+t)^{2b-2}dt$$

$$=-2^{1-2b}\int_0^{\frac{1-r}{1+r}} t^{\left(\frac{k}{2}-1\right)b}(1+t)^{2(b-1)}dt+2^{1-2b}\int_1^0 t^{\left(\frac{k}{2}-1\right)b}(1+t)^{2(b-1)}dt=I_1+I_2.$$

Taking $\frac{1-r}{1+r}=r_1,\ t=r_1u$, we have

$$I_1=-2^{1-2b}r_1^l\int_0^1 u^{\left(\frac{k}{2}-1\right)b}(1+r_1u)^{2(b-1)}du$$

using (1.5) we obtain,

$$I_1 = r_1^l \left(\frac{-2^{m-1}}{l} \right) G(l, m, n, -r_1),$$

where $l = \left(\frac{k}{2} - 1\right) b + 1, \quad m = 2(1-b),\ n = \left(\frac{k}{2} - 1\right) b + 2.$

$$I_2 = 2^{1-2b} \int_0^1 t^{\left(\frac{k}{2}-1\right)b} (1+t)^{2(b-1)} dt = \left(\frac{2^{m-1}}{l} \right) G(l, m, n, -1),$$

where $l = \left(\frac{k}{2} - 1\right) b + 1, \quad m = 2(1-b), \quad n = \left(\frac{k}{2} - 1\right) b + 2.$

Therefore

$$|f(z)| \geq \left(\frac{2^{m-1}}{l} \right) G(l, m, n, -1) - r_1^l \left(\frac{2^{m-1}}{l} \right) G(l, m, n, -r_1).$$

On the other hand we have

$$|f'(z)| \leq \frac{(1+|z|)^{\left(\frac{k}{2}-1\right)b}}{(1-|z|)^{\left(\frac{k}{2}+1\right)b}}.$$

Therefore

$$|f(z)| \leq \int_0^{|z|} \frac{(1-s)^{\left(\frac{k}{2}-1\right)b}}{(1+s)^{\left(\frac{k}{2}+1\right)b}} ds \leq -2^{1-2b} \int_1^{\frac{1-|z|}{1+|z|}} \zeta^{\left(\frac{k}{2}-1\right)b} (1+\zeta)^{2(b-1)} d\zeta$$

$$= \frac{2^{m-1}}{l} [G(l, m, n, -1) - r_1^{-l} G(l, m, n, -r_1^{-1})],$$

where $l = \left(\frac{k}{2} - 1\right) b + 1, \quad m = 2(1-b), \quad n = \left(\frac{k}{2} - 1\right) b + 2.$ □

Corollary 2.26. *Let $f \in B_k(1, \beta)$. Then, with $|z| = r,\ r_1 = \frac{1-r}{1+r}$ we have*

$$\frac{2^{m-1}}{l} [G(l, m, n, -1) - r_1^l G(l, m, n, -r_1)] \leq |f(z)|$$

$$\leq \frac{2^{m-1}}{l} [G(l, m, n, -1) - r_1^{-l} G(l, m, n, -r_1^{-1})]$$

where $l = \left(\frac{k}{2} - 1\right) (1-\beta) + 1, \quad m = 2\beta, \quad n = \left(\frac{k}{2} - 1\right) (1-\beta) + 2.$

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Ningegowda Ravikumar
Department of Mathematics
Yuvaraja's College
University of Mysore
Mysore-570 006, India
e-mail: ravisn.kumar@gmail.com

Satyanarayana Latha
Department of Mathematics
Yuvaraja's College
University of Mysore
Mysore-570 006, India
e-mail: drlatha@gmail.com