Stud. Univ. Babeş-Bolyai Math. 57(2012), No. 1, 25–41

Some properties of certain class of multivalent analytic functions

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Abstract. In this paper we introduce a certain general class $\Phi_p^{\beta}(a, c, A, B)$ ($\beta \geq 0$, $a > 0$, $c > 0$, $-1 \leq B < A \leq 1$, $p \in N = \{1, 2, ...\}$) of multivalent analytic functions in the open unit disc $U = \{z : |z| < 1\}$ involving the linear operator $L_p(a, c)$. The aim of the present paper is to investigate various properties and characteristics of this class by using the techniques of Briot-Bouquet differential subordination. Also we obtain coefficient estimates and maximization theorem concerning the coefficients.

Mathematics Subject Classification (2010): 30C45.

Keywords: Analytic, multivalent, differential subordination.

1. Introduction

Let $A(p)$ denote the class of functions of the form:

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k} \qquad (p \in N = \{1, 2, ...\}), \tag{1.1}$$

which are analytic and p-valent in the open unit disc $U = \{z : |z| < 1\}$. Let Ω denotes the class of bounded analytic functions $w(0) = 0$ and $|w(z)| \leq |z|$ for $z \in U$. If f and g are analytic in U, we say that f subordinate to g, written symbolically as follows:

$$f \prec g \quad (z \in U) \text{ or } f(z) \prec g(z),$$

if there exists a Schwarz function w, which (by definition) is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$ such that $f(z) = g(w(z))$ $(z \in U)$. In particular, if the function $g(z)$ is univalent in U, then we have the following equivalence (cf., e.g., [5], [18]; see also [19, p. 4]):

$$f(z) \prec g(z) \ \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

For functions $f(z) \in A(p)$, given by (1.1), and $g(z) \in A(p)$ given by

$$g(z) = z^p + \sum_{k=1}^{\infty} b_{p+k} z^{p+k} \qquad (p \in N), \tag{1.2}$$

then the Hadamard product (or convolution) of $f(z)$ and $g(z)$ is defined by

$$(f * g)(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} b_{p+k} z^{p+k} = (g * f)(z)\,. \tag{1.3}$$

We now define the function $\varphi_p(a, c; z)$ by

$$\varphi_p(a, c; z) = z^p + \sum_{k=1}^{\infty} \frac{(a)_k}{(c)_k} z^{p+k} \ \left(z \in U; a \in R; c \in R \backslash Z_0^- : Z_0^- = \{0, -1, -2, ...\}\right), \tag{1.4}$$

where $(\lambda)_\nu$ denoted the Pochhammer symbol defined (for $\lambda, \nu \in C$ and in terms of the Gamma function) by

$$(\lambda)_\nu = \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0; \lambda \in C\backslash\{0\}), \\ \lambda(\lambda+1)...(\lambda+n-1) & (\nu \in N; \lambda \in C). \end{cases} \tag{1.5}$$

With the aid of the function $\varphi_p(a, c; z)$ defined by (1.4), we consider a function $\varphi_p^+(a, c; z)$ given by the following convolution:

$$\varphi_p(a, c; z) * \varphi_p^+(a, c; z) = \frac{z^p}{(1-z)^{\lambda+p}} \quad (\lambda > -p\ ;\ z \in U), \tag{1.6}$$

which yields the following family of linear operator $I_p^\lambda(a, c)$:

$$I_p^\lambda(a, c) f(z) = \varphi_p^+(a, c; z) * f(z) \qquad (a, c \in R\backslash Z_0^-; \lambda > -p; z \in U)\ . \tag{1.7}$$

For a function $f(z) \in A(p)$, given by (1.1), it is easily seen from (1.6) that

$$I_p^\lambda(a, c) f(z) = z^p + \sum_{k=1}^{\infty} \frac{(c)_k (\lambda+p)_k}{(a)_k (1)_k} a_{p+k} z^{p+k} \qquad (z \in U)\ . \tag{1.8}$$

It is readily verified from the definition (1.8) that

$$z\left(I_p^\lambda(a, c) f(z)\right)' = (a-1) I_p^\lambda(a-1, c) f(z) + (p+1-a) I_p^\lambda(a, c) f(z)\ . \tag{1.9}$$

The operator $I_p^\lambda(a, c)$ was recently introduced by Cho et al. [6].

We observe also that:

(i) $I_p^1(p+1, 1) f(z) = f(z)$ and $I_p^1(p, 1) f(z) = \frac{zf'(z)}{p}$;

(ii) $I_p^n(a, a) f(z) = D^{n+p-1} f(z)$ $(n > -p)$, where $D^{n+p-1} f(z)$ is the $(n+p-1)-th$ order Ruscheweyh derivative of a function $f(z) \in A(p)$ (see Kumar and Shukla [15]);

(iii) $I_p^\delta(\delta+p+1, 1) f(z) = F_{\delta,p}(f)(z)$ $(\delta > -p)$, where $F_{\delta,p}(f)(z)$ is the generalized Bernardi-Livingston operator (see [7]), defined by

$$F_{\delta,p}(f)(z) = \frac{\delta+p}{z^\delta} \int_0^z t^{\delta-1} f(t) dt = z^p + \sum_{k=1}^{\infty} \left(\frac{\delta+p}{\delta+p+k}\right) a_{p+k} z^{p+k} (\delta > -p; p \in N); \tag{1.10}$$

(iv) $I_p^1(n+p,1)f(z) = I_{n,p}f(z)$ $(n > -p)$, where the operator $I_{n,p}$ is the $(n+p-1)-th$ Noor operator, considered by Liu and Noor [16];

(v) $I_p^1(p+1-\mu,1)f(z) = \Omega_z^{(\mu,p)}f(z)(-\infty < \mu < p+1)$, where $\Omega_z^{(\mu,p)}$ $(-\infty < \mu < p+1)$ is the extended fractional differential integral operator (see [26]), defined by

$$\Omega_z^{(\mu,p)}f(z) = z^p + \sum_{k=1}^{\infty} \frac{\Gamma(k+p+1)\Gamma(p+1-\mu)}{\Gamma(p+1)\Gamma(k+p+1-\mu)} a_{p+k} z^{p+k}$$
$$= \frac{\Gamma(p+1-\mu)}{\Gamma(p+1)} z^{\mu} D_z^{\mu} f(z) \quad (-\infty < \mu < p+1; z \in U), \tag{1.11}$$

where $D_z^{\mu}f(z)$ is, respectively, the fractional integral of $f(z)$ of order $-\mu$ when $-\infty < \mu < 0$ and the fractional derivative of $f(z)$ of order μ when $0 \le \mu < p+1$ (see, for details [23], [25] and [26]). The fractional differential operator $\Omega_z^{(\mu,p)}$ with $0 \le \mu < 1$ was investigated by Srivastava and Aouf [29].

Making use of the operator $I_p^{\lambda}(a,c)$, we now introduce a subclass of $A(p)$ as follows:

Definition 1.1. *A function $f(z) \in A(p)$ is said to be in the class $\Phi_p^{\beta}(\lambda, a, c, A, B)$ $(\beta > 0,\ a, c \in R\backslash Z_0^-,\ a > 1;\ \lambda > -p,\ p \in N,\ -1 \le B < A \le 1)$ if and only if it satisfies*

$$(1-\beta)\frac{I_p^{\lambda}(a,c)f(z)}{z^p} + \beta\frac{I_p^{\lambda}(a-1,c)f(z)}{z^p} \prec \frac{1+Az}{1+Bz} \qquad (z \in U). \tag{1.12}$$

By specializing the parameters β, λ, a, c, A and B, we obtain the following subclasses of analytic functions studied by various authors:

(i) $\Phi_p^1(1, p+1, 1, 1, \frac{1}{M}-1) = S_p(M)$ $(M > \frac{1}{2})$ *(Sohi [28]);*

(ii) $\Phi_p^1(1, p+1, 1, \beta[B+(A-B)(p-\alpha)], \beta B) = S_p(\alpha, \beta, A, B)$, $0 \le \alpha < p$, $p \in N$, $0 < \beta \le 1$ *(see Aouf [2]);*

(iii) $\Phi_p^1(1, p+1, 1, [B+(A-B)(p-\alpha)], B) = S_p(A, B, \alpha)$, $0 \le \alpha < p$, $p \in N$ *(see Aouf and Chen [4]);*

(iv) $\Phi_1^1(1, 2, 1, 1, \frac{1}{M}-1) = R(M)$ $(M > \frac{1}{2})$ *(see Goel [9]);*

(v) $\Phi_1^1(1, 2, 1, 2\alpha\beta-1, 2\beta-1) = R_1(\alpha, \beta)$ $(0 \le \alpha < 1,\ 0 < \beta \le 1)$ *(see Mogra [20]);*

(vi) $\Phi_1^1(1, 2, 1, (1-2\alpha)\beta, -\beta) = R(\alpha, \beta)$ $(0 \le \alpha < 1,\ 0 < \beta \le 1)$ *(see Juneja and Mogra [12]);*

(vii) $\Phi_p^1(1, 2, 1, (1-2\alpha)\beta, -\beta) = S_p(\alpha, \beta)$ $(0 \le \alpha < 1,\ 0 < \beta \le 1)$ *(see Owa [24]);*

(viii) $\Phi_1^1(n+1, a, a-1, A, B) = V_n(A, B)$ $(n \in N_0 = N \cup \{0\})$ *(see Kumar [14]);*

(ix) $\Phi_1^1(n+1, a, a-1, [B+(A-B)(1-\alpha)], B) = V_n(A, B, \alpha)$ $(n \in N_0,\ 0 \le \alpha < 1)$ *(see Aouf [3]);*

(x) $\Phi_p^{\beta}(\lambda, a, c, 1, \frac{1}{M}-1) = \Phi_p^{\beta}[\lambda, a, c, M]$ $(M > \frac{1}{2})$*, where* $\Phi_p^{\beta}[\lambda, a, c, M]$ *denotes the class of functions $f(z) \in A(p)$ satisfying the condition:*

$$\left|\left[(1-\beta)\frac{I_p^{\lambda}(a,c)f(z)}{z^p}+\beta\frac{I_p^{\lambda}(a-1,c)f(z)}{z^p}\right]-M\right|<M \quad \left(M>\frac{1}{2}; z\in U\right);$$

(xi) $\Phi_p^1(1,p+1-\mu,1,1,\frac{1}{M}-1)=\Phi_p[\mu,M]$ $(M>\frac{1}{2},-\infty<\mu<p)$, *where* $\Phi_p[\mu,M]$ *denotes the class of functions* $f(z)\in A(p)$ *satisfying the condition:*

$$\left|\frac{\Omega_z^{(\mu,1+p)}f(z)}{z^p}-M\right|<M \quad \left(M>\frac{1}{2}; -\infty<\mu<p; z\in U\right).$$

2. Preliminaries

To establish our main results, we shall need the following lemmas.

Lemma 2.1. [11] *Let* h *be a convex (univalent) in* U *with* $h(0)=1$ *and let the function* φ *given by*

$$\varphi(z)=1+d_1z+d_2z^2+..., \tag{2.1}$$

is analytic in U. *If*

$$\varphi(z)+\frac{1}{\gamma}z\varphi'(z)\prec h(z) \quad (z\in U), \tag{2.2}$$

where $\gamma\neq 0$ *and* $\mathrm{Re}(\gamma)\geq 0$, *then*

$$\varphi(z)\prec\psi(z)=\frac{\gamma}{z^{\gamma}}\int_0^z t^{\gamma-1}h(t)dt\prec h(z) \qquad (z\in U),$$

and ψ *is the best dominant of (2.2).*

Lemma 2.2. [27] *Let* $\Phi(z)$ *be analytic in* U *with*

$$\Phi(0)=1 \text{ and } \mathrm{Re}\{\Phi(z)\}>\frac{1}{2} \quad (z\in U).$$

Then, for any $F(z)$ *analytic in* U, *the set* $(\Phi*F)(U)$ *is contained in the convex hull of* $F(U)$, *i.e.,* $(\Phi*F)U\subset co\ F(U)$.

For complex numbers a,b and $c(c\neq 0,-1,-2,...)$, the Gaussian hypergeometric function is defined by

$${}_2F_1(a,b;c;z)=1+\frac{a.b}{c}\frac{z}{1!}+\frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!}+..., \ z\in U. \tag{2.3}$$

We note that the above series converges absolutely for $z\in U$ and hence represents an analytic function in U (see, for details, [30, Chapter 14]).

Each of the identities (asserted by Lemmas below) is well-known (cf., e.g., [30, Chapter 14]).

Lemma 2.3. [30] *For complex numbers* a, b *and* $c\,(c \neq 0, -1, -2, ...)$, *the next equalities hold:*

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}dt = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\ {}_2F_1(a, b; c; z), \tag{2.4}$$

$$(\mathrm{Re}(c) > \mathrm{Re}(b) > 0),$$

$${}_2F_1(a, b; c; z) \ = (1-z)^{-a}\ {}_2F_1(a, c-b; c; \frac{z}{z-1}); \tag{2.5}$$

and

$$(b+1)\ {}_2F_1(1, b; b+1; z) = (b+1) + bz\ \ {}_2F_1(1, b+1; b+2; z)\ . \tag{2.6}$$

Lemma 2.4. [13] *Let* $w(z) = \sum_{k=1}^{\infty} d_k z^k \in \Omega$, *if* ν *is any complex number, then*

$$\left|d_2 - \nu d_1^2\right| \leq \max\{1, |\nu|\}\,. \tag{2.7}$$

Equality may be attained with the functions $w(z) = z^2$ *and* $w(z) = z$.

3. Main results

Unless otherwise mentioned, we assume throughout of this paper that $\beta > 0$, $a, c \in R\backslash Z_0^-$, $\lambda > -p$, $p \in N$ and $-1 \leq B < A \leq 1$.

Theorem 3.1. *Let the function* f *defined by (1.1) be in the class* $\Phi_p^{\beta}(\lambda, a, c, A, B)$. *Then*

$$\frac{I_p^{\lambda}(a,c)f(z)}{z^p} \prec Q(z) \prec \frac{1+Az}{1+Bz} \qquad (z \in U), \tag{3.1}$$

where the function $Q(z)$ *given by*

$$Q(z) = \begin{cases} \dfrac{A}{B} + (1-\dfrac{A}{B})(1+Bz)^{-1}{}_2F_1(1, 1, \dfrac{a-1}{\beta}+1, \dfrac{Bz}{Bz+1})\ , & B \neq 0, \\ 1 + \dfrac{a-1}{a-1+\beta}Az\ , & B = 0, \end{cases}$$

is the best dominant of (3.1). Furthermore,

$$\mathrm{Re}\left\{\frac{I_p^{\lambda}(a,c)f(z)}{z^p}\right\} > \eta(\beta, a, A, B) \qquad (z \in U), \tag{3.2}$$

where

$$\eta(\beta, a, A, B) = \begin{cases} \dfrac{A}{B} + (1-\dfrac{A}{B})(1-B)^{-1}{}_2F_1(1, 1, \dfrac{a-1}{\beta}+1, \dfrac{B}{B-1})\ , & B \neq 0, \\ 1 - \dfrac{a-1}{a-1+\beta}A\ , & B = 0. \end{cases}$$

The estimate in (3.2) is the best possible.

Proof. Consider the function $\varphi(z)$ defined by

$$\varphi(z) = \frac{I_p^\lambda(a,c)f(z)}{z^p} \quad (z \in U) . \tag{3.3}$$

Then $\varphi(z)$ is of the form (2.1) and is analytic in U. Differentiating (3.3) logarithmically with respect to z and using the identity (1.9) in the resulting equation, we obtain

$$(1-\beta)\frac{I_p^\lambda(a,c)f(z)}{z^p} + \beta\frac{I_p^\lambda(a-1,c)f(z)}{z^p} = \varphi(z) + \frac{z\varphi'(z)}{(a-1)/\beta} \prec \frac{1+Az}{1+Bz} \qquad (z \in U).$$

Now, by using Lemma 2.1 for $\gamma = \frac{a-1}{\beta}$, we deduce that

$$\frac{I_p^\lambda(a,c)f(z)}{z^p} \prec Q(z) = \frac{a-1}{\beta} z^{\frac{1-a}{\beta}} \int_0^z t^{\frac{a-1}{\beta}-1} \left(\frac{1+At}{1+Bt}\right) dt$$

$$= \begin{cases} \frac{A}{B} + (1-\frac{A}{B})(1+Bz)^{-1} {}_2F_1(1,1,\frac{a-1}{\beta}+1;\frac{Bz}{Bz+1}) , & B \neq 0, \\ 1 + \frac{a-1}{a-1+\beta} Az , & B = 0, \end{cases}$$

by change of variables followed by use of the identities (2.4), (2.5) and (2.6) (with $a = 1,\ c = b+1,\ b = \frac{a-1}{\beta}$). This proves the assertion (3.1) of Theorem 3.1.

Next, in order to prove the assertion (3.2) of Theorem 3.1, it suffices to show that

$$\inf_{|z|<1} \{\mathrm{Re}(Q(z)\} = Q(-1) . \tag{3.4}$$

Indeed we have, for $|z| \le r < 1$,

$$\mathrm{Re}\left(\frac{1+Az}{1+Bz}\right) \ge \frac{1-Ar}{1-Br} .$$

Upon setting

$$g(s,z) = \frac{1+Asz}{1+Bsz} \text{ and } d\nu(s) = \left(\frac{a-1}{\beta}\right) s^{\frac{a-1}{\beta}} ds \ (0 \le s \le 1) ,$$

which is a positive measure on the closed interval $[0,1]$, we get

$$Q(z) = \int_0^1 g(s,z) d\nu(s) ,$$

so that

$$\mathrm{Re}\{Q(z)\} \ge \int_0^1 \left(\frac{1-Asr}{1-Bsr}\right) dv(s) = Q(-r) \quad (|z| \le r < 1).$$

Letting $r \to 1^-$ in the above inequality, we obtain the assertion (3.2) of Theorem 3.1. Finally, the estimate in (3.2) is the best possible as the function $Q(z)$ is the best dominant of (3.1).

Corollary 3.2. *For* $0 < \beta_2 < \beta_1$, *we have*

$$\Phi_p^{\beta_1}(\lambda, a, c, A, B) \subset \Phi_p^{\beta_2}(\lambda, a, c, A, B) \ .$$

Proof. Let $f \in \Phi_p^{\beta_1}(\lambda, a, c, A, B)$. Then by Theorem 3.1, we have

$$\frac{I_p^\lambda(a,c)f(z)}{z^p} \prec \frac{1+Az}{1+Bz} \qquad (z \in U).$$

Since

$$\begin{aligned} &(1-\beta_2)\frac{I_p^\lambda(a,c)f(z)}{z^p} + \beta_2\frac{I_p^\lambda(a-1,c)f(z)}{z^p} \\ = &\left(1-\frac{\beta_2}{\beta_1}\right)\frac{I_p^\lambda(a,c)f(z)}{z^p} + \frac{\beta_2}{\beta_1}\left\{(1-\beta_1)\frac{I_p^\lambda(a,c)f(z)}{z^p} + \beta_1\frac{I_p^\lambda(a-1,c)f(z)}{z^p}\right\} \\ \prec &\ \frac{1+Az}{1+Bz} \qquad (z \in U) \ , \end{aligned}$$

we see that $f \in \Phi_p^{\beta_2}(\lambda, a, c, A, B)$. This proves Corollary 3.2.

Taking $\beta = c = 1,\ a = \delta + p + 1\ (\delta > -p),\ \lambda = \delta,\ A = 1 - \frac{2\alpha}{p}\ (0 \le \alpha < p)$ and $B = -1$ in Theorem 3.1, we obtain the the following corollary.

Corollary 3.3. *If* $f \in A(p)$ *satisfies*

$$\mathrm{Re}\left\{\frac{f(z)}{z^p}\right\} > \frac{\alpha}{p} \qquad (0 \le \alpha < p; z \in U) \ ,$$

then the function $F_{\delta,p}(f)(z)$ *defined by (1.10) satisfies*

$$\mathrm{Re}\left\{\frac{F_{\delta,p}(f)(z)}{z^p}\right\} > \frac{\alpha}{p} + \left(1-\frac{\alpha}{p}\right)\left[{}_2F_1(1,1;p+\delta+1;\frac{1}{2}) - 1\right] \qquad (z \in U) \ .$$

The result is the best possible.

Remark 3.4. We note that Corollary 3.3 improves the corresponding result obtained by Obradovic [22] for $p = 1$.

Taking $\lambda = \beta = c = 1$, $a = p + 1 - \mu$, $-\infty < \mu < p$, $A = 1 - \frac{2\alpha}{p}$ $(0 \le \alpha < p)$ $B = -1$ in Theorem 3.1, we obtain the following corollary.

Corollary 3.5. *Let the function* $f(z)$ *given by (1.1) satisfy*

$$\mathrm{Re}\left\{\frac{\Omega_z^{(1+\mu,p)}f(z)}{z^p}\right\} > \frac{\alpha}{p} \quad (-\infty < \mu < p; 0 \le \alpha < p; p \in N; z \in U) \ .$$

Then

$$\mathrm{Re}\left\{\frac{\Omega_z^{(\mu,p)}f(z)}{z^p}\right\} > \frac{\alpha}{p} + \left(1-\frac{\alpha}{p}\right)\left[{}_2F_1(1,1;p+1-\mu;\frac{1}{2}) - 1\right] \quad (z \in U) \ .$$

The result is the best possible.

Taking $\mu = 0$ in Corollary 3.5, we obtain the following corollary.

Corollary 3.6. *Let the function $f(z)$ given by (1.1) satisfy*

$$\mathrm{Re}\left\{\frac{f^{'}(z)}{z^{p-1}}\right\} > \alpha \quad (0 \leq \alpha < p; z \in U) \ ,$$

Then

$$\mathrm{Re}\left\{\frac{f(z)}{z^{p}}\right\} > \frac{\alpha}{p} + \left(1 - \frac{\alpha}{p}\right)\left[{}_2F_1(1, 1; p + 1; \frac{1}{2}) - 1\right] \quad (z \in U) \ .$$

The result is the best possible.

Remark 3.7. We note that Corollary 3.6 improves the corresponding result obtained by Lee and Owa [17, Theorem 1] with $n = 1$.

Remark 3.8. If $f \in A(p)$ *satisfies* $\mathrm{Re}\left\{f^{'}(z)/z^{p-1}\right\} > \alpha$ $(0 \leq \alpha < p; z \in U)$, *then with the aid of Corollaries 2 and 4, we deduce that*

$$\mathrm{Re}\left\{\frac{F_{\delta,p}(f)(z)}{z^{p}}\right\} > \frac{\alpha}{p} + \left(1 - \frac{\alpha}{p}\right)\left[\left({}_2F_1(1, 1; p + 1; \frac{1}{2}) - 1\right)\right.$$
$$\left. + \left({}_2F_1(1, 1; p + \delta + 1; \frac{1}{2}) - 1\right)\left(2 - \left({}_2F_1(1, 1; p + 1; \frac{1}{2})\right)\right)\right],$$

which improve the result of Fukui et al. [8] for $p = 1$.

Corollary 3.9. *Let the function $f(z)$ given by (1.1) satisfy*

$$\mathrm{Re}\left\{\frac{I_p^n(n-1, n)f(z)}{z^{p}}\right\} > \frac{\alpha}{p} \quad (0 \leq \alpha < p; z \in U) \ ,$$

Then

$$\mathrm{Re}\left\{\frac{D^{n+p-1}f(z)}{z^{p}}\right\} > \frac{\alpha}{p} + \left(1 - \frac{\alpha}{p}\right)\left[{}_2F_1(1, 1; n; \frac{1}{2}) - 1\right] \quad (z \in U) \ .$$

The result is the best possible.

Theorem 3.10. *Let $f(z) \in \Phi_p^0(\lambda, a, c, A, B)$ and let the function $F_{\delta,p}(f)(z)$ defined by (1.10). Then*

$$\frac{I_p^{\lambda}(a, c)F_{\delta,p}(f)(z)}{z^{p}} \prec q(z) \prec \frac{1 + Az}{1 + Bz} \ , \tag{3.5}$$

where the function $q(z)$ given by

$$q(z) = \begin{cases} \dfrac{A}{B} + (1 - \dfrac{A}{B})(1 + Bz)^{-1}{}_2F_1(1, 1, p + \delta + 1; \dfrac{Bz}{Bz + 1}) \ , & B \neq 0 \\ 1 + \dfrac{p + \delta}{p + \delta + 1}Az \ , & B = 0. \end{cases}$$

is the best dominant of (3.5). Furthermore,

$$\mathrm{Re}\left\{\frac{I_p^{\lambda}(a, c)F_{\delta,p}(f)(z)}{z^{p}}\right\} > \zeta^{*} \quad (z \in U) \ , \tag{3.6}$$

where

$$\zeta^* = \begin{cases} \frac{A}{B} + (1 - \frac{A}{B})(1-B)^{-1} {}_2F_1(1,1;p+\delta+1;\frac{B}{B-1}) \ , & B \neq 0 \ , \\ 1 - \frac{p+\delta}{p+\delta+1} A \ , & B = 0 \ . \end{cases}$$

The estimate in (3.6) is the best possible.

Proof. From (1.10) it follows that

$$z\left(I_p^\lambda(a,c)F_{\delta,p}(f)(z)\right)' = (\delta+p)I_p^\lambda(a,c)f(z) - \delta I_p^\lambda(a,c)F_{\delta,p}(f)(z) \ . \tag{3.7}$$

By setting

$$\varphi(z) = \frac{I_p^\lambda(a,c)F_{\delta,p}(f)(z)}{z^p} \qquad (z \in U) \ , \tag{3.8}$$

we note that $\varphi(z)$ is of the form (2.1) and is analytic in U. Differentiating (3.8) with respect to z and using the identity (3.7) in the resulting equation, we get

$$\varphi(z) + \frac{z\varphi'(z)}{\delta+p} = \frac{I_p^\lambda(a,c)f(z)}{z^p} \prec \frac{1+Az}{1+Bz} \quad (z \in U) \ ,$$

which with the aid of Lemma 2.1 with $\gamma = \delta + p$, yields

$$\frac{I_p^\lambda(a,c)F_{\delta,p}(f)(z)}{z^p} \prec q(z) = (\delta+p)z^{-(\delta+p)} \int_0^z t^{\delta+p-1}\left(\frac{1+At}{1+Bt}\right) dt \ . \tag{3.9}$$

Now the remaining part of Theorem 3.10 follows by employing the techniques that we used in proving Theorem 3.1 above.

Taking $A = 1 - \frac{2\alpha}{p}$ $(0 \leq \alpha < p)$ and $B = -1$ in Theorem 3.10, we obtain the following corollary.

Corollary 3.11. *If $f \in A(p)$ satisfies*

$$\mathrm{Re}\left\{\frac{I_p^\lambda(a,c)f(z)}{z^p}\right\} > \frac{\alpha}{p} \quad (0 \leq \alpha < p; z \in U) \ ,$$

then

$$\mathrm{Re}\left\{\frac{I_p^\lambda(a,c)F_{\delta,p}(f)(z)}{z^p}\right\} > \frac{\alpha}{p} + \left(1 - \frac{\alpha}{p}\right)\left\{{}_2F_1(1,1;p+\delta+1;\frac{1}{2}) - 1\right\} \quad (z \in U) \ .$$

The result is the best possible.

Taking $\lambda = c = 1$ and $a = p$ in Corollary 3.11, we get the following corollary which in turn improves the corresponding result of Fukui et al. [8] for $p = 1$.

Corollary 3.12. *If $f \in A(p)$ satisfies*

$$\mathrm{Re}\left\{\frac{f'(z)}{z^{p-1}}\right\} > \alpha \quad (0 \leq \alpha < p; z \in U) \ ,$$

then

$$\operatorname{Re}\left\{\frac{F'_{\delta,p}(f)(z)}{z^{p-1}}\right\} > \alpha + (p-\alpha)\left\{{}_2F_1(1,1;p+\delta+1;\frac{1}{2}) - 1\right\} \quad (z \in U)\ .$$

The result is the best possible.

Taking $\lambda = c = 1$ and $a = p+1-\mu$ $(-\infty < \mu < p+1, p \in N)$ in Corollary 3.11, we obtain the following corollary.

Corollary 3.13. *If $f(z) \in A(p)$ satisfies*

$$\operatorname{Re}\left\{\frac{\Omega_z^{(\mu,p)} f(z)}{z^p}\right\} > \frac{\alpha}{p} \quad (0 \le \alpha < p; -\infty < \mu < p+1; p \in N; z \in U)\ ,$$

then

$$\operatorname{Re}\left\{\frac{\Omega_z^{(\mu,p)} F_{\delta,p}(f)(z)}{z^p}\right\} > \frac{\alpha}{p} + \left(1 - \frac{\alpha}{p}\right)\left\{{}_2F_1(1,1;p+\delta+1;\frac{1}{2}) - 1\right\} \quad (z \in U)\ .$$

The result is the best possible.

Theorem 3.14. *We have*

$$f \in \Phi_p^0(a,c,A,B) \Leftrightarrow F_{a-p-1}(f)(z) \in \Phi_p^1(a,c,A,B)$$

Proof. Using the identity (3.7) and

$$z\left(I_p^\lambda(a,c)F_{\delta,p}(f)(z)\right)' = (a-1)I_p^\lambda(a-1,c)F_{\delta,p}(f)(z) + (p+1-a)I_p^\lambda(a,c)F_{\delta,p}(f)(z)\ ,$$

for $\delta = a-p-1$, we deduce that

$$I_p^\lambda(a,c)f(z) = I_p^\lambda(a-1,c)F_{a-p-1}(f)(z)$$

and the assertion of Theorem 3.14 follows by using the definition of the class $\Phi_p^\beta(a,c,A,B)$.

Theorem 3.15. *If f, given by (1.1), belongs to the class $\Phi_p^\beta(a,c,A,B)$, then*

$$|a_{p+k}| \le \frac{(A-B)(a-1)_{k+1}}{(a-1+\beta k)(c)_k}\frac{(1)_k}{(\lambda+p)_k} \quad (k \ge 1)\ . \tag{3.10}$$

The result is sharp.

Proof. Since $f \in \Phi_p^\beta(a,c,A,B)$, we have

$$(1-\beta)\frac{I_p^\lambda(a,c)f(z)}{z^p} + \beta\frac{I_p^\lambda(a-1,c)f(z)}{z^p} = p(z)\ , \tag{3.11}$$

where $p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k \in P(A,B)$. Substituting the power series expansion of $I_p^\lambda(a,c)f(z)$, $I_p^\lambda(a-1,c)f(z)$ and $p(z)$ in (3.11) and equating the coefficients of z^k on both sides of the resulting equation, we obtain

$$\frac{(a-1+\beta k)(\lambda+k)_k}{(a-1)_{k+1}}\frac{(c)_k}{(1)_k}a_{p+k} = p_k \quad (k \ge 1)\ . \tag{3.12}$$

Using the well-known [1] coefficient estimates

$$|p_k| \leq (A-B) \quad (k \geq 1)$$

in (3.12), we get the required estimate (3.10).

In order to establish the sharpness of (3.10), consider the functions $f_k(z)$ defined by

$$(1-\beta)\frac{I_p^\lambda(a,c)f(z)}{z^p} + \beta\frac{I_p^\lambda(a-1,c)f(z)}{z^p} = \frac{1+Az^k}{1+Bz^k} \quad (k \geq 1)\ .$$

Clearly, $f_k(z) \in \Phi_p^\beta(\lambda, a, c, A, B)$ for each $k \geq 1$. It is easy to see that the functions $f_k(z)$ have the expansion

$$f_k(z) = z^p + \frac{(A-B)(a-1)_{k+1}}{(a-1+\beta k)(\lambda+p)_k}\frac{(1)_k}{(c)_k}z^{p+k} + ...$$

showing that the estimates in (3.10) are sharp.

Taking $\beta = \lambda = c = A = 1,\ a = p+1-\mu,\ -\infty < \mu < p$ and $B = \frac{1}{M} - 1$ $(M > \frac{1}{2})$ in Theorem 3.15, we obtain the following corollary.

Corollary 3.16. *If f, given by (1.1), belongs to the class $\Phi_p[\mu, M]$, then*

$$|a_{p+k}| \leq \frac{\left(\frac{2M-1}{M}\right)(p-\mu)_k}{(p+1)_k} \quad (k \geq 1)\ .$$

The result is sharp.

Theorem 3.17. *Let f, given by (1.1), belongs to the class $\Phi_p^\beta(\lambda, a, c, A, B)$ and ζ is any complex number. Then*

$$\left|a_{p+2} - \zeta a_{p+1}^2\right| \leq \frac{(A-B)(a-1)_3(1)_2}{(c)_2(\lambda+p)_2(a-1+2\beta)} \max\left\{1\ ,\right.$$
$$\left.\left|B + \zeta\frac{(A-B)(a-1)_2(\lambda+p+1)(c+1)(a-1+2\beta)}{2c(a+1)(\lambda+p)(a-1+\beta)^2}\right|\right\}\ . \tag{3.13}$$

The result is sharp.

Proof. From (1.12), we have

$$(1-\beta)\frac{I_p^\lambda(a,c)f(z)}{z^p} + \beta\frac{I_p^\lambda(a-1,c)f(z)}{z^p} - 1$$
$$= \left[A - B\left\{(1-\beta)\frac{I_p^\lambda(a,c)f(z)}{z^p} + \beta\frac{I_p^\lambda(a-1,c)f(z)}{z^p}\right\}\right]w(z), \tag{3.14}$$

where

$$w(z) = \sum_{k=1}^{\infty} d_k z^k \in \Omega.$$

Substituting the power series expansion of $I_p^\lambda(a,c)f(z)$, $I_p^\lambda(a-1,c)f(z)$ and $w(z)$ in (3.14), and equating the coefficients of z and z^2 we obtain

$$a_{p+1} = \frac{(A-B)(a-1)_2}{(a-1+\beta)(c)(\lambda+p)}d_1 \tag{3.15}$$

and

$$a_{p+2} = \frac{2(A-B)(a-1)_3}{(a-1+2\beta)(c)_2(\lambda+p)_2}(d_2 - Bd_1^2) \; . \tag{3.16}$$

Using (2.7), (3.15) and (3.16), we get:

$$\left|a_{p+2} - \zeta a_{p+1}^2\right| = \frac{(A-B)(a-1)_3}{(c)_2(\lambda+p)_2(a-1+2\beta)} \left|d_2 - \nu d_1^2\right| \; ,$$

where

$$\nu = B + \zeta \frac{(A-B)(a-1+2\beta)(c+1)(\lambda+p+1)(a-1)_2}{2c(a+1)(a-1+\beta)^2(\lambda+p)}$$

Since (2.7) is sharp, (3.13) is also sharp.

Taking $\beta = \lambda = c = A = 1,\ a = p+1-\mu\ (-\infty < \mu < p)$ and $B = \frac{1}{M} - 1\ (M > \frac{1}{2})$ in Theorem 3.17, we obtain the following corollary.

Corollary 3.18. *If f, given by (1.1), belongs to the class $\Phi_p[\mu, M]$, then*

$$\left|a_{p+2} - \zeta a_{p+1}^2\right| \leq \frac{\left(\frac{2M-1}{M}\right)(p-\mu)_3}{(1+p)_2(p+2-\mu)} \max\left\{1, \left|\frac{1}{M} - 1 + \zeta \frac{\left(\frac{2M-1}{M}\right)(p-\mu)(p+2)}{(p+1-\mu)(p+1)}\right\}\right| \; .$$

The result is sharp.

Theorem 3.19. *Let $f \in \Phi_p^{\beta}(a, c, A, B)$ and $g \in A(p)$ with* $\operatorname{Re}\left(\frac{g(z)}{z^p}\right) > \frac{1}{2}$ $(z \in U)$. *Then $h = f * g \in \Phi_p^{\beta}(a, c, A, B)$.*

Proof. We have

$$\begin{aligned} &(1-\beta)\frac{I_p^{\lambda}(a,c)h(z)}{z^p} + \beta\frac{I_p^{\lambda}(a-1,c)h(z)}{z^p} \\ &= \left\{(1-\beta)\frac{I_p^{\lambda}(a,c)f(z)}{z^p} + \beta\frac{I_p^{\lambda}(a-1,c)f(z)}{z^p}\right\} * \frac{g(z)}{z^p} \quad (z \in U). \end{aligned} \tag{3.17}$$

Since $\operatorname{Re}\left\{\frac{g(z)}{z^p}\right\} > \frac{1}{2}$ $(z \in U)$ and the function $\frac{1+Az}{1+Bz}$ is convex (univalent) in U, it follows from (3.17) and Lemma 2.2 that $h(z) = (f * g)(z) \in \Phi_p^{\beta}(a, c, A, B)$. This completes the proof of Theorem 3.19.

Corollary 3.20. *Let $f \in \Phi_p^{\beta}(a, c, A, B)$ and $g(z) \in A(p)$ satisfy*

$$\operatorname{Re}\left\{(1-\mu)\frac{g(z)}{z^p} + \mu\frac{g'(z)}{pz^{p-1}}\right\} > \frac{3 - 2\ {}_2F_1(1, 1; \frac{p}{\mu} + 1; \frac{1}{2})}{2\left[2 - \ {}_2F_1(1, 1; \frac{p}{\mu} + 1; \frac{1}{2})\right]}, \ (\mu > 0;\ z \in U). \tag{3.18}$$

Then $f * g \in \Phi_p^{\beta}(a, c, A, B)$.

Proof. From Theorem 3.1 (for $a = p+1$, $c = 1$, $\beta = \mu > 0$, $A = \dfrac{{}_2F_1(1,1;\frac{p}{\mu}+1;\frac{1}{2})-1}{2-\ {}_2F_1(1,1;\frac{p}{\mu}+1;\frac{1}{2})}$ and $B = -1$), condition (3.18) implies

$$\operatorname{Re}\left\{\frac{g(z)}{z^p}\right\} > \frac{1}{2} \qquad (z \in U)\ .$$

Using this, it follows from Theorem 3.19, that $(f * g)(z) \in \Phi_p^\beta(a, c, A, B)$.

Theorem 3.21. *If each of the functions $f(z)$ given by (1.1) and*

$$g(z) = z^p + \sum_{k=1}^{\infty} b_{p+k} z^{p+k}$$

belongs to the class $\Phi_p^\beta(\lambda, a, c, A, B)$, then so does the function

$$h(z) = (1-\beta) I_p^\lambda(a,c)(f*g)(z) + \beta I_p^\lambda(a-1,c)(f*g)(z)\ .$$

Proof. Since $f \in \Phi_p^\beta(a, c, A, B)$, it follows from (3.14) that

$$\left|(1-\beta)\frac{I_p^\lambda(a,c)f(z)}{z^p} + \beta\frac{I_p^\lambda(a-1,c)f(z)}{z^p} - 1\right|$$
$$< \left|A - B\left\{(1-\beta)\frac{I_p^\lambda(a,c)f(z)}{z^p} + \beta\frac{I_p^\lambda(a-1,c)f(z)}{z^p}\right\}\right| ,$$

which is equivalent to

$$\left|(1-\beta)\frac{I_p^\lambda(a,c)f(z)}{z^p} + \beta\frac{I_p^\lambda(a-1,c)f(z)}{z^p} - \xi\right| < \eta \quad (z \in U)\ , \tag{3.19}$$

where $\xi = \dfrac{1-AB}{1-B^2}$ and $\eta = \dfrac{A-B}{1-B^2}$. It is known [21] that $H(z) = \sum_{k=0}^{\infty} h_k z^k$ is analytic in U and $|H(z)| \le M$, then

$$\sum_{k=0}^{\infty} |h_k|^2 \le M^2\ . \tag{3.20}$$

Applying (3.18) to (3.19), we get

$$(1-\xi)^2 + \sum_{k=1}^{\infty}\left\{\frac{(a-1+\beta k)(c)_k(\lambda+p)_k}{(a-1)_{k+1}(1)_k}\right\}^2 |a_{p+k}|^2 \le \eta^2\ ,$$

that is, that

$$\sum_{k=1}^{\infty}\left\{\frac{(a-1+\beta k)(c)_k(\lambda+k)_k}{(a-1)_{k+1}(1)_k}\right\}^2 |a_{p+k}|^2 \le \frac{(A-B)^2}{1-B^2}\ . \tag{3.21}$$

Similarly, we have

$$\sum_{k=1}^{\infty}\left\{\frac{(a-1+\beta k)(c)_k(\lambda+k)_k}{(a-1)_{k+1}(1)_k}\right\}^2 |b_{p+k}|^2 \le \frac{(A-B)^2}{1-B^2}\ . \tag{3.22}$$

Now, for $|z| = r < 1$, by applying Cauchy-Schwarz inequality, we find that

$$\left|(1-\beta)\frac{I_p^\lambda(a,c)h(z)}{z^p} + \beta\frac{I_p^\lambda(a-1,c)h(z)}{z^p} - \xi\right|^2$$

$$= \left|(1-\xi) + \sum_{k=1}^{\infty}\left\{\frac{(a-1+\beta k)(c)_k(\lambda+p)_k}{(a-1)_{k+1}(1)_k}\right\}^2 a_{p+k}b_{p+k}z^k\right|^2$$

$$\leq (1-\xi)^2 + 2(1-\xi)\sum_{k=1}^{\infty}\left\{\frac{(a-1+\beta k)(c)_k(\lambda+p)_k}{(a-1)_{k+1}(1)_k}\right\}^2 |a_{p+k}|\,|b_{p+k}|\,r^k$$

$$+\left|\sum_{k=1}^{\infty}\left\{\frac{(a-1+\beta k)(c)_k(\lambda+p)_k}{(a-1)_{k+1}(1)_k}\right\}^2 a_{p+k}b_{p+k}z^k\right|^2$$

$$\leq (1-\xi)^2 + 2(1-\xi)\left[\sum_{k=1}^{\infty}\left\{\frac{(a-1+\beta k)(c)_k(\lambda+p)_k}{(a-1)_{k+1}(1)_k}\right\}^2 |a_{p+k}|^2 r^k\right]^{\frac{1}{2}} \cdot$$

$$\cdot\left[\sum_{k=1}^{\infty}\left\{\frac{(a-1+\beta k)(c)_k(\lambda+p)_k}{(a-1)_{k+1}(1)_k}\right\}^2 |b_{p+k}|^2 r^k\right]^{\frac{1}{2}} +$$

$$\left[\sum_{k=1}^{\infty}\left\{\frac{(a-1+\beta k)(c)_k(\lambda+p)_k}{(a-1)_{k+1}(1)_k}\right\}^2 |a_{p+k}|^2 r^k\right] \cdot$$

$$\cdot\left[\sum_{k=1}^{\infty}\left\{\frac{(a-1+\beta k)(c)_k(\lambda+p)_k}{(a-1)_{k+1}(1)_k}\right\}^2 |b_{p+k}|^2 r^k\right]$$

$$\leq (1-\xi)^2 + 2(1-\xi)\left[\sum_{k=1}^{\infty}\left\{\frac{(a-1+\beta k)(c)_k(\lambda+p)_k}{(a-1)_{k+1}(1)_k}\right\}^2 |a_{p+k}|^2\right]^{\frac{1}{2}} \cdot$$

$$\cdot\left[\sum_{k=1}^{\infty}\left\{\frac{(a-1+\beta k)(c)_k(\lambda+p)_k}{(a-1)_{k+1}(1)_k}\right\}^2 |b_{p+k}|^2\right]^{\frac{1}{2}} +$$

$$\left[\sum_{k=1}^{\infty}\left\{\frac{(a-1+\beta k)(c)_k(\lambda+p)_k}{(a-1)_{k+1}(1)_k}\right\}^2 |a_{p+k}|^2\right] \cdot$$

$$\cdot\left[\sum_{k=1}^{\infty}\left\{\frac{(a-1+\beta k)(c)_k(\lambda+p)_k}{(a-1)_{k+1}(1)_k}\right\}^2 |b_{p+k}|^2\right]$$

$$\leq (1-\xi)^2 + 2(1-\xi)\frac{(A-B)^2}{1-B^2} + \frac{(A-B)^4}{(1-B^2)^2}$$

$$= \left\{\frac{B(A-B)}{1-B^2}\right\}^2 + 2\frac{B(A-B)^3}{(1-B^2)^2} + \frac{(A-B)^4}{(1-B^2)^2} = \frac{A^2(A-B)^2}{(1-B^2)^2} < \eta^2,$$

by using (3.21) and (3.22).
Thus, again with the aid of (3.20), we have $h \in \Phi_p^\beta(\lambda, a, c, A, B)$.

Theorem 3.22. *Let* $f \in \Phi_p^\beta(\lambda, a, c, A, B)$ $(\beta > 0)$ *and*

$$S_n(z) = z^p + \sum_{k=1}^{n-1} a_{p+k} z^{p+k} \ (n \geq 2).$$

Then for $z \in U$*, we have*

$$\operatorname{Re}\left\{ \frac{\int\limits_0^z t^{-p}(I_p^\lambda(a,c)S_n(t))dt}{z} \right\} > \eta(\beta, a, A, B) \ ,$$

where $\eta(\beta, a, A, B)$ *is defined as in Theorem 3.1.*

Proof. Singh and Singh [27] prove that

$$\operatorname{Re}\left\{ 1 + \sum_{k=1}^{n-1} \frac{z^k}{k+1} \right\} > \frac{1}{2} \quad (z \in U) \ . \tag{3.23}$$

Writing

$$\frac{\int\limits_0^z t^{-p} I_p^\lambda(a,c)S_n(t)dt}{z} = \frac{I_p^\lambda(a,c)f(z)}{z^p} * \left\{ 1 + \sum_{k=1}^{n-1} \frac{z^k}{k+1} \right\}$$

and making use of (3.23), Theorem 3.1 and Lemma 2.2, the assertion of Theorem 3.22 follows at once.

Taking $\beta = \lambda = c = 1$, $a = p+1$, $A = 1 - \frac{2\alpha}{p}$ $(0 \leq \alpha < p)$ and $B = -1$ in Theorem 3.22, we obtain the following corollary.

Corollary 3.23. *Let* $f \in A(p)$ *satisfies* $\operatorname{Re}\left\{ \frac{f'(z)}{z^{p-1}} \right\} > \alpha$ $(0 \leq \alpha < p)$ *in* U*, then*

$$\operatorname{Re}\left[\frac{\int\limits_0^z t^{-p} S_n(t)dt}{z} \right] > \frac{\alpha}{p} + \left(1 - \frac{\alpha}{p}\right) \left\{ {}_2F_1\left(1, 1; p+1; \frac{1}{2}\right) - 1 \right\} \quad (z \in U) \ .$$

Acknowledgments. The authors thank the referees for their valuable suggestions to improve the paper.

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