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On skew group algebras and symmetric algebras

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Abstract. We identify and define a class of algebras which we call inv-symm algebras and prove that are principally symmetric. Two important examples are given, and we prove that the skew group algebra associated to these algebras remains inv-symm.

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1. Inv-symm algebras

Following [2] we recall the concept of an inverse semigroup and we use basic results without comments. A semigroup $(S, \cdot)$ is *inverse* if for any $s \in S$ there is a unique $\widehat{s}$ (named inverse) such that $s \cdot \widehat{s} \cdot s = s$ and $\widehat{s} \cdot s \cdot \widehat{s} = \widehat{s}$. By [2, 1.1, Theorem 3], if $(S, \cdot)$ is inverse then all idempotents of S commutes and we have $\widehat{\widehat{s}} = s$ and $\widehat{s \cdot t} = \widehat{t} \cdot \widehat{s}$ for any $s \in S$. We denote usually by k a commutative ring and by A a k-algebra. If B is a subset of A with $0 \notin B$, we denote by $B^\sharp$ the set $B \cup \{0\}$ and by $\mathrm{Idemp}(B)$ the set of all idempotents of B. The following definition is suggested by the ideas from [3] and by methods used to prove that the group algebra is a symmetric algebra.

Definition 1.1. *A k-algebra A is inv-symm if there is a finite k-basis B such that:*

(1) $(B^\sharp, \cdot)$ *is an inverse semigroup.*
(2) *For $t, s \in B$ we have $t \cdot s \neq 0$ if and only if $s \cdot \widehat{s} = \widehat{t} \cdot t$.*

Example 1.2. If $A = kG$ is the group algebra over a finite group G then the finite set $B = G$ is a k-basis which satisfies conditions from Definition 1.1. We have in this case $\widehat{s} = s^{-1}$, $t \cdot s \neq 0$ and $s \cdot \widehat{s} = \widehat{t} \cdot t$ for any $t, s \in B$.

Example 1.3. If $A = \mathrm{End}_k(M)$, where M is a kG-lattice (that is a finitely generated, free k-module with a G-stable finite basis X), then $B = \{b_{x,y} \mid x, y \in X\}$ with $b_{x,y} : M \to M$, $b_{x,y}(z) = x$ if $z = y$, and $b_{x,y}(z) = 0$ if $z \neq y$, satisfies the conditions from 1.1. It requires some computation to verify that $b_{x,y} \circ b_{x_1,y_1} = 0$ if $y \neq x_1$, and $b_{x,y} \circ b_{x_1,y_1} = b_{x,y_1}$ if $y = x_1$. We have that $b_{x,y} \in \mathrm{Idemp}(B)$ if and only if $x = y$.

Remark 1.4. Moreover the above two examples are also G-algebras with G-stable basis. This suggest that we can define a class of symmetric G-algebras and to analyze the skew group algebra in this case.

Lemma 1.5. *Let A be an inv-symm k-algebra with basis B satisfying Definition* 1.1 *and $t, s \in B$. The following statements are true:*

a) *For $0 \in B^\sharp$ we have $\widehat{0} = 0$ and $s \in B$ if and only if $\widehat{s} \in B$.*
b) *For all $s \in B$ we have $s \cdot \widehat{s} \in \mathrm{Idemp}(B)$ and $\widehat{s} \cdot s \in \mathrm{Idemp}(B)$. Particularly* $\mathrm{Idemp}(B) \neq \emptyset$.
c) *If $t \cdot s \neq 0$ and $t \cdot s \in \mathrm{Idemp}(B)$ then $t = \widehat{s}$.*

Proof. a) For 0 is easy to check. Let $s \in B$, then there is a unique $\widehat{s} \in B^\sharp$ with the properties of the inverse element. Suppose that $\widehat{s} = 0$ then $\widehat{\widehat{s}} = \widehat{0}$, which gives $s = 0$, a contradiction.

b) For $s \in B$ we have $\widehat{s} \in B^\sharp$ such that $s \cdot \widehat{s} \cdot s = s$ and $\widehat{s} \cdot s \cdot \widehat{s} = \widehat{s}$. Now $s \cdot \widehat{s} \in B$ (since if $s \cdot \widehat{s} = 0 \Rightarrow s = 0 \notin B$) and $(s \cdot \widehat{s}) \cdot (s \cdot \widehat{s}) = (s \cdot \widehat{s} \cdot s) \cdot \widehat{s} = s \cdot \widehat{s}$.

c) Suppose that $t \cdot s \neq 0$ and $t \cdot s \in \mathrm{Idemp}(B)$. Then $s \cdot \widehat{s} = \widehat{t} \cdot t$ and $t \cdot s \cdot t \cdot s = t \cdot s$. We multiply the last relation with $\widehat{s}$ on the right and obtain

$$t \cdot s \cdot t \cdot s \cdot \widehat{s} = t \cdot s \cdot \widehat{s} \Rightarrow t \cdot s \cdot t \cdot \widehat{t} \cdot t = t \cdot \widehat{t} \cdot t \Rightarrow t \cdot s \cdot t = t.$$

Similarly we obtain $s \cdot t \cdot s = t$, thus $t = \widehat{s}$.

□

From [1] we recall the definition of a symmetric algebra. A k-algebra A is called *symmetric* if it is finitely generated and projective as k-module and there is $\tau : A \to k$ a central form (that is k-linear map with $\tau(a \cdot a') = \tau(a' \cdot a)$ for all $a, a' \in A$), which induces an isomorphism of $A - A$-bimodules

$$\widehat{\tau} : A \to A^*, \ \widehat{\tau}(a)(b) = \tau(a \cdot b),$$

where $a, b \in A$ and A^* is the k-dual. τ is called *symmetric form* of A and A is *principally symmetric* if τ is onto.

Theorem 1.6. *If A is an inv-symm k-algebra then A is principally symmetric. In particular it is symmetric.*

Proof. By Definition 1.1 A is a finitely generated k-module and free, thus projective. We define the following k-linear form on the basis B

$$\tau_B : A \to k, \quad \tau_B(s) = \begin{cases} 1_k, s \in \mathrm{Idemp}(B) \\ 0, \ s \notin \mathrm{Idemp}(B) \end{cases}$$

From Lemma 1.5, b) it follows that τ_B is not the zero map and τ_B is a k-linear form. We prove that it is a central form, that is $\tau_B(s \cdot t) = \tau_B(t \cdot s)$ where $t, s \in B$, by considering the cases:

- If $t \cdot s \neq 0$ and $t \cdot s \in \mathrm{Idemp}(B)$, by Lemma 1.5, c) it follows that $\widehat{s} = t$ and then

$$\tau_B(s \cdot \widehat{s}) = 1_k = \tau_B(\widehat{s} \cdot s).$$

- If $t \cdot s \neq 0$ and $t \cdot s \in B \setminus \mathrm{Idemp}(B)$ then $\tau_B(t \cdot s) = 0$. Now, if $s \cdot t \neq 0$ and $s \cdot t \in \mathrm{Idemp}(B)$ by Lemma 1.5, c) we get that $s = \widehat{t}$, which is a contradiction with

$\tau_B(t\cdot s)=0$. So we have two possibilities: $s\cdot t=0$, or $s\cdot t\neq 0$ and $s\cdot t\notin \mathrm{Idemp}(B)$. In both subcases $\tau_B(s\cdot t)=0$.

- If $t\cdot s=0$ then $\tau_B(t\cdot s)=0$, and the same analyze to the second case gives us equality.

τ_B induces the following $A-A$-bimodule homomorphism $\widehat{\tau_B}: A\to A^*$ defined by

$$\widehat{\tau_B}(t)(s)=\tau_B(t\cdot s)$$

for any $t,s\in B$.

First we prove that $\widehat{\tau_B}$ is injective. Let $t_1,t_2\in B$ such that $\tau_B(t_1\cdot s)=\tau_B(t_2\cdot s)$ for any $s\in B$. We choose $s=\widehat{t_1}$ and obtain that $\tau_B(t_2\cdot\widehat{t_1})=1_k$. It follows that $t_2\cdot\widehat{t_1}\neq 0$ and $t_2\cdot\widehat{t_1}\in\mathrm{Idemp}(B)$. By Lemma 1.5, c) we obtain that $t_2=\widehat{\widehat{t_1}}=t_1$.

For surjectivity let $\lambda\in A^*$ and define $a=\sum_{t\in B}\lambda(t)\cdot\widehat{t}\in A$. Then for $s\in B$

$$\widehat{\tau_B}(a)(s)=\sum_{t\in B}\lambda(t)\tau_B(\widehat{t}\cdot s).$$

Since $\tau_B(\widehat{t}\cdot s)=1_k$ if and only if $s=t$ we obtain that

$$\widehat{\tau_B}(a)(s)=\lambda(s)\cdot\tau_B(\widehat{s}\cdot s)=\lambda(s).$$

This concludes the proof. □

2. Skew group algebras

In this section we will investigate the skew group algebra associated to a G-algebra which is an inv-symm algebra, where G is a finite group. The Remark 1.4 is the starting point of the next definition.

Definition 2.1. *A G-algebra A is called G-inv-symm if it is inv-symm, with the basis B (from Definition* 1.1*) G-stable.*

It is easy to show, using Theorem 1.6, that any G-inv-symm algebra is G-permutation and principally symmetric. If A is a G-algebra we denote the action of an $g\in G$ on $a\in A$ by ${}^g a$.

Theorem 2.2. *Let G be a finite group and A a G-algebra. If A is G-inv-symm then the skew group algebra, denoted $A\star G$, is inv-symm. In particular it is principally symmetric.*

Proof. We remind the definition of a skew group algebra. The skew group algebra $A\star G$ is the free A-module of basis

$$\{a\star g\mid a\in A, g\in G\},$$

where $a\star g$ is a notation and the product is given by

$$(a\star g)(b\star h)=a\cdot {}^g b\star gh.$$

Since B is the k-basis of A it is easy to check that the set

$$B\star G=\{s\star g\mid s\in B, g\in G\}$$

is a k-basis of the skew group algebra. Moreover it is a finite semigroup with zero, with the product defined above, since B is G-stable. Next we verify the conditions from Definition 1.1:

(1). We prove that the inverse of $s\star g\in B\star G$ is the element

$$\widehat{s\star g}={}^{g^{-1}}\widehat{s}\star g^{-1}\in B\star G.$$

We have

$$(s\star g)({}^{g^{-1}}\widehat{s}\star g^{-1})(s\star g)=(s\cdot\widehat{s}\star 1_G)(s\star g)=s\cdot\widehat{s}\cdot{}^{1_G}s\star g=s\star g.$$

Similarly we prove the other statement. Suppose now that there is $t\star h\in B\star G$ such that $(s\star g)(t\star h)(s\star g)=s\star g$. Then we have that

$$(s\cdot{}^{g}t\star gh)(s\star g)=s\star g\Rightarrow s\cdot{}^{g}t\cdot{}^{gh}s\star ghg=s\star g.$$

We have that $h=g^{-1}$ and $t={}^{g^{-1}}\widehat{s}$, thus it is unique.

(2). Let $s\star g, t\star h\in B\star G$. We have that $(t\star h)(s\star g)\neq 0$ if and only if $t\cdot{}^{h}s\neq 0$. We also have that

$$(s\star g)({}^{g^{-1}}\widehat{s}\star g^{-1})=({}^{h^{-1}}\widehat{t}\star h^{-1})(t\star h)\Leftrightarrow s\cdot\widehat{s}\star g={}^{h^{-1}}\widehat{t}\cdot{}^{h^{-1}}t\star 1_G\Leftrightarrow$$

$$s\cdot\widehat{s}={}^{h^{-1}}(\widehat{t}\cdot t)\Leftrightarrow{}^{h}s\cdot{}^{h}\widehat{s}=\widehat{t}\cdot t.$$

But since A is G-inv-symm the last condition is equivalent to $t\cdot{}^{h}s\neq 0$, by Definition 1.1, statement(2). □

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