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$\mathcal{N}$-structures applied to associative-$\mathcal{I}$-ideals in IS-algebras

Ali H. Handam

Abstract. In this paper the notion of $\mathcal{N}$-$\mathcal{I}$-ideals and $\mathcal{N}$-associative $\mathcal{I}$-ideals in IS-algebra is introduced, as well as some of their properties are investigated. The relations between $\mathcal{N}$-$\mathcal{I}$-ideals and $\mathcal{N}$-associative $\mathcal{I}$-ideals are discussed. A characterization of $\mathcal{N}$-associative $\mathcal{I}$-ideals is provided.

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1. Introduction

Imai and Iséki [1] in 1966 introduced the notion of a BCK-algebra. In the same year, Iséki [2] introduced BCI-algebras as a super class of the class of BCK-algebras. In 1993, Jun et al. [3] introduced a new class of algebras related to BCI-algebras and semigroups, called a BCI-semigroup/BCI-monoid/BCI-group. In 1998, for the convenience of study, Jun et al. [8] renamed the BCI-semigroup (respectively, BCI-monoid and BCI-group) as the IS-algebra (respectively, IM-algebra and IG-algebra) and studied further properties of these algebras (see [7]).

A (crisp) set A in a universe X can be defined in the form of its characteristic function $\mu_A : X \to \{0, 1\}$ yielding the value 1 for elements belonging to the set A and the value 0 for elements excluded from the set A. So far most of the generalization of the crisp set have been conducted on the unit interval $[0, 1]$ and they are consistent with the asymmetry observation. In other words, the generalization of the crisp set to fuzzy sets relied on spreading positive information that fit the crisp point $\{1\}$ into the interval $[0, 1]$. Because no negative meaning of information is suggested, we now feel a need to deal with negative information. To do so, we also feel a need to supply mathematical tool. To attain such object, Jun et al. [5] introduced a new function which is called negative-valued function, and constructed $\mathcal{N}$-structures. They applied $\mathcal{N}$-structures to BCK/BCI-algebras, and discussed $\mathcal{N}$-subalgebras and $\mathcal{N}$-ideals in BCK/BCI-algebras. Jun et al. [6] considered closed ideals in BCH-algebras based on

$\mathcal{N}$-structures. Jun et al. [4] introduced the notion of a (created) $\mathcal{N}$-ideal of subtraction algebras, and investigated several characterizations of $\mathcal{N}$-ideals.

In this paper, we introduced the notion of $\mathcal{N}$-$\mathcal{I}$-ideals and $\mathcal{N}$-associative $\mathcal{I}$-ideals in IS-algebras, and studied several related properties.

2. Basic results on IS-algebras

The following necessary elementary aspects of IS-algebras will be used throughout this paper.

By a BCI-algebra we mean an algebra $(X, *, 0)$ of type $(2, 0)$ satisfying the following axioms: for every $x, y, z \in X$ [2],
(I) $((x * y) * (x * z)) * (z * y) = 0$,
(II) $(x * (x * y)) * y = 0$,
(III) $x * x = 0$,
(IV) $x * y = 0$ and $y * x = 0$ imply $x = y$.
A BCI-algebra X satisfying $0 \leq x$ for all $x \in X$ is called a BCK-algebra. In any BCI-algebra X one can define a partial order "$\preceq$" by putting $x \preceq y$ if and only if $x * y = 0$.
A BCI-algebra X has the following properties for any $x, y, z \in X$ [2]:
$(A1)$ $x * 0 = x$,
$(A2)$ $(x * y) * z = (x * z) * y$,
$(A3)$ $x \preceq y$ implies that $(x * z) \preceq (y * z)$ and $(z * y) \preceq (z * x)$,
$(A4)$ $(x * z) * (y * z) \preceq x * y$,
$(A5)$ $x * (x * (x * y)) = x * y$,
$(A6)$ $0 * (x * y) = (0 * x) * (0 * y)$,
$(A7)$ $0 * (0 * ((x * z) * (y * z))) = (0 * y) * (0 * x)$.

A non-empty subset I of a BCI-algebra X is called an ideal of X if $(S1)$: $0 \in I$, $(S2)$: $x * y \in I$ and $y \in I$ imply that $x \in I$. A non-empty subset I of X is called a associative ideal of X if it satisfies $(S1)$ and $(S3)$: $((x * y) * z) \in I$, $(y * z) \in I$ imply that $x \in I$.

Definition 2.1. [8]. *An IS-algebra is a non-empty set X with two binary operations "$*$" and "$\cdot$" and constant 0 satisfying the axioms*

$(B1)$ $(X, *, 0)$ is a BCI-algebra,
$(B2)$ $(X, \cdot)$ is a semigroup,
$(B3)$ the operation "$\cdot$" is distributive (on both sides) over the operation "$*$", that is,

$$x \cdot (y * z) = (x \cdot y) * (x \cdot z) \text{ and } (x * y) \cdot z = (x \cdot z) * (y \cdot z) \text{ for all } x, y, z \in X.$$

Note that, the IS-algebra is a generalization of the ring (see [8]).

Proposition 2.2. [3]. *Let X be an IS-algebra. Then we have*

(1) $0 \cdot x = x \cdot 0 = 0$,
(2) $x \preceq y$ implies that $x \cdot z \preceq y \cdot z$ and $z \cdot x \preceq z \cdot y$, for all $x, y, z \in X$.

Definition 2.3. [8]. *A non-empty subset A of an IS-algebra X is called a left (resp. right) $\mathcal{I}$-ideal of X if*
(1) $x \cdot a \in A$ *(resp. $a \cdot x \in A$) whenever $x \in X$ and $a \in A$,*

(2) for any $x, y \in X$, $x * y \in A$ and $y \in A$ imply that $x \in A$.
Both a left and right $\mathcal{I}$-ideal is called $\mathcal{I}$-ideal.

Definition 2.4. [9]. *A non-empty subset A of an IS-algebra X is called a left (resp. right) associative $\mathcal{I}$-ideal of X if*
(1) $x \cdot a \in A$ *(resp. $a \cdot x \in A$) whenever $x \in X$ and $a \in A$,*
(2) *for any $x, y, z \in X$, $(x * y) * z \in A$ and $y * z \in A$ imply that $x \in A$.*

3. $\mathcal{N}$-associative $\mathcal{I}$-ideals

Denote by $\mathcal{F}(X, [-1, 0])$ the collection of functions from a set X to $[-1, 0]$. We say that, an element of $\mathcal{F}(X, [-1, 0])$ is a negative-valued function from X to $[-1, 0]$ (briefly, $\mathcal{N}$-function on X). By an $\mathcal{N}$-structure we mean an ordered pair (X, ξ), where ξ is an $\mathcal{N}$-function on X. In what follows, let X be an IS-algebra and ξ an $\mathcal{N}$-function on X unless otherwise specified.

Definition 3.1. *Let X be an IS-algebra. An $\mathcal{N}$-structure (X, ξ) is called a left $\mathcal{N}$-$\mathcal{I}$-ideal (resp. a right $\mathcal{N}$-$\mathcal{I}$-ideal) of X if*
$(C1)$ $(\xi(xy) \leq \xi(y))$ *(resp. $\xi(xy) \leq \xi(x)$) for all $x, y \in X$;*
$(C2)$ $\xi(x) \leq \max\{\xi(x * y), \xi(y)\}$ *for all $x, y \in X$.*

An $\mathcal{N}$-structure (X, ξ) is called an $\mathcal{N}$-$\mathcal{I}$-ideal of X if it is both a left $\mathcal{N}$-$\mathcal{I}$-ideal and a right $\mathcal{N}$-$\mathcal{I}$-ideal of X.

Definition 3.2. *Let X be an IS-algebra. An $\mathcal{N}$-structure (X, ξ) is called a left $\mathcal{N}$-associative $\mathcal{I}$-ideal (resp. a right $\mathcal{N}$-associative $\mathcal{I}$-ideal) of X if it satisfies $(C1)$ and*
$(C3)$ $\xi(x) \leq max\{\xi((x * y) * z), \xi(y * z)\}$ *for all $x, y, z \in X$.*

An $\mathcal{N}$-structure (X, ξ) is called an $\mathcal{N}$-associative $\mathcal{I}$-ideal of X if it is both a left $\mathcal{N}$-associative $\mathcal{I}$-ideal and a right $\mathcal{N}$-associative $\mathcal{I}$-ideal of X.

Example 3.3. Consider an IS-algebra $X = \{0, a, b, c\}$ with Cayley tables as follows:

$*$	0	a	b	c
0	0	0	b	b
a	a	0	c	b
b	b	b	0	0
c	c	b	a	0

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
c	0	a	b	c

(1) Let (X, ξ) be an $\mathcal{N}$-structure in which ξ is given by

$$\xi = \begin{pmatrix} 0 & a & b & c \\ t_0 & t_1 & t_0 & t_1 \end{pmatrix}, \text{ where } t_0 < t_1 \text{ in } [-1, 0].$$

Then (X, ξ) is an $\mathcal{N}$-$\mathcal{I}$-ideal of X.
(2) Let (X, ζ) be an $\mathcal{N}$-structure in which ζ is given by

$$\zeta = \begin{pmatrix} 0 & a & b & c \\ t_0 & t_0 & t_1 & t_1 \end{pmatrix}, \text{ where } t_0 < t_1 \text{ in } [-1, 0].$$

Then (X, ζ) is an $\mathcal{N}$-associative $\mathcal{I}$-ideal of X.

Proposition 3.4. *Every left (resp. right) $\mathcal{N}$-associative $\mathcal{I}$-ideal (X, ξ) satisfies the following inequality:*

$$(\forall x \in X)\ (\xi(0) \leq \xi(x)) \tag{3.1}$$

Theorem 3.5. *Every left (resp. right) $\mathcal{N}$-associative $\mathcal{I}$-ideal is a left (resp. right) $\mathcal{N}$-$\mathcal{I}$-ideal.*

Proof. Let (X, ξ) be a left (resp. right) $\mathcal{N}$-associative $\mathcal{I}$-ideal of X. Then, $\xi(xy) \leq \xi(y)$ (resp. $\xi(xy) \leq \xi(x)$) for all $x, y \in X$. Now, let $z = 0$ in $(C3)$, we have $\xi(x) \leq max\{\xi((x * y) * 0), \xi(y * 0)\}$ for all $x, y \in X$. So, $\xi(x) \leq max\{\xi((x * y)), \xi(y)\}$. Therefore, (X, ξ) is a left (resp. right) $\mathcal{N}$-$\mathcal{I}$-ideal of X. □

The next example shows that the converse of Theorem 3.5 is not always true.

Example 3.6. Consider the $\mathcal{N}$-$\mathcal{I}$-ideal (X, ξ) given in Example 3.3. By routine calculations, it is easy to check that (X, ξ) is not an $\mathcal{N}$-associative $\mathcal{I}$-ideal of X.

Proposition 3.7. *Every left (resp. right) $\mathcal{N}$-associative $\mathcal{I}$-ideal (X, ξ) satisfies the following inequality:*

$$(\forall x, y \in X)\ (\xi(x) \leq \xi((x * y) * y)) \tag{3.2}$$

Proof. Let (X, ξ) be a left (resp. right) $\mathcal{N}$-associative $\mathcal{I}$-ideal of X. If we let $z := y$ in $(C3)$, then we have $\xi(x) \leq max\{\xi((x * y) * y), \xi(y * y)\}$ for all $x, y \in X$. Using 3.1 and (III), it follows that, $\xi(x) \leq \xi((x * y) * y)$ for all $x, y \in X$. □

Proposition 3.8. *If (X, ξ) is a left (resp. right) $\mathcal{N}$-associative $\mathcal{I}$-ideal of X, then*

$$(\forall x, y \in X)\ (x \preceq y \Rightarrow \xi(x) \leq \xi(y)) \tag{3.3}$$

Proof. Let $x, y \in X$ be such that $x \preceq y$. If we let $z := 0$ in $(C3)$, then we have $\xi(x) \leq max\{\xi((x * y) * 0), \xi(y * 0)\}$ for all $x, y \in X$. Since, $x \preceq y$ implies $x * y = 0$, $\xi(x) \leq max\{\xi(0 * 0), \xi(y * 0)\}$. It follows from axiom (III) and $(A1)$ that $\xi(x) \leq \xi(y)$. □

Proposition 3.9. *Let (X, ξ) be a left (resp. right) $\mathcal{N}$- $\mathcal{I}$-ideal of X. Then, $x * y \preceq z$ implies $\xi(x) \leq max\{\xi(z), \xi(y)\}$ for all $x, y, z \in X$.*

Theorem 3.10. *Let (X, ξ) be a left (resp. right) $\mathcal{N}$-associative $\mathcal{I}$-ideal of X. Then, for any $x, y, z \in X$,*

(i) $x * y \preceq z$ implies $\xi(x) \leq \xi(y * z)$.
(ii) $\xi(x) \leq \xi(0 * x)$.
(iii) $\xi((x \cdot y) * (x \cdot z)) \leq \xi(y * z)$ (resp. $\xi((x \cdot z) * (y \cdot z)) \leq \xi(x * y)$).

Proof. (i) Suppose that (X, ξ) is a left (resp. right) $\mathcal{N}$-associative $\mathcal{I}$-ideal of X, by ($C3$) we have $\xi(x) \leq max\{\xi((x*y)*w), \xi(y*w)\}$ for all $x, y, w \in X$. Since, $x*y \preceq z$ implies $(x*y)*w \preceq z*w$, by (3.3), it follows that $\xi((x*y)*w) \leq \xi(z*w)$. Hence, $\xi(x) \leq max\{(\xi(z*w), \xi(y*w)\}$. If we let $w = z$, then we have, $\xi(x) \leq max\{(\xi(0), \xi(y*z)\} = \xi(y*z)$.
(ii) Let $z = x*y$ in ($C3$), then

$$\xi(x) \leq max\{\xi(0), \xi(y*(x*y)\} = \xi(y*(x*y) \tag{3.4}$$

If we let $y = 0$ in (3.4), then we obtain also

$$\begin{aligned} \xi(x) &\leq \xi(0*(x*0)) \\ &= \xi(0*x) \qquad \text{by } (A1) \end{aligned}$$

(iii) It follows directly from ($B3$) and ($C1$). □

Definition 3.11. [5]. *Let (X, ξ) and (X, ζ) be two $\mathcal{N}$-structures.*

(1) The union, $\xi \cup \zeta$ of ξ and ζ is defined by $(\xi \cup \zeta)(x) = \max\{\xi(x), \zeta(x)\}$ for all $x \in X$.
(2) The intersection, $\xi \cap \zeta$ of ξ and ζ is defined by $(\xi \cap \zeta)(x) = \min\{\xi(x), \zeta(x)\}$ for all $x \in X$.

Obviously, $(X, \xi \cup \zeta)$ and $(X, \xi \cap \zeta)$ are $\mathcal{N}$-structures which are called the union and the intersection of (X, ξ) and (X, ζ), respectively.

Proposition 3.12. *If (X, ξ) and (X, ζ) are left (resp. right) $\mathcal{N}$-associative $\mathcal{I}$-ideals of X, then the union $(X, \xi \cup \zeta)$ is a left (resp. right) $\mathcal{N}$-associative $\mathcal{I}$-ideal of X.*

Now, we give an example to show that the intersection of two $\mathcal{N}$-$\mathcal{I}$-ideals may not be an $\mathcal{N}$-$\mathcal{I}$-ideal.

Example 3.13. Consider the two $\mathcal{N}$-$\mathcal{I}$-ideals (X, ξ) and (X, ζ) given in Example 3.3. The intersection $\xi \cap \zeta$ is given by

$$\xi \cap \zeta = \begin{pmatrix} 0 & a & b & c \\ t_0 & t_0 & t_0 & t_1 \end{pmatrix}, \text{ where } t_0 < t_1 \text{ in } [-1, 0].$$

$\xi \cap \zeta$ is not an $\mathcal{N}$-$\mathcal{I}$-ideal of X, since $(\xi \cap \zeta)(c) = t_1 \nleq \max\{(\xi \cap \zeta)(c*b), (\xi \cap \zeta)(b)\} = t_0$.

For any $\mathcal{N}$-function ξ on X and $t \in [-1, 0)$, define the set $\mathcal{C}(\xi, t)$ as

$$\mathcal{C}(\xi, t) = \{x \in X \mid \xi(x) \leq t\}.$$

Theorem 3.14. *An $\mathcal{N}$-structure (X, ξ) is a left (resp. right) $\mathcal{N}$-associative $\mathcal{I}$-ideal of X if and only if every non-empty set $\mathcal{C}(\xi, t)$ is a left (resp. right) associative $\mathcal{I}$-ideal of X for all $t \in [-1, 0)$.*

Proof. Assume that (X, ξ) is a left (resp. right) $\mathcal{N}$-associative $\mathcal{I}$-ideal of X and let $t \in [-1, 0)$ be such that $\mathcal{C}(\xi, t) \neq \emptyset$. Let $x \in X$ and $a \in \mathcal{C}(\xi, t)$. Then, $\xi(a) \leq t$. It follows from ($C1$) that $\xi(x \cdot a) \leq \xi(a) \leq t$ (resp. $\xi(a \cdot x) \leq \xi(a) \leq t$). Hence, $x \cdot a \in \mathcal{C}(\xi, t)$ (resp. $a \cdot x \in \mathcal{C}(\xi, t)$). Now, let $(x*y)*z \in \mathcal{C}(\xi, t)$ and $(y*z) \in \mathcal{C}(\xi, t)$. Then, $\xi((x*y)*z) \leq t$

and $\xi(y*z) \leq t$. Using $(C3)$ we obtain, $\xi(x) \leq max\{\xi((x*y)*z), \xi(y*z)\} \leq t$. Thus $x \in \mathcal{C}(\xi, t)$. Therefore, $\mathcal{C}(\xi, t)$ is a left (resp. right) associative $\mathcal{I}$-ideal of X for all $t \in [-1, 0)$.

Conversely, suppose that every non-empty set $\mathcal{C}(\xi, t)$ is a left (resp. right) associative $\mathcal{I}$-ideal of X for all $t \in [-1, 0)$. If there are $a, b \in X$ such that $\xi(a \cdot b) > \xi(b)$ (resp. $\xi(a \cdot b) > \xi(a)$), then, $\xi(a \cdot b) > t_0 \geq \xi(b)$ (resp. $\xi(a \cdot b) > t_0 \geq \xi(a)$) for some $t_0 \in [-1, 0)$. Hence, $b \in \mathcal{C}(\xi, t_0)$ (resp. $a \in \mathcal{C}(\xi, t_0)$) and $a \cdot b \notin \mathcal{C}(\xi, t_0)$. This is a contradiction. Thus, $\xi(x \cdot y) \leq \xi(y)$ (resp. $\xi(x \cdot y) \leq \xi(x)$) for all $x, y \in X$. Now, assume that there exist $a, b, c \in X$ such that $\xi(a) > max\{\xi((a*b)*c), \xi(b*c)\}$. Then, $\xi(a) > t_1 \geq \max\{\xi((a*b)*c), \xi(b*c)\}$ for some $t_1 \in [-1, 0)$. Hence, $(a*b)*c, b*c \in \mathcal{C}(\xi, t_1)$ and $a \notin \mathcal{C}(\xi, t_1)$, which is a contradiction. Therefore, (X, ξ) is a left (resp. right) $\mathcal{N}$-associative $\mathcal{I}$-ideal of X. □

Theorem 3.15. *Let A be a left (resp. right) associative $\mathcal{I}$-ideal of X and let (X, ξ) be an $\mathcal{N}$-structure in X defined by*

$$\xi(x) = \begin{cases} t_0 & \text{if x} \in A \\ t_1 & \text{otherwise} \end{cases},$$

where $t_0 < t_1$ in $[-1, 0]$. Then, the $\mathcal{N}$-structure (X, ξ) is a left (resp. right) $\mathcal{N}$-associative $\mathcal{I}$-ideal of X.

Proof. It follows directly from Theorem 3.14. □

For any $\mathcal{N}$-structure (X, ξ) and any element $w \in X$, consider the set

$$\mathcal{D}_w := \{x \in X \mid \xi(x) \leq \xi(w)\}.$$

Then, $\mathcal{D}_w$ is non-empty subset of X.

Theorem 3.16. *If an $\mathcal{N}$-structure (X, ξ) is a left (resp. right) $\mathcal{N}$-associative $\mathcal{I}$-ideal of X, then $\mathcal{D}_w$ is a left (resp. right) associative $\mathcal{I}$-ideal of X for all $w \in X$.*

Proof. Let $a \in \mathcal{D}_w$ and $x \in X$. Then, $\xi(a) \leq \xi(w)$. By $(C1)$ it follows that $\xi(x \cdot a) \leq \xi(a) \leq \xi(w)$ (resp. $\xi(a \cdot x) \leq \xi(a) \leq \xi(w)$). Hence $x \cdot a \in \mathcal{D}_w$ (resp. $a \cdot x \in \mathcal{D}_w$). Now, let $x, y, z \in X$ be such that $(x*y)*z \in \mathcal{D}_w$ and $y*z \in \mathcal{D}_w$. Then, $\xi((x*y)*z) \leq \xi(w)$ and $\xi(y*z) \leq \xi(w)$. By $(C3)$ it follows that $\xi(x) \leq max\{\xi((x*y)*z), \xi(y*z)\} \leq \xi(w)$. Hence, $x \in \mathcal{D}_w$. Therefore, $\mathcal{D}_w$ is a left (resp. right) associative $\mathcal{I}$-ideal of X for all $w \in X$. □

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Ali H. Handam
Department of Mathematics
Al al-Bayt University
P.O. Box: 130095, Al Mafraq, Jordan
e-mail: `ali.handam@windowslive.com`