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Generalizations of Krasnoselskii's fixed point theorem in cones

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Abstract. We give some generalizations of Krasnoselskii's fixed point theorem in cones.

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1. Introduction

Firstly we will present the definition of a cone.

Definition 1.1. *Let X be a normed linear space. A nonempty closed, convex set $P \subset X$ is called a cone if it satisfies the following two conditions:*
(i) $x \in P, \lambda \geq 0$ implies $\lambda x \in P$;
(ii) $x \in P, -x \in P$ implies $x = 0$.

After the well known paper of Legget and Williams(see [6]), many authors have given generalizations of the following Krasnoselskii's fixed point theorem:

Theorem 1.2. *(Krasnoselskii) Let $(X, | \,.\, |)$ be a normed linear space, $K \subset X$ a cone and " $\prec$ " the order relation induced by K. Let be $r, R \in R_+$, $0 < r < R$, $K_{r,R} := \{u \in K : r \leq | u | \leq R\}$ and let $N : K_{r,R} \to K$ be a completely continuous map. Assume that one of the following conditions is satisfied:*
(i) $| Nu | \geq | u |$ if $| u | = r$ and $| Nu | \leq | u |$ if $| u | = R$
(ii) $| Nu | \leq | u |$ if $| u | = r$ and $| Nu | \geq | u |$ if $| u | = R$.
Then N has a fixed point u^* in K with $r \leq | u^* | \leq R$.

For example, in [8], the author gives the following result. Before to state it, we introduce a few notations. We shall consider two wedges K_1, K_2 of X and the corresponding wedge $K := K_1 \times K_2$ of $X^2 := X \times X$. For

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$r, R \in R_+^2$, $r = (r_1, r_2)$, $R = (R_1, R_2)$, we write $0 < r < R$ if $0 < r_1 < R_1$ and $0 < r_2 < R_2$, and we use the notations:

$$(K_i)_{r_i R_i} := \{u \in K_i : r_i \leq\mid u \mid\leq R_i\} \ (i = 1, 2)$$

$$K_{rR} := \{u \in K : r_i \leq\mid u_i \mid\leq R_i \text{ for } i = 1, 2\}.$$

Clearly, $K_{rR} = (K_1)_{r_1 R_1} \times (K_2)_{r_2 R_2}$.

Theorem 1.3. *([8]) Let $(X, \mid . \mid)$ be a normed linear space; $K_1, K_2 \subset X$ two wedges; $K := K_1 \times K_2$; $\alpha_i, \beta_i > 0$ with $\alpha_i \neq \beta_i$ for $i = 1, 2$, and let $r_i = \min\{\alpha_i, \beta_i\}, R_i = \max\{\alpha_i, \beta_i\}$ for $i = 1, 2$. Assume that $N : K_{rR} \to K$, $N = (N_1, N_2)$, is a compact map and there exist $h_i \in K_i \setminus \{0\}$, $i = 1, 2$ such that for each $i \in \{1, 2\}$ the following condition is satisfied in K_{rR} :*

$$N_i u \neq \lambda u_i \textit{ for } \mid u_i \mid= \alpha_i \textit{ and } \lambda > 1;$$

$$N_i u + \mu h_i \neq u_i \textit{ for } \mid u_i \mid= \beta_i \textit{ and } \mu > 0.$$

Then N has a fixed point $u = (u_1, u_2)$ in K with $r_i \leq\mid u_i \mid\leq R_i$ for $i = 1, 2$.

Also, in [9], the author gives the following result (Here $(E, \mid . \mid)$ is a normed linear space and $\| . \|$ is another norm on E, $C \subset E$ is a nonempty convex, not necessarily closed set with $0 \notin C$ and $\lambda C \subset C$ for all $\lambda > 0$), assuming that there exist constants $c_1, c_2 > 0$ such that the norms $\mid . \mid$ and $\| . \|$ are topologically equivalent, which is

$$c_1 \mid x \mid\leq\| x \|\leq c_2 \mid x \mid \text{ for all } x \in C.$$

Also assume that $\| . \|$ is increasing with respect to C , that is $\| x+y \|>\| x \|$ for all $x, y \in C$.

Theorem 1.4. *([9]) Assume $0 < c_2\rho < R$, $\| . \|$ is increasing with respect to C, and the map $N : D = \{x \in C :\| x \|\leq R\} \to C$ is compact. In addition assume that the following conditions are satisfied:*

(H1) $\| N(x) \|\geq\| x \|$ for all $x \in C$ with $\mid x \mid= \rho$,

(H2) $\mid N(x) \mid<\mid x \mid$ for all $x \in C$ with $\| x \|= R$.

Then N has at least one fixed point $x \in C$ with $\rho \leq\mid x \mid$ and $\| x \|< R$.

For other generalizations and applications of Krasnoselskii's fixed point theorem in cone the reader may see the papers [7] and [1]-[4].

In this paper we are interested to give some new abstract results and we use conditions of type

$$\varphi(u) \geq \varphi(Nu) \text{ if } \mid u \mid= r$$

instead of condition

$$\mid u \mid\geq\mid Nu \mid \text{ if } \mid u \mid= r$$

which is assumed in Krasnoselskii's fixed point theorem in cone.

2. The main results

Throughout this paper we consider $(X, |\ .\ |)$ be a normed linear space, $K \subset X$ a positive cone, "$\preceq$" the order relation induced by K and "$\prec$" the strict order relation induced by K.

Theorem 2.1. *Let be* $K_{r,R} = \{x \in K : r \leq| x |\leq R\}$, *where* $r, R \in R_+$, $r < R$. *We assume that* $N : K_{r,R} \to K$ *is a completely continuous operator and* $\varphi : K \to R_+, \psi : K \to R$. *Also, assume that the following conditions are satisfied:*

$$(i.1) \begin{cases} \varphi(0) = 0 \text{ and there exists } h \in K - \{0\} \text{ such that} \\ \varphi(\lambda h) > 0, \text{ for all } \lambda \in (0, 1], \\ \varphi(x + y) \geq \varphi(x) + \varphi(y) \text{ for all } x, y \in K, \end{cases}$$

$(i.2)$ $\psi(\alpha x) > \psi(x)$ *for all* $\alpha > 1$ *and for all* $x \in K$ *with* $| x |= R$,

$$(i.3) \begin{cases} \varphi(u) \leq \varphi(Nu) & \text{if } | u |= r \\ \psi(u) \geq \psi(Nu) & \text{if } | u |= R. \end{cases}$$

Then N *has a fixed point in* $K_{r,R}$

Proof. Let $N^* : K \to K$ be given by

$$N^*(u) = \begin{cases} h, & \text{if } u = 0, \\ (1 - \frac{|u|}{r})h + \frac{|u|}{r} N(\frac{r}{|u|}u), & \text{if } 0 <| u |< r, \\ Nu, & \text{if } r \leq| u |\leq R, \\ N(\frac{R}{|u|}u), & \text{if } | u |\geq R. \end{cases}$$

N is completely continuous, so N^* is completely continuous too. From our hypothesis we have that $N^*(K) \subset K$ is a convex and relatively compact set, so from Schauder' s fixed point theorem it follows that there exists $u^* \in K$ with $N^*(u^*) = u^*$. We have to consider three cases.

Case 1. Suppose that $u^* = 0$. We have $0 = N^*(0) = h$, a contradiction with $h \in K \setminus \{0\}$.

Case 2. Suppose that $0 <| u^* |< r$. We obtain

$$\left(1 - \frac{| u^* |}{r}\right) h + \frac{| u^* |}{r} N\left(\frac{r}{| u^* |} u^*\right) = u^*,$$

$$\left(\frac{r}{| u^* |} - 1\right) h + N(\frac{r}{| u^* |} u^*) = \frac{r}{| u^* |} u^*.$$

Let $\lambda := \frac{r}{|u^*|} - 1$ and $u_0 := \frac{r}{|u^*|} u^*$. Since $| u^* |< r$ we have that $\frac{r}{|u^*|} > 1$, so $\lambda > 0$. Also, $| u_0 |=| \frac{r}{|u^*|} u^* |= \frac{r}{|u^*|} | u^* |= r$, so $| u_0 |= r$. We obtain

$$\lambda h + N(u_0) = u_0 \tag{2.1}$$

For $\lambda > 0$, from (i1) we obtain that

$$\varphi(N(u_0) + \lambda h) \geq \varphi(N(u_0)) + \varphi(\lambda h) > \varphi(N(u_0)).$$

Then, from (2.1) we obtain $\varphi(u_0) > \varphi(N(u_0))$, a contradiction with (i3).

Case 3. Suppose that $| u^* |> R$. We have $N(\frac{R}{|u^*|} u^*) = u^*$. Let $u_1 := \frac{R}{|u^*|} u^*$ and $\beta := \frac{|u^*|}{R} > 1$. We have $| u_1 |= R$ and $N(u_1) = u^* = u_1 \frac{|u^*|}{R}$, so $N(u_1) = \beta u_1$. From (i.2) we obtain $\psi(N(u_1)) = \psi(\beta u_1) > \psi(u_1)$, a contradiction with (i.3). So $r \leq| u^* |\leq R$ and the conclusion follows. □

Remark 2.2. (1) If $X := C[0,1], \eta > 0, I \subset [0,1]$, $I \neq [0,1]$, $\| x \|:= \max\limits_{t\in[0,1]} x(t)$ and $K := \{x \in C[0,1] : x \geq 0 \text{ on } [0,1], x(t) \geq \eta \| x \| \text{ for all } t \in I\}$ is a cone, a functional that satisfies (i1) is

$$\varphi(x) := \min_{t\in I} x(t).$$

Indeed, $\varphi(0) = 0$,there exists $h \in K - \{0\}$ such that $\varphi(\lambda h) > 0$, for all $\lambda \in (0,1]$ and

$$\varphi(x+y) = \min_{t\in I}[x(t)+y(t)] \geq \min_{t\in I} x(t) + \min_{t\in I} y(t) = \varphi(x) + \varphi(y).$$

(2) The norm is an example of functional that satisfies (i2).

Theorem 2.3. *Let $K_{r,R} = \{x \in K : r \leq| x |\leq R\}$, where $r, R \in R_+$, $r < R$. We assume that $N : K_{r,R} \to K$ is a completely continuous operator and $\varphi, \psi : K \to R$. Also, we assume that the following conditions are satisfied:*

$(ii.1)$ φ is strictly decreasing,

$(ii.2)$ $\psi(\alpha x) < \psi(x)$ for all $\alpha > 1$ and for all $x \in K$ with $| x |= R$,

$$(ii.3) \begin{cases} \varphi(u) \geq \varphi(Nu) & \text{if} \quad | u |= r, \\ \psi(u) \leq \psi(Nu) & \text{if} \quad | u |= R. \end{cases}$$

Then N has a fixed point in $K_{r,R}$.

Proof. Let $h \succ 0$ and $N^* : K \to K$,

$$N^*(u) = \begin{cases} h, & \text{if} \quad u = 0 \\ (1 - \frac{|u|}{r})h + \frac{|u|}{r} N(\frac{r}{|u|}u), & \text{if} \quad 0 <| u |< r \\ Nu, & \text{if} \quad r \leq| u |\leq R \\ N(\frac{R}{|u|}u), & \text{if} \quad | u |\geq R. \end{cases}$$

Since N^* is completely continuous, we have, like in Theorem 2.1, that there exists $u^* \in K$ so that $N^*(u^*) = u^*$. We consider three cases.

Case 1. If $u^* = 0$ we obtain $0 = N^*(0) = h$, a contradiction with $h \succ 0$.

Case 2. If $0 <| u^* |< r$. We obtain (2.1) with $\lambda > 0$ and $| u_0 |= r$, like in Theorem 2.1. From $\lambda h \succ 0$, we have that

$$N(u_0) + \lambda h \succ N(u_0),$$

so, from (ii.1), we have that

$$\varphi(N(u_0) + \lambda h) < \varphi(N(u_0))$$

and from (2.1) we obtain

$$\varphi(u_0) < \varphi(N(u_0)) \text{ for } | u_0 |= r,$$

a contradiction with (ii.3).

Case 3. If $| u^* |> R$, we have that

$$N\left(\frac{R}{| u^* |}u^*\right) = u^*,$$

so

$$N\left(\frac{R}{\mid u^* \mid}u^*\right) = \left(\frac{R}{\mid u^* \mid}u^*\right)\frac{\mid u^* \mid}{R}.$$

Let be $u_1 := \frac{R}{|u^*|}u^*$, so $\mid u_1 \mid= R$ and let be $\beta := \frac{|u^*|}{R} > 1$. We obtain $N(u_1) = \beta u_1$, so

$$\psi\left(N(u_1)\right) = \psi\left(\beta u_1\right) \tag{2.2}$$

From (ii.2) we obtain

$$\psi\left(\beta u_1\right) < \psi\left(u_1\right)$$

and from (2.2) we have

$$\psi\left(N(u_1)\right) < \psi\left(u_1\right) \text{ for } \mid u_1 \mid= R,$$

a contradiction with (ii.3). So $r \leq\mid u^* \mid\leq R$ and the conclusion follows. □

Remark 2.4. $\psi(x) := \frac{1}{|x|+1}$ is an example of functional that satisfies (ii.2). Indeed, for $\alpha > 1$ and $\mid x \mid= R$, we have that

$$\psi(\alpha x) = \frac{1}{\alpha \mid x \mid +1} < \frac{1}{\mid x \mid +1} = \psi(x).$$

Also, if $\mid . \mid$ is strictly increasing, i.e., $x < y$ implies $\mid x \mid<\mid y \mid$, then $\varphi(x) := \frac{1}{|x|+1}$ is strictly decreasing, so it satisfies (ii.1).

Theorem 2.5. *Let* $K_{r,R} := \{x \in K : r \leq\mid x \mid\leq R\}$, *where* $r, R \in R_+, r < R$. *We assume that* $N : K_{r,R} \to K$ *is a completely continuous operator and* $\varphi, \psi : K \to R_+$. *Also, we assume that the following conditions are satisfied:*

$(iii.1)\begin{cases} \varphi(\alpha x) = \alpha\varphi(x), \textit{ for all } \alpha > 0 \textit{ and for all } x \in K, \\ \varphi(\alpha x) > \varphi(x), \textit{ for all } \alpha > 1 \textit{ and for all } x \in K \textit{ with } \mid x \mid= R, \end{cases}$

$(iii.2)\begin{cases} \psi(0) = 0 \textit{ and there exists } h \in K \setminus \{0\} \textit{ such that} \\ \psi(\lambda h) > 0 \textit{ for all } \lambda \in (0, 1], \\ \psi(\alpha x) = \alpha\psi(x) \textit{ for all } \alpha > 0 \textit{ and for all } x \in K, \\ \psi(x + y) \geq \psi(x) + \psi(y) \textit{ for all } x, y \in K, \end{cases}$

$(iii.3)$

$$\begin{cases} \varphi(u) \geq \varphi(Nu) & \textit{if} \quad \mid u \mid= r, \\ \psi(u) \leq \psi(Nu) & \textit{if} \quad \mid u \mid= R. \end{cases}$$

Then N *has a fixed point in* $K_{r,R}$.

Proof. Define $N^* : K_{r,R} \to K$ by

$$N^*(u) := (\frac{R}{\mid u \mid} + \frac{r}{\mid u \mid} - 1)^{-1}N((\frac{R}{\mid u \mid} + \frac{r}{\mid u \mid} - 1)u).$$

Since N is completely continuous, it follows that N^* is completely continuous too. Let

$$\alpha := \frac{R}{\mid u \mid} + \frac{r}{\mid u \mid} - 1$$

and

$$u_0 := \alpha u.$$

We have now,

$$\alpha N^*(u) = N(\alpha u).$$

If $| u |= r,$ then

$$\alpha = \frac{R}{r} \text{ and } | u_0 |=| \alpha u |= \frac{R}{r} r = R.$$

So, from (iii.2),

$$\psi(N(u_0)) = \psi(N(\alpha u)) = \psi(\alpha N^*(u)) = \alpha\psi(N^*(u)) \tag{2.3}$$

and from (iii.3),

$$\psi(N(u_0)) \geq \psi(u_0) = \psi(\alpha u) = \alpha\psi(u). \tag{2.4}$$

From (2.3) and (2.4) we obtain that

$$\psi(N^*(u) \geq \psi(u) \text{ if } | u |= r. \tag{2.5}$$

If $| u |= R$, then

$$\alpha = \frac{r}{R} \text{ and } | u_0 |=| \alpha u |= \frac{r}{R} R = r.$$

Using (iii.3) we obtain that

$$\varphi(\alpha u) = \varphi(u_0) \geq \varphi(N(u_0)) = \varphi(N(\alpha u)) = \varphi(\alpha N^*(u)) \tag{2.6}$$

and from (iii.1),

$$\varphi(\alpha u) = \alpha\varphi(u) \text{ and } \varphi(\alpha N^*(u)) = \alpha\varphi(N^*(u)). \tag{2.7}$$

From (2.6) and (2.7) we deduce that

$$\varphi(u) \geq \varphi(N^*(u)) \text{ if } | u |= R. \tag{2.8}$$

So, (2.5) and (2.8) imply that φ, ψ and N^* satisfy all the conditions of Theorem 2.1 (with φ and ψ changing their places and N^* instead of N). So N^* has a fixed point u^* in $K_{r,R}$. It follows that

$$N^*(u^*) = u^*, \text{ with } r \leq| u^* |\leq R,$$

so

$$\frac{1}{\alpha} N(\alpha u^*) = u^*.$$

Making the notation $u_1 := \alpha u^*$, where $\alpha = \frac{R}{|u^*|} + \frac{r}{|u^*|} - 1$, we obtain

$$N(u_1) = u_1 \tag{2.9}$$

and

$$| u_1 |= \alpha | u^* |= R + r- | u^* | .$$

Since

$$\begin{aligned} R + r- \quad | \quad u^* |\geq r, \text{ for } r \leq| u^* |\leq R, \\ R + r- \quad | \quad u^* |\leq R, \text{ for } r \leq| u^* |\leq R, \end{aligned}$$

we have that

$$r \leq| u_1 |\leq R, \quad \text{that is } u_1 \in K_{r,R}. \tag{2.10}$$

From (2.9) and (2.10) the conclusion follows. □

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