Stud. Univ. Babeş-Bolyai Math. 56(2011), No. 4, 157–164

Some results on the solutions of a functional-integral equation

Viorica Mureşan

Abstract. In this paper we give existence, uniqueness, data dependence and comparison theorems for the solutions of a functional-integral equation of the same type as that considered by L. Olszowy [6]. We apply some results from Picard and weakly Picard operators' theory (see I.A. Rus, [7]).

Mathematics Subject Classification (2010): Primary 34K05, Secondary 34K15, 47H10.

Keywords: functional-integral equations, fixed points, Picard operators, weakly Picard operators.

1. Introduction

The fixed point theory has a lot of applications in the field of functional-differential equations (see for example [1]-[6], [8]). In the paper [6] has been given theorems on the existence and asymptotic characterization of the solutions of the following problem:

$$y'(t) = f(t, y(H(t)), y'(h(t))), t \in [0, \infty) \tag{1.1}$$

$$y(0) = 0. \tag{1.2}$$

Technique linking measures of noncompactness with the Tichonov' fixed point principle in suitable Fréchet space was used.

As it was shown in [6], the problem (1.1)+(1.2) is equivalent with the following functional- integral equation:

$$x(t) = f(t, \int_0^{H(t)} x(s)ds, x(h(t))), t \in [0, \infty) \tag{1.3}$$

The aim of this paper is to give existence, uniqueness, data dependence and comparison theorems for the solutions of a functional-integral equation

This paper was presented at the International Conference on Nonlinear Operators, Differential Equations and Applications, July 5-8, 2011, Cluj-Napoca, Romania.

of the same type as that considered in [6]. We apply some results from Picard and weakly Picard operators' theory (see [7] and [8]).

2. Weakly Picard operators

Here, first we present some notions and results from the weakly Picard operators' theory.

Let (X, d) be a metric space and $A : X \longrightarrow X$ an operator.

We denote by $A^0 := 1_X$, $A^1 := A, ..., A^{n+1} := A \circ A^n$, $n \in \mathbb{N}$, the iterate operators of the operator A. Also:

$$P(X) := \{Y \subset X \ / \ Y \neq \emptyset\},$$

$$I(A) := \{Y \in P(X) \ / \ A(Y) \subset Y\},$$

the family of all nonempty invariant subsets of A,

$$F_A = \{x \in X / A(x) = x\},$$

the fixed point set of the operator A.

Following Rus I.A. [7] and [8], we have:

Definition 2.1. *The operator A is a Picard operator if there exists $x^* \in X$ such that*

1) $F_A = \{x^*\}$*;*

2) *the successive approximation sequence $(A^n(x_0))_{n\in\mathbb{N}}$ converges to x^*, for all $x_0 \in X$.*

Definition 2.2. *A is a weakly Picard operator if the sequence $(A^n(x_0))_{n\in\mathbb{N}}$ converges for all $x_0 \in X$ and the limit (which generally depends on x_0) is a fixed point of A.*

Definition 2.3. *For an weakly Picard operator $A : X \to X$ we define the operator A^∞ as follows:*

$$A^\infty : X \to X, \quad A^\infty(x) := \lim_{n\to\infty} A^n(x), \ \textit{for all } x \in X.$$

Remark 2.4. $A^\infty(X) = F_A$.

We have

Theorem 2.5. (Data dependence theorem) *Let (X, d) be a complete metric space and $A, B : X \longrightarrow X$ two operators. We suppose that:*

(i) A is an α-contraction and let $F_A = \{x_A^*\}$;

(ii) $F_B \neq \emptyset$ and let $x_B^* \in F_B$;

(iii) there exists $\delta > 0$, such that $d(A(x), B(x)) \leq \delta$, for all $x \in X$.

Then

$$d(x_A^*, x_B^*) \leq \frac{\delta}{1-\alpha}.$$

Theorem 2.6. (Characterization theorem) *Let* (X, d) *be a metric space and* $A : X \to X$ *an operator. The operator* A *is a weakly Picard operator if and only if there exists a partition of* X, $X = \cup_{\lambda \in \Lambda} X_\lambda$,*such that:*

(i) $X_\lambda \in I(A)$;

(ii) $A|_{X_\lambda} : X_\lambda \to X_\lambda$ *is a Picard operator, for all* $\lambda \in \Lambda$.

Lemma 2.7. *Let* $(X, \leq)$ *be an ordered metric space and* $A : X \to X$ *an operator. We suppose that:*

(i) *A is a weakly Picard operator;*

(ii) *A is increasing.*

Then the operator A^∞ *is increasing.*

Lemma 2.8. (Abstract Gronwall lemma) *Let* $(X, \leq)$ *be an ordered metric space and* $A : X \to X$ *an operator. We suppose that:*

(i) *A is a Picard operator;*

(ii) *A is increasing.*

If we denote by x_A^* *the unique fixed point of* A, *then:*

(a) $x \leq A(x)$ *implies* $x \leq x_A^*$;

(b) $x \geq A(x)$ *implies* $x \geq x_A^*$.

Lemma 2.9. (Abstract comparison lemma) *Let* $(X, \leq)$ *be an ordered metric space and the operators* $A, B, C : X \to X$ *be such that:*

(i) $A \leq B \leq C$;

(ii) *the operators* A, B, C *are weakly Picard operators;*

(iii) *the operator* B *is increasing.*

Then $x \leq y \leq z$ *implies* $A^\infty(x) \leq B^\infty(y) \leq C^\infty(z)$.

3. Existence, uniqueness and data dependence results

Let us consider the following functional-integral equation:

$$x(t) = \alpha + f(t, \int_0^{g(t)} x(s)ds, x(h(t))), t \in [0, T] \tag{3.1}$$

under the following assumptions:

(A_1) $f \in C([0, T] \times \mathbb{R}^2)$;

(A_2) $g, h \in C([0, T], [0, T])$ and $g(t) \leq t$, $h(t) \leq t$, for all $t \in [0, T]$;

(A_3) $\alpha \in \mathbb{R}$ and $f(0, 0, \alpha) = 0$;

(A_4) there exists $k_1 > 0$ and $0 < k_2 < 1$, such that

$$|f(t, u_1, v_1) - f(t, u_2, v_2)| \leq k_1 |u_1 - u_2| + k_2 |v_1 - v_2| ,$$

$$\text{for all } t \in [0, T] \text{ and all } u_i, v_i \in \mathbb{R},\ i = 1, 2.$$

We have

Theorem 3.1. *If all the conditions* $(A_1) - (A_4)$*are satisfied, then the equation (3.1) has in* $C[0, T]$ *a unique solution.*

Proof. On $C[0,T]$, we consider a Bielecki norm $||\cdot||_\tau$, defined by

$$||x||_\tau = \max_{t\in[0,T]} |x(t)|\ e^{-\tau t},$$

where $\tau > 0$, and the operator

$$A : (C[0,T], ||\cdot||_\tau) \to (C[0,T], ||\cdot||_\tau),$$

defined by

$$A(x)(t) := \alpha + f(t, \int_0^{g(t)} x(s)ds, x(h(t))), t \in [0,T].$$

So, we have a fixed point equation:

$$x = A(x).$$

Let $x, z \in C[0,T]$ be. We obtain

$$|A(x)(t) - A(z)(t)| =$$

$$= |f(t, \int_0^{g(t)} x(s)ds, x(h(t))) - f(t, \int_0^{g(t)} z(s)ds, z(h(t)))| \leq$$

$$\leq k_1 |\int_0^{g(t)} (x(s) - z(s))ds| + k_2|x(h(t)) - z(h(t))| \ \leq$$

$$\leq k_1 \int_0^{g(t)} |x(s) - z(s)|e^{-\tau s}e^{\tau s}ds + k_2|x(h(t)) - z(h(t))|e^{-\tau h(t)}e^{\tau h(t)} \leq$$

$$\leq (k_1 \int_0^{g(t)} e^{\tau s}ds + k_2 e^{\tau h(t)})||x - z||_\tau \leq$$

$$\leq (k_1 \int_0^{t} e^{\tau s}ds + k_2 e^{\tau t})||x - z||_\tau \leq$$

$$\leq (\frac{k_1}{\tau} + k_2)e^{\tau t}||x - z||_\tau,$$

for all $t \in [0,T]$.

So,

$$|A(x)(t) - A(z)(t)|e^{-\tau t} \leq (\frac{k_1}{\tau} + k_2)||x - z||_\tau,$$

for all $t \in [0,T]$.

It follows that

$$||A(x) - A(z)||_\tau \leq (\frac{k_1}{\tau} + k_2)||x - z||_\tau,$$

for all $x, z \in C[0,T]$.

We choose τ large enough, such that $\frac{k_1}{\tau} + k_2 < 1$. By applying Contraction mapping principle, we obtain that A is a Picard operator. □

Now, together with (3.1), we consider the following equation:

$$x(t) = \alpha + F(t, \int_0^{g(t)} x(s)ds, x(h(t))), t \in [0, T], \tag{3.2}$$

where $F \in C([0, T] \times \mathbb{R}^2)$ and α, g, h are the same as in (3.1).

We have

Theorem 3.2. *We suppose that:*

(i) the conditions $(A_1)-(A_4)$ are satisfied and $x^* \in C[0, T]$ is the unique solution of the equation (3.1);

(ii) the equation (3.2) has solutions in $C[0, T]$ and $z^* \in C[0, T]$ is a solution of (3.2);

(iii) there exists $\eta > 0$ such that

$$|f(t, u, v) - F(t, u, v)| \leq \eta, \text{ for all } t \in [0, T] \text{ and all } u, v \in \mathbb{R}.$$

Then

$$||x^* - z^*||_\tau \leq \frac{\eta}{1 - (\frac{k_1}{\tau} + k_2)},$$

where τ is large enough such that $\frac{k_1}{\tau} + k_2 < 1$.

Proof. Consider

$$A_F : (C[0, T], || \cdot ||_\tau) \to (C[0, T], || \cdot ||_\tau),$$

$$A_F(x)(t) := \alpha + F(t, \int_0^{g(t)} x(s)ds, x(h(t))), t \in [0, T],$$

the corresponding operator of (3.2).

We have

$$|A(x)(t) - A_F(x)(t)| \leq \eta,$$

for all $t \in [0, T]$, and consequently

$$||A(x) - A_F(x)||_\tau \leq \eta,$$

for all $x \in C[0, T]$. □

Now, we apply Data dependence theorem (Theorem 2.5).

Theorem 3.3. *We suppose that:*

(i) the conditions $(A_1) - (A_4)$ are satisfied and $x^* \in C[0, T]$ is the unique solution of the equation (3.1);

(ii) $u_i, v_i \in \mathbb{R}$, $i = 1, 2$ and $u_1 \leq u_2$, $v_1 \leq v_2$ implies $f(t, u_1, v_1) \leq f(t, u_2, v_2)$, for all $t \in [0, T]$.

Then

$$x \leq A(x) \quad \textit{implies} \quad x \leq x^*$$

and

$$x \geq A(x) \quad \textit{implies} \quad x \geq x^*.$$

Proof. The operator A is a Picard operator and A is increasing. So, we apply Abstract Gronwall lemma (Lemma 2.8). □

4. Comparison results

Consider the following functional-integral equation:

$$x(t) = x(0) + f(t, \int_0^{g(t)} x(s)ds, x(h(t))), t \in [0,T]. \tag{4.1}$$

The corresponding operator,

$$A_f : (C[0,T], ||\cdot||_\tau) \to (C[0,T], ||\cdot||_\tau),$$

$$A_f\ (x)(t) := x(0) + f(t, \int_0^{g(t)} x(s)\ ds,\ x(h(t))), \quad t \in [0,T],$$

is a continuous operator but it isn't a contraction.

We denote

$$S_f = \{\alpha \in \mathbb{R} / f(0,0,\alpha) = 0\} \text{ and } X_\alpha := \{x \in C[0,T] / x(0) = \alpha\}.$$

Then

$$\cup_{\alpha \in S_f} X_\alpha \text{ is a partition of } C[0,T]$$

and X_α is an invariant subset of A_f if and only if $\alpha \in S_f$.

We have

Theorem 4.1. *We suppose that:*

(i) the conditions $(A_1) - (A_4)$ are satisfied for (4.1);

(ii) $S_f \neq \emptyset$.

Then

$$A_f|_{\cup_{\alpha \in S_f} X_\alpha} : \cup_{\alpha \in S_f} X_\alpha \to \cup_{\alpha \in S_f} X_\alpha$$

is a weakly Picard operator and $card\ F_{A_f} = card S_f$.

Proof. By using the result of Theorem 3.1, we have that

$$A_f|_{X_\alpha} : X_\alpha \to X_\alpha \text{ is a Picard operator, for all } \alpha \in S_f.$$

So, we apply Characterization theorem of the weakly Picard operators (Theorem 2.6). □

Remark 4.2. If the conditions $(A_1) - (A_4)$ are satisfied and $S_f = \{\alpha^*\}$, then the equation (4.1) has in $C[0,T]$ a unique solution.

We have

Theorem 4.3. *We suppose that:*

(i) all the conditions of Theorem 4.1 are satisfied;

(ii) $u_i, v_i \in \mathbb{R}$, $i = 1,2$ and $u_1 \leq u_2$, $v_1 \leq v_2$ implies $f(t,u_1,v_1) \leq f(t,u_2,v_2)$, for all $t \in [0,T]$.

Let x^* be a solution of the equation (4.1) and x^{**} a solution of the following inequality:

$$x(t) \leq x(0) + f(t, \int_0^{g(t)} x(s)ds, x(h(t))), t \in [0,T].$$

Then

$$x^{**}(0) \leq x^*(0) \text{ implies } x^{**} \leq x^*.$$

Proof. We remark that

$$x^* = A_f(x^*) \quad \text{and} \quad x^{**} \leq A_f(x^{**}).$$

From Lemma 2.7 and the condition (ii) we have that the operator A_f^∞ is increasing. If $\beta \in \mathbb{R}$ then we consider $\widetilde{\beta} \in C[0,T]$ defined by $\widetilde{\beta}(t) = \beta$, for all $t \in [0,T]$. By using the previous considerations and because the operator A_f^∞ is increasing, we obtain:

$$x^{**} \leq A_f^\infty(x^{**}(0)) = A_f^\infty(\widetilde{x^{**}(0)}) \leq A_f^\infty(\widetilde{x^*(0)}) = x^*.$$

□

Now, we consider the following functional-integral equations:

$$x(t) = x(0) + f_i(t, \int_0^{g(t)} x(s)ds, \ x(h(t))), t \in [0,T], \tag{4.2}$$

$i = \overline{1,3}$, where g, h are the same in all three equations.

We have

Theorem 4.4. *We suppose that:*

(i) *the corresponding conditions of Theorem 4.1 are satisfied for all equations (4.2);*

(ii) $f_2(t, \cdot, \cdot) : \mathbb{R}^2 \to \mathbb{R}$ *is increasing for all* $t \in [0,T]$;

(iii) $f_1 \leq f_2 \leq f_3$.

Let x_i^* *be a solution of the corresponding equation (4.2),* $i = \overline{1,3}$. *Then*

$$x_1^*(0) \leq x_2^*(0) \leq x_3^*(0) \ \ \textit{implies} \ \ x_1^* \leq x_2^* \leq x_3^*.$$

Proof. First we remark that the operators A_{f_i}, $i = \overline{1,3}$ are weakly Picard operators (Theorem 4.1). From (ii) we have that the operator A_{f_2} is increasing. From the condition (iii) we have that $A_{f_1} \leq A_{f_2} \leq A_{f_3}$. On the other hand, $x_i^* = A_{f_i}^\infty(\widetilde{x_i^*(0)})$, $i = \overline{1,3}$. Now, the proof follows from Abstract comparison lemma (Lemma 2.9). □

References

[1] Azbelev, N.V., Maksimov, V.P., Rahmatulina, L.F. , *Introduction to the theory of functional-differential equations*, MIR, Moscow, 1991 (in Russian).

[2] Hale, J.K., *Theory of functional differential equations,* Springer Verlag, 1977.

[3] Hale, J.K., Verduyn Lunel, S.M., *Introduction to functional-differential equations,* Springer Verlag, New York, 1993.

[4] Mureşan, V., *Differential equations with affine modification of the argument,* Transilvania Press, Cluj-Napoca, 1997 (in Romanian).

[5] Mureşan, V., *Functional-integral equations*, Ed. Mediamira, Cluj-Napoca, 2003.

[6] Olszowy, L., *On existence of solutions of a neutral differential equation with deviating argument*, Collect. Math., **61**(2010), no. 1, 37-47.

[7] Rus, I.A., *Picard operators and applications*, Sc. Math. Japonicae, **58**(2003), no. 1, 191-229.

[8] Rus, I.A., *Some nonlinear functional differential and integral equations, via weakly Picard operator theory: a survey*, Carpathian J. Math., **26**(2010), no. 2, 230-258.

Viorica Mureşan
Faculty of Computer Sciences and Automation
Technical University of Cluj-Napoca
28 Memorandumului Street
400114 Cluj-Napoca, Romania
e-mail: vmuresan@math.utcluj.ro