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On (h,k)–trichotomy for skew-evolution semiflows in Banach spaces

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Abstract. In this paper we define the notion of (h,k)–trichotomy for skew-evolution semiflows and we emphasize connections between various other concepts of trichotomy on infinite dimensional spaces, as uniform exponential trichotomy, exponential trichotomy and Barreira-Valls exponential trichotomy. The approach is motivated by various examples. Some characterizations for the newly introduced concept are also provided.

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1. Preliminaries

As the dynamical systems that are modelling processes issued from engineering, economics or physics are extremely complex, of great interest is to study the solutions of differential equations by means of associated skew-evolution semiflows, introduced in [10]. They are appropriate to study the asymptotic properties of the solutions for evolution equations of the form

$$\begin{cases} \dot{u}(t) = A(t)u(t), \ t > t_0 \geq 0 \\ u(t_0) = u_0, \end{cases}$$

where $A : \mathbb{R} \to \mathcal{B}(V)$ is an operator, $\mathrm{Dom}A(t) \subset V$, $u_0 \in \mathrm{Dom}A(t_0)$. The case of stability for skew-evolution semiflows is emphasized in [16] and the study of dichotomy for evolution equations is given in [9], where we generalize some concepts given in [1], as well as in [15].

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The exponential dichotomy for evolution equations is one of the domains of the stability theory with an impressive development due to its role in approaching several types of differential equations (see [2], [3], [4], [5], [7] and [8]). Hence, the techniques that describe the stability and instability in Banach spaces have been improved to characterize the dichotomy and its natural generalization, the trichotomy, studied for the case of linear differential equations in the finite dimensional setting in [12]. In fact, the trichotomy supposes the continuous splitting of the state space, at any moment, into three subspaces: the stable one, the instable one and the central manifold. The study of the trichotomy for evolution operators is given in [11]. Some concepts for the stability, instability, dichotomy and trichotomy of skew-evolution semiflows are studied in [14].

In this paper, beside other types of trichotomy, as uniform exponential trichotomy, Barreira-Valls exponential trichotomy, exponential trichotomy, we define, exemplify and characterize the concept of (h,k)–trichotomy for skew-evolution semiflows, as a generalization of the (h,k)–dichotomy given in [6] for evolution operators and in [13] for skew-evolution semiflows. Connections between the trichotomy classes are also emphasized.

2. Notations. Definitions. Examples

Let us denote by X a metric space, by V a Banach space and by $\mathcal{B}(V)$ the space of all bounded linear operators from V into itself. We consider the sets $\Delta = \left\{(t,t_0) \in \mathbb{R}_+^2,\ t \geq t_0\right\}$ and $T = \left\{(t,s,t_0) \in \mathbb{R}_+^3,\ (t,s),(s,t_0) \in \Delta\right\}$. Let I be the identity operator on V. We denote $Y = X \times V$ and $Y_x = \{x\} \times V$, where $x \in X$. Let us define the set $\mathcal{E}$ of all mappings $f : \mathbb{R}_+ \to [1,\infty)$ for which there exists a constant $\alpha \in \mathbb{R}_+$ such that $f(t) = e^{\alpha t}$, $\forall t \geq 0$.

Definition 2.1. A mapping $\varphi : \Delta \times X \to X$ is called *evolution semiflow* on X if following relations hold:

(s_1) $\varphi(t,t,x) = x,\ \forall (t,x) \in \mathbb{R}_+ \times X$;

(s_2) $\varphi(t,s,\varphi(s,t_0,x)) = \varphi(t,t_0,x), \forall (t,s,t_0) \in T, x \in X$.

Definition 2.2. A mapping $\Phi : \Delta \times X \to \mathcal{B}(V)$ is called *evolution cocycle* over an evolution semiflow φ if:

(c_1) $\Phi(t,t,x) = I, \forall (t,x) \in \mathbb{R}_+ \times X$;

(c_2) $\Phi(t,s,\varphi(s,t_0,x))\Phi(s,t_0,x) = \Phi(t,t_0,x), \forall (t,s,t_0) \in T, x \in X$.

Definition 2.3. The mapping $C : \Delta \times Y \to Y$ defined by the relation

$$C(t,s,x,v) = (\varphi(t,s,x), \Phi(t,s,x)v),$$

where Φ is an evolution cocycle over an evolution semiflow φ, is called *skew-evolution semiflow* on Y.

Example 2.4. Let $\mathcal{C} = \mathcal{C}(\mathbb{R},\mathbb{R})$ be the metric space of all continuous functions $x : \mathbb{R} \to \mathbb{R}$, with the topology of uniform convergence on compact subsets of

$\mathbb{R}$. $\mathcal{C}$ is metrizable relative to the metric

$$d(x,y)=\sum_{n=1}^{\infty}\frac{1}{2^n}\frac{d_n(x,y)}{1+d_n(x,y)},\ \text{where } d_n(x,y)=\sup_{t\in[-n,n]}|x(t)-y(t)|.$$

Let $f:\mathbb{R}_+\to(0,\infty)$ be a decreasing function. We denote by X the closure in $\mathcal{C}$ of the set $\{f_t,\ t\in\mathbb{R}_+\}$, where $f_t(\tau)=f(t+\tau)$, $\forall\tau\in\mathbb{R}_+$. We obtain that (X,d) is a metric space and that the mapping

$$\varphi:\Delta\times X\to X,\ \varphi(t,s,x)(\tau)=x_{t-s}(\tau)=x(t-s+\tau)$$

is an evolution semiflow on X. Let $V=\mathbb{R}$. The mapping $\Phi:\Delta\times X\to\mathcal{B}(\mathbb{R})$ given by

$$\Phi(t,s,x)v=e^{\int_s^t x(\tau-s)d\tau}v$$

is an evolution cocycle. Hence, $C=(\varphi,\Phi)$ is a skew-evolution semiflow on Y.

Two classic asymptotic properties for evolution cocycles are given, as in [14], by the next

Definition 2.5. A evolution cocycle Φ is said to have:

(i) *uniform exponential growth* if there exist some constants $M\geq 1$ and $\omega>0$ such that:

$$\|\Phi(t,t_0,x)v\|\leq Me^{\omega(t-s)}\|\Phi(s,t_0,x)v\|,\tag{2.1}$$

for all $(t,s,t_0)\in T$ and all $(x,v)\in Y$.

(ii) *uniform exponential decay* if there exist some constants $M\geq 1$ and $\omega>0$ such that:

$$\|\Phi(s,t_0,x)v\|\leq Me^{\omega(t-s)}\|\Phi(t,t_0,x)v\|,\tag{2.2}$$

for all $(t,s,t_0)\in T$ and all $(x,v)\in Y$.

3. Concepts of trichotomy

Definition 3.1. A continuous mapping $P:Y\to Y$ defined by

$$P(x,v)=(x,P(x)v),\ \forall(x,v)\in Y,\tag{3.1}$$

where $P(x)$ is a linear projection on Y_x, is called *projector* on Y.

Remark 3.2. The mapping $P(x):Y_x\to Y_x$ is linear and bounded and satisfies the relation $P(x)P(x)=P^2(x)=P(x)$ for all $x\in X$.

Definition 3.3. A projector P on Y is called *invariant* relative to a skew-evolution semiflow $C=(\varphi,\Phi)$ if following relation holds:

$$P(\varphi(t,s,x))\Phi(t,s,x)=\Phi(t,s,x)P(x),\tag{3.2}$$

for all $(t,s)\in\Delta$ and all $x\in X$.

Definition 3.4. Three projectors $\{P_k\}_{k\in\{1,2,3\}}$ are said to be *compatible* with a skew-evolution semiflow $C = (\varphi, \Phi)$ if:

(t_1) each of the projectors P_k, $k \in \{1, 2, 3\}$ is invariant on Y;

(t_2) $\forall x \in X$, the projections $P_1(x)$, $P_2(x)$ and $P_3(x)$ verify the relations

$$P_1(x) + P_2(x) + P_3(x) = I \text{ and } P_i(x)P_j(x) = 0, \ \forall i, j \in \{1, 2, 3\}, \ i \neq j.$$

In what follows we will denote $C_k(t, s, x, v) = (\varphi(t, s, x), \Phi_k(t, s, x)v)$, $(t, t_0, x, v) \in \Delta \times Y$, $\forall k \in \{1, 2, 3\}$, where $\Phi_k(t, t_0, x) = \Phi(t, t_0, x)P_k(x)$. Let us remind the definitions for various classes of trichotomy, as in [14] and [17].

Definition 3.5. A skew-evolution semiflow $C = (\varphi, \Phi)$ is called *uniformly exponentially trichotomic* if there exist some constants $N \geq 1$, $\nu > 0$ and three projectors $\{P_k\}_{k\in\{1,2,3\}}$ compatible with C such that:

(uet_1)
$$e^{\nu(t-s)} \|\Phi_1(t, t_0, x)v\| \leq N \|\Phi_1(s, t_0, x)v\|; \tag{3.3}$$

(uet_2)
$$e^{\nu(t-s)} \|\Phi_2(s, t_0, x)v\| \leq N \|\Phi_2(t, t_0, x)v\|; \tag{3.4}$$

(uet_3)
$$\begin{aligned} \|\Phi_3(s, t_0, x)v\| &\leq Ne^{\nu(t-s)} \|\Phi_3(t, t_0, x)v\| \leq \\ &\leq N^2 e^{2\nu(t-s)} \|\Phi_3(s, t_0, x)v\|, \end{aligned} \tag{3.5}$$

for all $(t, s, t_0) \in T$ and all $(x, v) \in Y$.

Remark 3.6. The constants N and ν are called *trichotomic characteristics* and P_1, P_2, P_3 *associated trichotomic projectors.*

Definition 3.7. A skew-evolution semiflow $C = (\varphi, \Phi)$ is called *exponentially trichotomic* if there exist a mapping $N : \mathbb{R}_+ \to [1, \infty)$, a constant $\nu > 0$ and three projectors $\{P_k\}_{k\in\{1,2,3\}}$ compatible with C such that:

(et_1)
$$e^{\nu(t-s)} \|\Phi_1(t, t_0, x)v\| \leq N(s) \|\Phi_1(s, t_0, x)v\|; \tag{3.6}$$

(et_2)
$$e^{\nu(t-s)} \|\Phi_2(s, t_0, x)v\| \leq N(t) \|\Phi_2(t, t_0, x)v\|; \tag{3.7}$$

(et_3)
$$\begin{aligned} \|\Phi_3(s, t_0, x)v\| &\leq N(t)e^{\nu(t-s)} \|\Phi_3(t, t_0, x)v\| \leq \\ &\leq N(t)N(s)e^{2\nu(t-s)} \|\Phi_3(s, t_0, x)v\|, \end{aligned} \tag{3.8}$$

for all $(t, s, t_0) \in T$ and all $(x, v) \in Y$.

Definition 3.8. A skew-evolution semiflow $C = (\varphi, \Phi)$ is called *Barreira-Valls exponentially trichotomic* if there exist some constants $N \geq 1$, $\alpha, \beta, \mu, \rho > 0$ and three projectors $\{P_k\}_{k\in\{1,2,3\}}$ compatible with C such that:

$(BVet_1)$
$$e^{\alpha(t-s)} \|\Phi_1(t, t_0, x)v\| \leq Ne^{\beta s} \|\Phi_1(s, t_0, x)v\|; \tag{3.9}$$

$(BVet_2)$
$$\|\Phi_2(s, t_0, x)v\| \leq Ne^{-\alpha t}e^{\beta s} \|\Phi_2(t, t_0, x)v\|; \tag{3.10}$$

$(BVet_3)$

$$\|\Phi_3(t, t_0, x)v\| \leq Ne^{\mu t}e^{-\rho s} \|\Phi_3(s, t_0, x)v\| \leq$$
$$\leq N^2 e^{2\mu t}e^{-2\rho s} \|\Phi_3(t, t_0, x)v\|, \tag{3.11}$$

for all $(t, s, t_0) \in T$ and all $(x, v) \in Y$.

Further, let us introduce a more general concept of trichotomy for skew-evolution semiflows, given by the next

Definition 3.9. A skew-evolution semiflow $C = (\varphi, \Phi)$ is (h, k)*–trichotomic* if there exist a constant $N \geq 1$, two continuous mappings $h, k : \mathbb{R}_+ \to \mathbb{R}_+^*$ and three projectors families $\{P_k\}_{k\in\{1,2,3\}}$ compatible with C such that:

(t_1)

$$h(t - s) \|\Phi_1(t, t_0, x)v\| \leq Nk(s) \|\Phi_1(s, t_0, x)v\|; \tag{3.12}$$

(t_2)

$$h(t - s) \|\Phi_2(s, t_0, x)v\| \leq Nk(t) \|\Phi_2(t, t_0, x)v\|; \tag{3.13}$$

(t_3)

$$\|\Phi_3(t, t_0, x)v\| \leq Nk(s)h(t - s) \|\Phi_3(s, t_0, x)v\|; \tag{3.14}$$

$$\|\Phi_3(s, t_0, x)v\| \leq Nk(t)h(t - s) \|\Phi_3(t, t_0, x)v\|; \tag{3.15}$$

for all $(t, s, t_0) \in T$ and all $(x, v) \in Y$.

The concept of (h, k)–trichotomy generalizes the notions of uniform exponential trichotomy, exponential trichotomy and Barreira-Valls exponential trichotomy, as shown in

Remark 3.10. 1) If $h \in \mathcal{E}$ and k is constant in Definition 3.9, then C is uniformly exponentially trichotomic;

2) If $h \in \mathcal{E}$ then C is exponentially trichotomic;

3) If $h, k \in \mathcal{E}$ then C is Barreira-Valls exponentially trichotomic.

In the next particular cases, other (h, k)–asymptotic properties for skew-evolution semiflows are emphasized.

Remark 3.11. (i) For $P_2 = P_3 = 0$ we obtain in Definition 3.9 the property of (h, k)–exponential stability;

(ii) For $P_1 = P_3 = 0$ in Definition 3.9 the property of (h, k)–exponential instability is obtained;

(iii) For $P_3 = 0$ we obtain in Definition 3.9 the property of (h, k)–exponential dichotomy. On the other hand, for $P_3 = 0$, in Definition 3.5, Definition 3.7 and Definition 3.8 the properties of uniform exponential dichotomy, exponential dichotomy, respectively Barreira-Valls exponential dichotomy are obtained (see [13]).

We have following connections between the previously defined classes of trichotomy, given by

Remark 3.12. A uniformly exponentially trichotomic skew-evolution semiflow is Barreira-Valls exponentially trichotomic, which also implies that it is exponentially trichotomic.

The converse statements are not always true, as shown in the next examples.

Example 3.13. Let $f : \mathbb{R}_+ \to (0,\infty)$ be a decreasing function with the property that there exists $\lim\limits_{t\to\infty} f(t) = l > 0$. We will consider $\lambda > f(0)$. We define the metric space (X, d) and the evolution semiflow as in Example 2.4.

Let us consider $V = \mathbb{R}^3$ with the norm $\|v\| = |v_1| + |v_2| + |v_3|$, where $v = (v_1, v_2, v_3) \in V$. The mapping $\Phi : \Delta \times X \to \mathcal{B}(V)$, defined by

$$\Phi(t,s,x)v =$$
$$=\left(\frac{e^{t\sin t-2t}}{e^{s\sin s-2s}}e^{-\int_s^t x(\tau-s)d\tau}v_1, \frac{e^{3t-2t\cos t}}{e^{3s-2s\cos s}}e^{\int_s^t x(\tau-s)d\tau}v_2, e^{(t-s)x(0)-\int_s^t x(\tau-s)d\tau}v_3\right)$$

is an evolution cocycle over the evolution semiflow φ. We consider the projectors $P_1,\ P_2,\ P_3 : Y \to Y$, $P_1(x,v) = (v_1,0,0)$, $P_2(x,v) = (0,v_2,0)$ and $P_3(x,v) = (0,0,v_3)$, where $x \in X$ and $v = (v_1,v_2,v_3) \in V$, compatible with the skew-evolution semiflow $C = (\varphi,\Phi)$.

We obtain

$$|\Phi_1(t,s,x)v| = e^{t\sin t-s\sin s+2s-2t}e^{-\int_s^t x(\tau-s)d\tau}|v_1| \leq$$
$$\leq e^{-t+3s}e^{-l(t-s)}|v_1| = e^{-(1+l)t}e^{(3+l)s}|v_1|,$$

for all $(t,s,x,v) \in \Delta \times Y$ and

$$|\Phi_2(t,s,x)v| = e^{3t-3s-2t\cos t+2s\cos s+\int_s^t x(\tau-s)d\tau}|v_2| \geq$$
$$\geq e^{t-s}e^{l(t-s)}|v_2| = e^{(1+l)t}e^{-(1+l)s}|v_2|,$$

for all $(t,s,x,v) \in \Delta \times Y$.

We also have, for all $(t,s,x,v) \in \Delta \times Y$,

$$|\Phi_3(t,s,x)v| \leq e^{[\lambda-x(0)]t}e^{-[\lambda-x(0)]s}|v_3|$$

and

$$|\Phi_3(t,s,x)v| \geq e^{[l-x(0)]t}e^{-[l-x(0)]s}|v_3|.$$

Hence, the skew-evolution semiflow $C = (\varphi,\Phi)$ is Barreira-Valls exponentially trichotomic with the characteristics

$$N = 1,\ \alpha = \beta = 3 + l,\ \mu = \rho = \min\{\lambda - x(0), x(0) - l\}.$$

Let us suppose now that $C = (\varphi,\Phi)$ is uniformly exponentially trichotomic. According to Definition 3.5, there exist $N \geq 1$ and $\nu > 0$ such that

$$e^{t\sin t-s\sin s+2s-2t}e^{-\int_s^t x(\tau-s)d\tau}|v_1| \leq Ne^{-\nu(t-s)}|v_1|,\ \forall t \geq s \geq 0$$

and If we consider $t = 2n\pi + \frac{\pi}{2}$ and $s = 2n\pi$, $n \in \mathbb{N}$, we have

$$e^{2n\pi-\frac{\pi}{2}} \leq Ne^{-\nu\frac{\pi}{2}}e^{\int_{2n\pi}^{2n\pi+\frac{\pi}{2}} x(\tau-2n\pi)d\tau} \leq Ne^{(-\nu+\lambda)\frac{\pi}{2}},$$

which, for $n \to \infty$, leads to a contradiction.

Hence, we obtain that $C = (\varphi,\Phi)$ is not uniformly exponentially trichotomic.

Example 3.14. We consider the metric space (X,d), the Banach space V, the projectors P_1, P_2, P_3 and the evolution semiflow φ defined as in Example 2.4. Let $g:\mathbb{R}_+\to[1,\infty)$ be a continuous function with

$$g(n)=e^{n\cdot 2^{2n}} \text{ and } g\left(n+\frac{1}{2^{2n}}\right)=e^4, \text{ for all } n\in\mathbb{N}.$$

The mapping $\Phi:\Delta\times X\to\mathcal{B}(V)$, defined by

$$\Phi(t,s,x)v=$$

$$=\left(\frac{g(s)}{g(t)}e^{-(t-s)-\int_s^t x(\tau-s)d\tau}v_1,\frac{g(s)}{g(t)}e^{t-s+\int_s^t x(\tau-s)d\tau}v_2,e^{-(t-s)x(0)+\int_s^t x(\tau)d\tau}v_3\right)$$

is an evolution cocycle over the evolution semiflow φ.

We have that,

$$e^{(1+l)(t-s)}\left\|\Phi_1(t,s,x)v\right\|\leq g(s)\left\|v_1\right\|$$

and

$$e^{(1+l)(t-s)}\left\|v_2\right\|\leq g(s)e^{(1+l)(t-s)}\left\|v_2\right\|\leq g(t)\left\|\Phi_2(t,s,x)v\right\|,$$

for all $(t,s,x,v)\in\Delta\times Y$. We also have

$$|\Phi_3(t,s,x)v|\leq e^{x(0)(t-s)}|v_3|$$

and

$$|\Phi_3(t,s,x)v|\geq e^{-x(0)(t-s)}|v_3|,$$

for all $(t,s,x,v)\in\Delta\times Y$. Thus, $C=(\varphi,\Phi)$ is exponentially trichotomic with

$$\nu=\max\{1+l,\lambda\} \text{ and } N(t)=\sup_{s\in[0,t]}g(s).$$

If we suppose that C is Barreira-Valls exponentially trichotomic, then there exist $N\geq 1$, $\alpha>0$ and $\beta\geq 0$ such that

$$g(s)e^{\alpha t}\leq Ng(t)e^{\beta s+t-s+\int_s^t x(\tau-s)d\tau},$$

for all $(t,s,x)\in\Delta\times X$.

From here, for $t=n+\frac{1}{2^{2n}}$ and $s=n$, it follows that

$$e^{n(2^{2n}+\alpha-\beta)}\leq 81Ne^{\frac{1-\alpha+x(0)}{2^{2n}}},$$

which, for $n\to\infty$, leads to a contradiction.

4. Main results

Let $C:\Delta\times Y\to Y$, $C(t,s,x,v)=(\Phi(t,s,x)v,\varphi(t,s,x))$ be a skew-evolution semiflow on Y. Some characterizations for the concept of (h,k)–trichotomy are obtained. Therefore, let us suppose that $h:\mathbb{R}_+\to\mathbb{R}_+^*$ is a nondecreasing function such that

$$h(u+v)\leq h(u)h(v),\ u,v\in\mathbb{R}_+. \qquad (\chi)$$

Theorem 4.1. *Let $C = (\varphi, \Phi)$ be skew-evolution semiflow such that there exist three projectors $\{P_k\}_{k\in\{1,2,3\}}$ compatible with C such that Φ_1 has uniform exponential growth and Φ_2 has uniform exponential decay. If there exist a constant $K \geq 1$ and two mappings $h, k : \mathbb{R}_+ \to \mathbb{R}_+^*$, where h satisfies condition (χ), such that:*

(*i*)

$$\int_s^t h(\tau - s) \|\Phi_1(\tau, t_0, x)v\| \, d\tau \leq Kk(s) \|\Phi_1(s, t_0, x)v\| ; \tag{4.1}$$

(*ii*)

$$\int_s^t h(t - \tau) \|\Phi_2(\tau, t_0, x)v\| \, d\tau \leq Kk(t) \|\Phi_2(t, t_0, x)v\| ; \tag{4.2}$$

(*iii*)

$$\int_s^t \frac{1}{h(\tau - s)} \|\Phi_3(\tau, t_0, x)v\| \, d\tau \leq Kk(s) \|\Phi_3(s, t_0, x)v\| ; \tag{4.3}$$

$$\int_s^t \frac{1}{h(t - \tau)} \|\Phi_3(\tau, t_0, x)v\| \, d\tau \leq Kk(t) \|\Phi_3(s, t_0, x)v\| , \tag{4.4}$$

for all $(t, s, t_0) \in T$ and all $(x, v) \in Y$, then C is (h, k)–trichotomic.

Proof. Let us suppose that (*i*) holds. As a first step, we consider $s \in [t-1, t]$. We obtain

$$h(t-s) \|\Phi_1(t, t_0, x)v\| = \int_{t-1}^t h(t-s) \|\Phi_1(t, t_0, x)v\| \, d\tau \leq$$

$$\leq \int_{t-1}^t h(t-\tau)h(\tau - s) \|\Phi_1(t, \tau, \varphi(\tau, t_0, x))\Phi_1(s, t_0, x)v\| \, d\tau \leq$$

$$\leq Me^{\omega} h(1) \int_s^t h(\tau - s) \|\Phi_1(\tau, t_0, x)v\| \, d\tau \leq KMe^{\omega} h(1)k(s) \|\Phi_1(s, t_0, x)v\| ,$$

for all $(x, v) \in Y$, where M and ω are given by Definition 2.5, as Φ_1 has uniform exponential growth.

As a second step, if $t \in [s, s+1)$, we have

$$h(t-s) \|\Phi_1(t, t_0, x)v\| \leq Me^{\omega} h(1) \|\Phi_1(s, t_0, x)v\| ,$$

for all $(x, v) \in Y$. Hence, relation (3.12) is obtained, for $N = Me^{\omega}h(1)(K+1)$.

Now, as Φ_2 has uniform exponential decay, an equivalent definition (see [14]) assures the existence of a nondecreasing function $g : [0, \infty) \to [1, \infty)$ with the property $\lim_{t\to\infty} g(t) = \infty$ such that

$$\|\Phi(s, t_0, x)v\| \leq g(t-s) \|\Phi(t, t_0, x)v\| ,$$

for all $(t, s, t_0) \in T$ and all $(x, v) \in Y$. Let us denote $D = \int_0^1 g(\tau)d\tau$.

If (*ii*) holds, we obtain

$$Dh(t-s) \|\Phi(s, t_0, x)v\| = \int_0^1 h(t-s)g(\tau) \|\Phi(s, t_0, x)v\| \, d\tau \leq$$

$$\leq \int_0^1 h(t-\tau)h(\tau-s)g(\tau)\,\|\Phi(s,t_0,x)v\|\,d\tau \leq$$
$$\leq h(t)\int_0^1 h(\tau)g(\tau)\,\|\Phi(s,t_0,x)v\|\,d\tau =$$
$$= \int_s^{s+1} h(u-t_0)g(u-s)\,\|\Phi(s,t_0,x)v\|\,du \leq$$
$$\leq \int_0^t h(u-s)\,\|\Phi_2(u,t_0,x)v\|\,du \leq Kk(t)\,\|\Phi_2(t,t_0,x)v\|\,,$$

for all $t \geq s+1 > s \geq 0$ and all $(x,v) \in Y$.

On the other hand, for $t \in [s,s+1)$ we obtain for all $(x,v) \in Y$

$$\|\Phi_2(t,t_0,x)v\| \geq g(t-s)\,\|\Phi(s,t_0,x)v\| \geq g(1)\,\|\Phi(s,t_0,x)(x)v\|\,.$$

We obtain thus relation (3.13).

A similar proof, based on the property (χ) of function h, shows that the inequalities from (iii) imply relations (3.14).

Hence, according to Definition 3.9, C is (h,k)–trichotomic. □

Remark 4.2. Relation (4.1) defines the (h,k)*–integral stability*, while relation (4.1) defines the (h,k)*–integral instability* for skew-evolution semiflow, similar to the notions defined in [17].

In the below mentioned particular cases, we obtain, as in [14], characterizations for other classes of trichotomy.

Corollary 4.3. *In the hypothesis of Theorem* 4.1,

(i) if $h,k \in \mathcal{E}$ and are given by $t \mapsto e^{\alpha t}$ respectively $t \mapsto Me^{\alpha t}$, $M \geq 1$, the skew-evolution semiflow C is uniformly exponentially trichotomic;

(ii) if $h \in \mathcal{E}$, the skew-evolution semiflow C is exponentially trichotomic;

(iii) if $h,k \in \mathcal{E}$ and are given by $t \mapsto e^{\alpha t}$ respectively $t \mapsto Me^{\beta t}$, $M \geq 1$ and $\beta > \alpha$, the skew-evolution semiflow C is Barreira-Valls exponentially trichotomic.

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