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On best simultaneous approximation in operator and function spaces

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Abstract. Let X be a Banach space, $(I, \sum, \mu)$ a finite measure space and $L^1(\mu, X)$ the Banach space of all X-valued μ-integrable functions on the unit interval I equipped with the usual 1-norm. In this paper we prove that for a closed subspace G of X, $L^1(\mu, G)$ is simultaneously Chebyshev in $L^1(\mu, X)$ if and only if G is simultaneously Chebyshev in X. Further results are obtained in the space of bounded linear operators $L(l^1, X)$ and in the space of continuous functions $C^1(I, l^p)$ with respect to the L^1 norm.

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1. Introduction

Let X be a Banach space and $(I, \sum, \mu)$ be a finite measure space. Let us denote by $L^1(\mu, X)$, the Banach space of all X-valued μ-integrable functions on the unit interval I equipped with the usual 1-norm.

For a closed subspace G of X, let us recall that G is simultaneously proximinal in X if for all m-tuples $(x_1, x_2, ..., x_m) \in X^m$, there exists $g \in G$ such that

$$\sum_{i=1}^{m} \|x_i - g\| = \text{dist}(x_1, x_2, ..., x_m, G) = \inf \left\{ \sum_{i=1}^{m} \|x_i - z\| : z \in G \right\}.$$

In this case, g is called a best simultaneous approximation of $(x_1, x_2, ..., x_m)$ in G. If this best approximation is unique for all $(x_1, x_2, ..., x_m) \in X^m$, then G is called simultaneously Chebyshev.

Of course for $m = 1$ the preceding concepts are just best approximation and proximinality.

The problem of best simultaneous approximation can be viewed as a special case of vector valued approximation. Recent results in this area are due to Pinkus [10], where he considered the problem when a finite dimensional subspace is a uniqueness space. Results on best simultaneous approximation in general Banach spaces may be found in [9] and [11]. Related results on $L^p(\mu, X)$, $1 \leq p < \infty$, are given in [12]. In [12], it is shown that if G is a reflexive subspace of a Banach space X, then $L^p(\mu, G)$ is simultaneously proximinal in $L^p(\mu, X)$. If $p = 1$, Abu Sarhan and Khalil [1], proved that if G is a reflexive subspace of the Banach space X or G is a 1-summand subspace of X, then $L^1(\mu, G)$ is simultaneously proximinal in $L^1(\mu, X)$.

It is the aim of this paper to give some sufficient conditions for $L^1(\mu, G)$ to be a Chebyshev subspace of $L^1(\mu, X)$. Further results are obtained in the space of bounded linear operators $L(l^1, X)$ and in the space of continuous functions $C^1(I, l^p)$ with respect to the L^1 norm.

Throughout this paper, X is a Banach space and G is a closed subspace of X.

2. Main results

In [1] it is shown that if $m = 1$ and G is a finite dimensional subspace of a Banach space X, then G is Chebyshev in X if and only if $L^1(\mu, G)$ is Chebyshev in $L^1(\mu, X)$. The main result in this section is: If G is a reflexive subspace of X, then G is simultaneously Chebyshev in X if and only if $L^1(\mu, G)$ is simultaneously Chebyshev in $L^1(\mu, X)$.

Theorem 2.1. *Let G be a reflexive subspace of X. Then G is simultaneously Chebyshev in X if and only if $L^1(\mu, G)$ is simultaneously Chebyshev in $L^1(\mu, X)$.*

Proof. Let $f_1, f_2, ..., f_m \in L^1(\mu, X)$. Since G is reflexive, it follows that [Th.4, 12], there exists $g \in L^1(\mu, G)$ such that

$$\sum_{i=1}^{m} \|f_i - g\|_1 = \text{dist}(f_1, f_2, ..., f_m, L^1(\mu, G)).$$

Thus by [Th.2.2, 2], we have:

$$\sum_{i=1}^{m} \|f_i(t) - g(t)\| = \text{dist}(f_1(t), f_2(t), ..., f_m(t), G),$$

for almost all $t \in I$. But G is simultaneously Chebyshev. So $g(t)$ is unique. Thus g is determined uniquely, and $L^1(\mu, G)$ is simultaneously Chebyshev in $L^1(\mu, X)$.

Conversely. Let $x_1, x_2, ..., x_m \in X$. For $i = 1, 2, ..., m$, consider the functions: $f_i : I \to X$, $f_i(t) = x_i$, for all $t \in I$. Since $L^1(\mu, G)$ is simultaneously Chebyshev in $L^1(\mu, X)$, there exists $g \in L^1(\mu, G)$ such that

$$\text{dist}(f_1, f_2, ..., f_m, L^1(\mu, G)) = \sum_{i=1}^{m} \|f_i - g\|_1 \leq \sum_{i=1}^{m} \|f_i - h\|_1$$

for all $h \in L^1(\mu, G)$. Thus by [Th.2.2, 2] , we have:

$$\sum_{i=1}^{m} \|f_i(t) - g(t)\| \leq \sum_{i=1}^{m} \|f_i(t) - h(t)\| \tag{2.1}$$

for almost all $t \in I$. But since G is reflexive, there exists $w \in G$ such that

$$\sum_{i=1}^{m} \|x_i - w\| \leq \sum_{i=1}^{m} \|x_i - z\|$$

for all $z \in G$, [Lemma 1.12] . Hence the function $b(t) = w$ for all $t \in I$ is a best simultaneous approximation of $f_1, f_2, ..., f_m$ in $L^1(\mu, G)$. Equation (2.1) and since $L^1(\mu, G)$ is simultaneously Chebyshev in $L^1(\mu, X)$ it follows that $g(t) = b(t) = w$ and w is unique. Hence G is simultaneously Chebyshev in X. □

For $0 < p < \infty$, let us denote by $l^p(X)$, the space of all sequences (x_n) in X such that $\sum\limits_{n=1}^{\infty} \|x_n\|^p < \infty$. For $x = (x_n) \in l^p(X)$, let

$$\|x\|_p = \begin{cases} \left(\sum\limits_{k=1}^{\infty} \|x_n\|^p \right)^{\frac{1}{p}} & 1 \leq p < \infty \\ \sum\limits_{k=1}^{\infty} \|x_n\|^p & 0 < p < 1 \end{cases}$$

In the space $l^1(X)$, we have the following result:

Theorem 2.2. *G is simultaneously Chebyshev in X if and only if $l^1(G)$ is simultaneously Chebyshev in $l^1(X)$.*

Proof. For $1 \leq i \leq m$, let $x_i = (x_{in}) \in l^1(X)$. If $g_n \in G$ is such that

$$\sum_{i=1}^{m} \|x_{in} - g_n\| \leq \sum_{i=1}^{m} \|x_{in} - z\| \tag{2.2}$$

for all $z \in G$. Using triangle inequality and taking $z = 0$ in (2.2) we get

$$\sum_{i=1}^{m} \|g_n\| - \|x_{in}\| \leq \sum_{i=1}^{m} \|x_{in} - g_n\| \leq \sum_{i=1}^{m} \|x_{in}\|$$

and this implies

$$m \|g_n\| = \sum_{i=1}^{m} \|g_n\| \leq 2 \sum_{i=1}^{m} \|x_{in}\| . \tag{2.3}$$

Thus

$$\sum_{n=1}^{\infty} \|g_n\| \leq \frac{2}{m} \sum_{i=1}^{m} \sum_{n=1}^{\infty} \|x_{in}\| < \infty.$$

Hence the element $g = (g_n) \in l^1(G)$ and g is a best simultaneous approximation of the m-tuple $((x_{in}))_{i=1}^{m}$ in $l^1(G)$. The uniqueness of g_n implies that $g = (g_n)$ is unique and $l^1(G)$ is simultaneously Chebyshev in $l^1(X)$.

Conversely. Let $x_1, x_2, ..., x_m \in X$. For each $i = 1, 2, ..., m$, consider the sequence $(x_i, 0, ...) \in l^1(X)$. Since $l^1(G)$ is simultaneously Chebyshev in $l^1(X)$, it follows that there exists a sequence of the form $(g, 0, ...)$ in $l^1(G)$ such that

$$\sum_{i=1}^{m} \|(x_i, 0, ...) - (g, 0, ...)\| < \sum_{i=1}^{m} \|(x_i, 0, ...) - (z_1, z_2, ...)\|$$

for all $(z_n) \in l^1(G) \backslash \{(g, 0, ...)\}$. This implies that

$$\sum_{i=1}^{m} \|x_i - g\| < \sum_{i=1}^{m} \|x_i - z\|$$

for all $z \in G \backslash \{g\}$. □

For the space of bounded linear operators, $L(l^1, X)$, from l^1 into X, where l^1 is the space of all summable real sequences it has been proved in [1] that G is proximinal in X if and only if $L(l^1, G)$ is proximinal in $L(l^1, X)$. For the case of simultaneous approximation we have the following result:

Theorem 2.3. *G is simultaneously proximinal in X if and only if $L(l^1, G)$ is simultaneously proximinal in $L(l^1, X)$.*

Proof. Let $T_1, T_2, ..., T_m \in L(l^1, X)$. If (δ_n) is the natural basis of l^1, then $T_i \delta_n \in X$, $i = 1, 2, ..., m$.

Since G is simultaneously proximinal, so for each n there exists $x_n \in G$ such that

$$\sum_{i=1}^{m} \|T_i(\delta_n) - x_n\| = \text{dist } (T_1(\delta_n), T_2(\delta_n), ..., T_m(\delta_n), G).$$

Define $S : l^1 \to G$, $S(\delta_n) = x_n$. Then S is a bounded linear operator from l^1 into G. It is clear that S is linear. To prove that S is bounded, let $y = (\alpha_n) \in l^1$, $\|y\|_1 = \sum_{n=1}^{\infty} |\alpha_n| \leq 1$. Then

$$\|S(y)\| = \left\| \sum_{n=1}^{\infty} \alpha_n S(\delta_n) \right\| \leq \sum_{n=1}^{\infty} |\alpha_n| \, \|S(\delta_n)\| = \sum_{n=1}^{\infty} |\alpha_n| \, \|x_n\| .$$

Using (2.3) in Theorem 2.2 we get

$$\|S(y)\| \leq \sum_{n=1}^{\infty} |\alpha_n| \frac{2}{m} \sum_{i=1}^{m} \|T_i(\delta_n)\| \leq \sum_{n=1}^{\infty} |\alpha_n| \frac{2}{m} \sum_{i=1}^{m} \|T_i\| = \frac{2}{m} \sum_{i=1}^{m} \|T_i\| \sum_{n=1}^{\infty} |\alpha_n|$$

Hence S is a bounded linear operator with $\|S\| \leq \frac{2}{m}\sum_{i=1}^{m}\|T_i\|$. Now for any $x=(\beta_n)\in l^1$ we have

$$\begin{aligned}
\sum_{i=1}^{m}\|T_i(x)-S(x)\| &= \sum_{i=1}^{m}\left\|T_i\left(\sum_{n=1}^{\infty}\beta_n\delta_n\right)-S\left(\sum_{n=1}^{\infty}\beta_n\delta_n\right)\right\| \\
&= \sum_{i=1}^{m}\left\|\sum_{n=1}^{\infty}\beta_n T_i(\delta_n)-\sum_{n=1}^{\infty}\beta_n S(\delta_n)\right\| \\
&= \sum_{i=1}^{m}\left\|\sum_{n=1}^{\infty}\beta_n\left(T_i(\delta_n)-S(\delta_n)\right)\right\| \\
&\leq \sum_{i=1}^{m}\sum_{n=1}^{\infty}|\beta_n|\,\|T_i(\delta_n)-S(\delta_n)\| \\
&= \sum_{n=1}^{\infty}|\beta_n|\operatorname{dist}\left(T_1(\delta_n),T_2(\delta_n),...,T_m(\delta_n),G\right). \\
&\leq \sum_{n=1}^{\infty}|\beta_n|\sum_{i=1}^{m}\|T_i(\delta_n)-g\|
\end{aligned}$$

for every $g\in G$. In particular for every $A\in L(l^1,G)$

$$\begin{aligned}
\sum_{i=1}^{m}\|T_i(x)-S(x)\| &\leq \sum_{n=1}^{\infty}|\beta_n|\sum_{i=1}^{m}\|T_i(\delta_n)-A(\delta_n)\| \\
&\leq \sum_{n=1}^{\infty}|\beta_n|\sum_{i=1}^{m}\|T_i-A\| \\
&= \sum_{i=1}^{m}\|T_i-A\|\sum_{n=1}^{\infty}|\beta_n| = \sum_{i=1}^{m}\|T_i-A\|\,\|x\|.
\end{aligned}$$

Taking supremum over all $x\in l^1$, $\|x\|=1$ we get

$$\sum_{i=1}^{m}\|T_i-S\| \leq \sum_{i=1}^{m}\|T_i-A\|.$$

Hence $L(l^1,G)$ is simultaneously proximinal in $L(l^1,X)$.

Conversely. Let $x_1,x_2,...,x_m\in X$. For each $i=1,2,...,m$, define $T_i: l^1\to X$,

$$T_i\delta_n = \begin{cases} x_i & n=1 \\ 0 & n\neq 1 \end{cases}.$$

Then $T_i\in L(l^1,X)$ and $\|T_i\|=\|x_i\|$. By assumption there exists $A\in L(l^1,G)$ such that

$$\sum_{i=1}^{m}\|T_i-A\| \leq \sum_{i=1}^{m}\|T_i-B\|$$

for all $B \in L(l^1, G)$. Hence

$$\sum_{i=1}^{m} \|x_i - A\delta_1\| = \sum_{i=1}^{m} \|(T_i - A)\,\delta_1\| \le \sum_{i=1}^{m} \|T_i - A\| \le \sum_{i=1}^{m} \|T_i - B\| .$$

If B runs over all functions of the form

$$B\delta_n = \begin{cases} w & n = 1 \\ 0 & n \neq 1 \end{cases}$$

for all $w \in G$, we obtain $\sum_{i=1}^{m} \|x_i - A\delta_1\| \le \sum_{i=1}^{m} \|x_i - w\|$ for all $w \in G$. Hence G is simultaneously proximinal in X. □

Theorem 2.4. *If $L(l^1, G)$ is simultaneously Chebyshev in $L(l^1, X)$, then G is simultaneously Chebyshev in X.*

Proof. Suppose G is not Chebyshev in X. Then there exist $g_1, g_2 \in G$ and $x_1, x_2, ..., x_m \in X$ such that

$$\sum_{i=1}^{m} \|x_i - g_1\| = \sum_{i=1}^{m} \|x_i - g_2\| = \mathrm{dist}(x_1, x_2, ..., x_m, G).$$

For $i = 1, 2, ..., m$, let

$$T_i\delta_n = \begin{cases} x_i & n = 1 \\ 0 & n \neq 1 \end{cases} ,$$

and

$$A_1\delta_n = \begin{cases} g_1 & n = 1 \\ 0 & n \neq 1 \end{cases} , \qquad A_2\delta_n = \begin{cases} g_2 & n = 1 \\ 0 & n \neq 1 \end{cases} .$$

Then

$$\sum_{i=1}^{m} \|T_i - A_1\| = \sum_{i=1}^{m} \|T_i - A_2\| = dist(T_1, T_2, ..., T_m, L(l^1, G)).$$

This contradict the fact that $L(l^1, G)$ is simultaneously Chebyshev. □

We remark that the converse of Theorem 2.4 is not true. To see this, let G be a Chebyshev subspace of X and $x_1, x_2, ..., x_m \in X$. For each $i = 1, 2, ..., m$, define $T_i : l^1 \to X$

$$T_i\delta_n = \begin{cases} x_i & n = 1 \\ 0 & n \neq 1 \end{cases} ,$$

then if $z \in G$ is such that $\sum_{i=1}^{m} \|x_i - z\| = \mathrm{dist}(x_1, x_2, ..., x_m, G)$, the operator $A : l^1 \to X$,

$$A\delta_n = \begin{cases} z & n = 1 \\ 0 & n \neq 1 \end{cases} ,$$

is a best simultaneous approximation of $T_1, T_2, ..., T_m$ in $L(l^1, G)$ that is

$$\sum_{i=1}^{m} \|T_i - A\| \le \sum_{i=1}^{m} \|T_i - B\|$$

for all $B \in L(l^1, G)$. Let $r = \min_{1 \le i \le m} \|x_i - z\|$. Consider the map

$$S : l^1 \to G,\ S\delta_n = \begin{cases} z & n = 1 \\ z_n & n \neq 1 \end{cases}$$

where $|z_n| < r$. Then $\sum_{i=1}^m \|T_i - S\| = \sum_{i=1}^m \|T_i - A\|$. Hence $L(l^1, G)$ is not a Chebyshev subspace of $L(l^1, X)$.

As a corollary from Theorem 2.2 for the Banach space c_0 we have:

Corollary 2.5. *G is simultaneously Chebyshev in X if and only if $L(c_0, G)$ is simultaneously Chebyshev in $L(c_0, X)$.*

Proof. By the result of Grothendieck [6] , page 86, we have $L(c_0, G) = l^1(G)$. the result follows from Theorem 2.2. □

3. Further results

An n-dimensional subspace V_n of $C(I)$, the space of continuous functions on a compact set I, is called a Haar subspace if any $f \in V_n \setminus \{0\}$, f has at most $n - 1$ zero's on I. Haar subspaces on intervals of real numbers are called T-Systems. For each natural number n, let M_n be an n-dimensional Haar subspace. Set

$$U = \left\{ g \in L^1(\mu, l^p) : g = (g_i),\ g_i \in M_i \right\}.$$

We remark that U is a closed subspace of $L^1(\mu, l^p)$, [1] .

On the space of continuous functions $C^1(I, l^p)$, we have the following result

Theorem 3.1. *For $1 \le p < \infty$, U is proximinal in $C^1(I, l^p)$ with respect to the L^1 norm.*

Proof. Let $p = 1$ and let $S_1, S_2, ...S_m \in C^1(I, l^1)$. Then for each $i = 1, 2, ..., m$, $S_i = (f_{i,k})_{k=1}^\infty$ and $\|S_i\| = \int_I \sum_{k=1}^\infty |f_{i,k}(t)|\, dt$. Hence $\sum_{i=1}^m \|S_i\| = \sum_{i=1}^m \int_I \sum_{k=1}^\infty |f_{i,k}(t)|\, dt$. Using the Monotone Convergence Theorem, we get:

$$\sum_{i=1}^m \|S_i\| = \sum_{k=1}^\infty \sum_{i=1}^m \int_I |f_{i,k}(t)|\, dt = \sum_{k=1}^\infty \sum_{i=1}^m \|f_{i,k}\|_1 .$$

Since for each k, M_k is finite dimensional, there exists $g_k \in M_k$ such that

$$\sum_{i=1}^m \|f_{i,k} - g_k\|_1 \le \sum_{i=1}^m \|f_{i,k} - h_k\|_1$$

for all $h_k \in M_k$. Note that

$$\sum_{i=1}^m \|f_{i,k} - h_k\|_1 \ge \sum_{i=1}^m \|f_{i,k} - g_k\|_1 \ge \sum_{i=1}^m \left| \|f_{i,k}\|_1 - \|g_k\|_1 \right| .$$

for all $h_k \in M_k$. Since $0 \in M_k$, we get:

$$m \|g_k\|_1 \leq 2 \sum_{i=1}^{m} \|f_{i,k}\|_1$$

and so

$$\sum_{k=1}^{\infty} \|g_k\|_1 \leq \frac{2}{m} \sum_{i=1}^{m} \sum_{k=1}^{\infty} \|f_{i,k}\|_1 = \frac{2}{m} \sum_{i=1}^{m} \|f_i\|$$

Hence $g = (g_k) \in U$ and

$$\sum_{i=1}^{m} \|S_i - g\| = \sum_{k=1}^{\infty} \sum_{i=1}^{m} \int_I |f_{i,k}(t) - g_k(t)|\, dt \leq \sum_{k=1}^{\infty} \sum_{i=1}^{m} \int_I |f_{i,k}(t) - h_k(t)|\, dt$$

for all $h_k \in M_k$. In particular we get $\sum_{i=1}^{m} \|S_i - g\| \leq \sum_{i=1}^{m} \|S_i - h\|$ for all $h \in U$. Hence U is proximinal in $C^1(I, l^1)$ with respect to the L^1 norm.

For $1 < p < \infty$, let $S_1, S_2, ...S_m \in C^1(I, l^p)$. Consider the operator

$$\begin{aligned} P_k &: L^1(\mu, l^p) \to L^1(\mu, l^p_k) \\ P_k f &= (f_1, f_2, ..., f_k) \end{aligned}$$

where $f = (f_i)_{i=1}^{\infty}$. Then P_k is continuous. For $1 \leq k < \infty$, set $U_k = \left\{ g = (g_i) \in \prod_{i=1}^{k} M_i \right\}$. Since U_k is finite dimensional, there exists some $\widehat{g} \in U_k$ such that

$$\sum_{i=1}^{m} \|P_k S_i - \widehat{g}\|_1 \leq \sum_{i=1}^{m} \|P_k S_i - h\|_1 \tag{3.1}$$

for all $h \in U_k$,. Let us write g^k for $\widehat{g}$. We shall prove that the sequence$\left(g^k\right)$ must have a subsequence that converges to some $g \in U$.

Since $P_k S_i \to S_i$, then the sets $E_i = \{P_1 S_i,\ P_2 S_i,\ P_3 S_i,\ ...,\ S_i\}$, $i = 1, 2, ..., m$ are weakly compact in $L^1(\mu, l^p)$. Set $\widehat{E} = \left\{g^1,\ g^2,\ g^3,\ ...,\ g^n,\ ...\right\}$. We want to prove that $\widehat{E}$ is weakly relatively compact. Since l^p is reflexive, then by the Dunford Theorem $[4, p.101]$, it is enough to prove that $\widehat{E}$ is bounded and uniformly integrable. Note that

$$\sum_{i=1}^{m} \|P_k S_i - h\|_1 \geq \sum_{i=1}^{m} \left\|P_k S_i - g^k\right\|_1 \geq \sum_{i=1}^{m} \left|\|P_k S_i\|_1 - \left\|g^k\right\|_1\right|.$$

for all $h \in U_k$. Since $0 \in U_k$, we get

$$m \left\|g^k\right\|_1 \leq 2 \sum_{i=1}^{m} \|P_k S_i\|_1$$

Hence $\widehat{E}$ is bounded.

To see that $\widehat{E}$ is uniformly integrable, first note that for each k

$$\|P_k S_i\|_1 \leq \|S_i\|_1$$

$i = 1, 2, ...m$. Thus $\lim\limits_{\mu(\Omega) \to 0} \int_\Omega \|h(t)\|\, d\mu(t) = 0$ uniformly for h in E_i, $i = 1, 2, ..., m$.

Now let $\epsilon > 0$ be given. By the uniform integrability of E_i there exists $\delta_i > 0$ such that $\int\limits_{\Omega} \|h(t)\|\, d\mu(t) < \frac{\epsilon}{2}$ whenever $\mu(\Omega) < \delta_i$ for all $h \in E_i$. Hence for $\mu(\Omega) < \delta = \min\limits_{1 \le i \le m} (\delta_i)$

$$\int\limits_{\Omega} \left\|g^k(t)\right\| d\mu(t) < \frac{2}{m} \sum_{i=1}^{m} \int\limits_{\Omega} \|P_k S_i\|\, d\mu(t) < \epsilon.$$

Since δ depends only on $E_1, E_2, ..., E_m$ and ϵ it follows that $\widehat{E}$ is uniformly integrable and hence weakly relatively compact. Thus there exists $g \in L^1(\mu, l^p)$ such that $g^k \to g$ weakly.

Since the sequence $\left(g^k\right)$ in U converges weakly to some $g \in L^1(\mu, l^p)$ and U is a closed subspace of $L^1(\mu, l^p)$, hence weakly closed, it follows that $g \in U$.

For $h \in U$, we have $\|P_k h - h\|_1 \to 0$. Hence for each $i = 1, 2, ..., m$, $\|P_k S_i - P_k h\|_1 \underset{k}{\to} \|S_i - h\|_1$. Now let $\varphi \in L^{\infty}(\mu, l^{p^*}) = \left(L^1(\mu, l^p)\right)^*$, the dual of $L^1(\mu, l^p)$. Then

$$\begin{aligned} \sum_{i=1}^{m} |\langle S_i - g, \varphi \rangle| &= \lim_{k \to \infty} \sum_{i=1}^{m} |\langle P_k S_i - g, \varphi \rangle| \\ &\le \underline{\lim} \sum_{i=1}^{m} \left\|P_k S_i - g^k\right\| \\ &\le \underline{\lim} \sum_{i=1}^{m} \|P_k S_i - P_k h\| \end{aligned}$$

for all $h \in U_k$, since U_k is proximinal. Hence

$$\sum_{i=1}^{m} |\langle S_i - g, \varphi \rangle| \le \sum_{i=1}^{m} \|S_i - h\|\, .$$

Consequently $\sum_{i=1}^{m} \|S_i - g\| \le \sum_{i=1}^{m} \|S_i - h\|$ for all $h \in U$.

Thus U is proximinal in $C^1(I, l^p)$, with the L^1-norm, $1 < p < \infty$. □

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