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$\alpha-$tauberian results

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Abstract. In this paper we consider problems that are analoguous to those on summability $(C,1)$ introduced and studied by Hardy. A series $\sum_n x_n$ is said to be summable $(C,1)$ (to sum $S \in \mathbb{C}$) if the sequence $n^{-1}\sum_{k=1}^{n} s_k$ where $s_k = \sum_{i=1}^{k} x_i$ tends to S. Here we extend the Hardy's *tauberian* theorem for *Cesàro means* where it is shown that if the sequence $(x_n)_n$ satisfies $\sup_n \{n\,|x_n - x_{n-1}|\} < \infty$, then $n^{-1}s_n \to \chi$ implies $x_n \to \chi$ for some $\chi \in \mathbb{C}$. In this work, for given sequences λ and μ, we give $\alpha-$tauberian theorems which consists in determining the set of all sequences α such that

$$\frac{1}{\lambda_n}\sum_{k=1}^{n}\mu_k\left(\sum_{i=k}^{\infty}x_i\right) \to l \text{ implies } \frac{x_n}{\alpha_n} \to l' \ \ (n \to \infty)$$

for all $X \in cs$? Then we give simplifications of these theorems in the cases when $\alpha \in \widehat{C_1}$, and $\alpha \in \widehat{\Gamma}$. Finally we deal with the converse of the last condition.

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1. Introduction

In this paper we study problems that are similar to those stated by Hardy [6], Móricz and Rhoades, (cf. [10]), de Malafosse and Rakočević (cf. [5]). In [6] it is said that a series $\sum_{k=1}^{\infty} x_k$ is *summable* $(C,1)$ (to sum $l \in \mathbb{C}$) if

$$\chi_n = \frac{1}{n}\sum_{k=1}^{n} s_k \to l$$

where $s_k = \sum_{i=1}^{k} x_i$. It was shown (cf. [6, p. 132, Theorem 77]) that if a series $\sum_{k=1}^{\infty} x_k$ is *summable* $(C,1)$ to sum S if and only if

$$S = \sum_{k=1}^{\infty}\left(\sum_{i=k}^{\infty}\frac{x_i}{i}\right). \tag{1.1}$$

Móricz and Rhoades gave a generalization of the Hardy theorem using the *weighted mean matrix* $\overline{N}$, (cf. [10, 11]). In de Malafosse and Rakočević (cf. [5]) the series $\sum_{k=1}^{\infty} x_k$ is said to be *summable* (C, λ, μ) (to sum $L \in \mathbb{C}$) for given sequences λ and μ if

$$\chi'_n = \frac{1}{\lambda_n} \sum_{k=1}^{n} \frac{1}{\mu_k} s_k \to L.$$

When $\lambda_n = n$ and $\mu_n = 1$ for all n, summability (C, λ, μ) reduces to summability $(C, 1)$. In the following we extend Hardy's *tauberian* theorem for *Cesàro means* where it is shown that if the sequence $X = (x_n)_n$ satisfies $\sup_n \{n |x_n - x_{n-1}|\} < \infty$, then $n^{-1} s_n \to \chi$ implies $x_n \to \chi$ for some $\chi \in \mathbb{C}$. In this way for given sequences λ and μ we determine the set of all the sequences α such that

$$\frac{1}{\lambda_n} \sum_{k=1}^{n} \mu_k \left(\sum_{i=k}^{\infty} x_i \right) \to l \text{ implies } \frac{x_n}{\alpha_n} \to l' \ \ (n \to \infty) \text{ for all } X \in cs$$

for some l, $l' \in \mathbb{C}$. This statement is called an $\alpha-$tauberian problem. The main result is given by Theorem 4.3.

This paper is organized as follows. In Section 2 we recall some results on the sets of sequences and matrix transformations. In Section 3 we give some properties of the operator Σ^+ defined by $[\Sigma^+ X]_n = \sum_{k=n}^{\infty} x_k$ for all n, on special sets of sequences. In Section 4 we state some α-tauberian theorems in the general case and in the case when $\lambda_n = n$ and $\mu_n = n^{\xi}$ where ξ is a real. Then we give simplifications of α-tauberian theorems when α belongs to special sets of sequences such as $\widehat{C_1}$, or $\widehat{\Gamma}$. Finally we deal with the converse of the previous tauberian results.

2. Preliminary results

In the following we write $A = (a_{nk})_{n,k \geq 1}$ for an infinite matrix of complex numbers. For a given sequence $X = (x_n)_{n \geq 1}$ of complex numbers we define $A_n(X) = \sum_{k=1}^{\infty} a_{nk} x_k$, (provided the series $A_n(X)$ converge) and $AX = (\sum_{k=1}^{\infty} a_{nk} x_k)_{n \geq 1}$. We write s, ℓ_∞, c_0 and c for the sets of all complex, bounded, naught and convergent sequences, respectively, furthermore cs is the set of all convergent series. For E, $F \subset s$, we write (E, F) for the *set of all matrix transformations* that map E to F. For given $\tau \in s$ we define $D_\tau = (\tau_n \delta_{nk})_{n,k \geq 1}$, (where $\delta_{nn} = 1$ for all n and $\delta_{nk} = 0$ otherwise). We define by U^+ the set of all sequences $(u_n)_{n \geq 1} \in s$ with $u_n > 0$ for all n and consider the spaces $s_\alpha = D_\alpha \ell_\infty$, $s_\alpha^0 = D_\alpha c_0$ and $s_\alpha^{(c)} = D_\alpha c$ for $\alpha \in U^+$, see [2, 3]. It can easily be seen that for α, $\beta \in U^+$ and E, $F \subset s$ we have $A \in (D_\alpha E, D_\beta F)$ if and only if $D_{1/\beta} A D_\alpha \in (E, F)$. If $e = (1, 1, ...)$ we put $s_1 = s_e$. Let E and F be any subsets of s. It is well known, see [1] that $(s_1, s_1) = (c_0, s_1) = (c, s_1) = S_1$, where S_1 is the set of all infinite matrices $A = (a_{nk})_{n,k \geq 1}$ such that $\sup_n (\sum_{k=1}^{\infty} |a_{nk}|) < \infty$. For any subset E of s, AE

is the set of all sequences Y such that $Y = AX$ for some $X \in E$. For any subset F of s, the matrix domain $F(A) = F_A$ of A is the set of all sequences X such that $AX \in F$.

In this paper we consider the operators represented by the infinite matrices $C(\lambda)$ and $\Delta(\lambda)$ for $\lambda \in U^+$, see [3]. Recall that $[C(\lambda)]_{n,k} = 1/\lambda_n$ for $k \leq n$ and 0 otherwise. In the following we will use the convention that any term with nonpositive subscript is equal to zero. It can be proved that the matrix $\Delta(\lambda)$ defined by $[\Delta(\lambda)]_{nn} = \lambda_n$, $[\Delta(\lambda)]_{n,n-1} = -\lambda_{n-1}$ and $[\Delta(\lambda)]_{nk} = 0$ for $k \neq n-1$, n, $n \geq 1$, is the inverse of $C(\lambda)$. If $\lambda = e$ we get the well-known operator of the first difference represented by $\Delta(e) = \Delta$ and it is usually written $\Sigma = C(e)$. We have $[\Delta X]_n = x_n - x_{n-1}$ for all $n \geq 1$. Then $\Delta = \Sigma^{-1}$ and Δ, $\Sigma \in S_R = \left(s_{(R^n)_n}, s_{(R^n)_n}\right)$ for $R > 1$. We also use the transpose of $C(\lambda)$ denoted by $C^+(\lambda)$. We easily see that $C^+(\lambda) = \Sigma^+ D_{1/\lambda}$ where Σ^+ is the transpose of Σ.

3. Some properties of the infinite matrix Σ^+ considered as operator in s_α, s_α^0, or $s_\alpha^{(c)}$

In this section we are interested in the study of the set of all sequences X such that

$$\frac{1}{\lambda_n} \sum_{k=1}^{n} \mu_k r_k \to l \text{ for some } l \in \mathbb{C},$$

where $r_k = \sum_{i=k}^{\infty} x_i$.

In the following we will use the characterizations of the sets (E, F), where E, F are either of the sets c or c_0.

We will consider the next conditions

$$A \in S_1, \tag{3.1}$$

$$\lim_{n\to\infty} a_{nk} = l_k \text{ for some } l_k \in \mathbb{C} \text{ and for all } k. \tag{3.2}$$

From [9, Theorem 1.36, p. 160] we immediately deduce the next lemma.

Lemma 3.1. *i) $A \in (c_0, c_0)$ if and only if (3.1) and (3.2) hold with $l_k = 0$;*

ii) $A \in (c, c_0)$ if and only if (3.1), (3.2) hold with $l_k = 0$ and

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} = 0.$$

iii) $A \in (c_0, c)$ if and only if (3.1) and (3.2) hold;

iv) a) $A \in (c, c)$ if and only if (3.1), (3.2) hold and

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} = l \text{ for some } l \in \mathbb{C}. \tag{3.3}$$

b) Let $A \in (c,c)$ and $x \in c$. If (3.3) and (3.2) hold with $l_k = 0$, then

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} x_k = l \lim_{n\to\infty} x_n .$$

Note that the statements given in iv) are direct consequences of Silverman Toeplitz theorem.

We will use the next lemma where $T = (t_{nk})_{n,k\geq 1}$ is called a lower triangular matrix if $t_{nk} = 0$ for $k > n$.

Lemma 3.2. *Let $A = (a_{nk})_{n,k\geq 1}$ be an infinite matrix and T a lower triangular matrix. Then*

$$T(AX) = (TA)X \text{ for all } X \in s(A).$$

Proof. Since $X \in s(A)$ the series $\sum_{k=1}^{\infty} a_{nk} x_k$ is convergent for all n. Then

$$[T(AX)]_n = \sum_{m=1}^{n} t_{nm} \left(\sum_{k=1}^{\infty} a_{mk} x_k \right) = \sum_{k=1}^{\infty} \left(\sum_{m=1}^{n} t_{nm} a_{mk} \right) x_k = [(TA)X]_n$$

for all n and for all $X \in s(A)$. □

In all that follows we use the operator represented by the infinite matrix Σ^+. For the convenience to the reader we note that

$$\Sigma^+ = \begin{pmatrix} 1 & 1 & . & . \\ & 1 & 1 & . \\ \mathbf{0} & & . & . \\ & & & . \end{pmatrix}.$$

We use the following results where Δ^+ is the transpose of Δ.

Lemma 3.3. *i) $\Sigma^+(\Delta^+ X) = X$ for all $X \in c_0$ and $\Delta^+(\Sigma^+ X) = X$ for all $X \in cs$,*

ii) the operator Σ^+ is bijective from cs to c_0 and Δ^+ is bijective from c_0 to cs.

Proof. i) comes from [1, Lemma 3, p. 19]. ii) is a direct consequence of i). □

Lemma 3.2 and Lemma 3.3 lead to define the product $T\Sigma^+$ by $(T\Sigma^+)X = T(\Sigma^+ X)$ for all $X \in cs$ where T is a triangle, that is a lower triangle with $[T]_{nn} \neq 0$ for all n. We note that T is bijective from s to itself and that T^{-1} is again a triangle matrix. In this way we have

Lemma 3.4. *Let T be a triangle, then $T\Sigma^+ \in (cs, Tc_0)$ is bijective and*

$$\left(T\Sigma^+\right)^{-1} = \Delta^+ T^{-1}.$$

Proof. Let $B \in Tc_0$ and consider the equation

$$\left(T\Sigma^+\right) X = B \text{ for } X \in cs. \tag{3.4}$$

Since

$$\left(T\Sigma^+\right) X = T\left(\Sigma^+ X\right) \text{ for all } X \in cs,$$

and $T : c_0 \to Tc_0$ is bijective, equation (3.4) is equivalent to $\Sigma^+ X = T^{-1}B$. Then since $T^{-1}B \in c_0$ and Σ^+ is bijective from cs to c_0, we deduce $T\Sigma^+$ is bijective and (3.4) has a unique solution given by $X = (T\Sigma^+)^{-1} B = \Delta^+ \left(T^{-1}B\right)$. Finally by Lemma 2 it can easily be seen that $X = \Delta^+ \left(T^{-1}B\right) = \left(\Delta^+ T^{-1}\right) B$ for all B and $(T\Sigma^+)^{-1} = \Delta^+ T^{-1}$. □

Let λ, $\mu \in U^+$. In the following we will use the notation $\sigma_n = \sum_{k=1}^n \mu_k$ and define the map

$$\phi_n(X) = \frac{1}{\lambda_n} \left(\sum_{k=1}^n \sigma_k x_k + \sigma_n r_{n+1} \right) \text{ for all } X \in cs \text{ and } n \geq 1.$$

Let us state the next result where $\mathbb{R}^{+*}$ is the set of all reals > 0.

Theorem 3.5. *Let E be a set of sequences.*

i) $c_0 \subset E$ implies $E(\Sigma^+) = cs$;

ii) a) $E(\Sigma^+) \subset \Delta^+ E$;

b) $E \subset c_0$ implies that $E(\Sigma^+) = \Delta^+ E$;

iii) $c_0(\Sigma^+) \subset s_\alpha^{(c)}$ if and only if $1/\alpha \in \ell_\infty$.

iv) a) Let E be either of the sets s_α, s_α^0, or $s_\alpha^{(c)}$. Then

$$E\left(\Sigma^+\right) = cs \text{ if and only if } 1/\alpha \in \ell_\infty.$$

b) $c_0(\Sigma^+) = c(\Sigma) = \Delta^+ c_0 = cs$.

v) a) Assume that

$$\sigma/\lambda \in l_\infty \text{ and } \lambda_n \to \infty \ (n \to \infty). \tag{3.5}$$

Then

$$c_0\left(C(\lambda) D_\mu \Sigma^+\right) = cs,$$

and

$$\frac{1}{\lambda_n} \sum_{k=1}^n \mu_k r_k \to 0 \text{ for all } X \in cs. \tag{3.6}$$

b) The condition $\sup_n (n/\lambda_n) < \infty$ implies $c_0(C(\lambda)\Sigma^+) = cs$.

vi) Assume that

$$\sigma/\lambda \in \ell_\infty \text{ and } \lambda_n \to l \ (n \to \infty) \text{ for some } l \in \mathbb{R}^{+*} \bigcup \{+\infty\}. \tag{3.7}$$

Then

$$c\left(C(\lambda) D_\mu \Sigma^+\right) = cs,$$

and

$$\frac{1}{\lambda_n} \sum_{k=1}^n \mu_k r_k \to L_X \text{ for some } L_X \in \mathbb{C} \text{ and for all } X \in cs.$$

Proof. i) Necessity. Let $X \in E(\Sigma^+)$. Then $\Sigma^+ X$ exists and $X \in cs$, so $E(\Sigma^+) \subset cs$. Sufficiency. Let $X \in cs$. Then the series $\sum_{k=n}^\infty x_k$ are convergent for all n and $\Sigma^+ X \in c_0$, but the inclusion $c_0 \subset E$ implies $\Sigma^+ X \in E$ and $X \in E(\Sigma^+)$. So we have shown $cs \subset E(\Sigma^+)$. We conclude $E(\Sigma^+) = cs$.

ii) a) If $E(\Sigma^+) = \emptyset$ trivially we have $E(\Sigma^+) \subset \Delta^+ E$. Now assume $E(\Sigma^+) \neq \emptyset$ and let $X \in E(\Sigma^+)$. Then $Y = \Sigma^+ X$ exists, $\Sigma^+ X \in E$ and $X \in cs$. Since $cs \subset c_0$, we have by Lemma 3.3

$$\Delta^+ \left(\Sigma^+ X\right) = \Delta^+ Y = X.$$

We conclude that $X \in \Delta^+ E$ and $E(\Sigma^+) \subset \Delta^+ E$. b) We show that $E \subset c_0$ implies $E(\Sigma^+) \supset \Delta^+ E$. For every $X \in E$ we have $Y = \Delta^+ X \in \Delta^+ E$ and from Lemma 3.3 we have $\Sigma^+ Y = \Sigma^+ (\Delta^+ X) = X$ since $X \in E \subset c_0$. Then $\Sigma^+ Y = X \in E$ and $Y \in E(\Sigma^+)$. So we have shown $\Delta^+ E \subset E(\Sigma^+)$. This result and a) imply b).

iii) Assume $c_0(\Sigma^+) \subset s_\alpha^{(c)}$. Then $I \in \left(c_0(\Sigma^+), s_\alpha^{(c)}\right)$ and since $c_0(\Sigma^+) = c(\Sigma)$ we deduce $\Delta \in \left(c, s_\alpha^{(c)}\right)$, $D_{1/\alpha}\Delta \in (c, c)$ and $1/\alpha \in \ell_\infty$.

iv) a) Using i) we see that it is enough to show that $c_0 \subset E$ if and only if $1/\alpha \in \ell_\infty$ for $E = s_\alpha$, s_α^0, or $s_\alpha^{(c)}$. We have $c_0 \subset s_\alpha$ if and only if $I \in (c_0, s_\alpha)$, that is $D_{1/\alpha} \in (c_0, s_1) = S_1$ and $1/\alpha \in \ell_\infty$. In the same way using the characterizations of (c_0, c_0) and (c_0, c) we deduce $c_0 \subset E$ if and only if $1/\alpha \in \ell_\infty$ for $E = s_\alpha^0$, or $s_\alpha^{(c)}$.

b) Let $X \in c_0(\Sigma^+)$. Then $\Sigma^+ X \in c_0$, $X \in cs$ and so $c_0(\Sigma^+) \subset cs$. Now $X \in cs$ implies $\Sigma^+ X = \left(\sum_{k=n}^\infty x_k\right)_{n \geq 1} \in c_0$ since $\sum_{k=n}^\infty x_k \to 0$ $(n \to \infty)$ and $X \in c_0(\Sigma^+)$. This shows $cs \subset c_0(\Sigma^+)$ and as we have just shown $c_0(\Sigma^+) \subset cs$, so $c_0(\Sigma^+) = cs$. Finally by ii) b) we have $c_0(\Sigma^+) = \Delta^+ c_0$.

v) a) We show $cs \subset c_0(C(\lambda) D_\mu \Sigma^+)$. By Lemma 3.2 we have

$$C(\lambda) D_\mu \left(\Sigma^+ X\right) = \left(C(\lambda) D_\mu \Sigma^+\right) X \quad \text{for all } X \in cs \tag{3.8}$$

since $C(\lambda) D_\mu$ is a triangle and $X \in s(\Sigma^+) = cs$. Now for every $X \in cs$ we have $\Sigma^+ X \in c_0$ and since (3.5) holds we have $C(\lambda) D_\mu \in (c_0, c_0)$ and then $C(\lambda) D_\mu (\Sigma^+ X) \in c_0$ for all $X \in cs$. Finally since (3.8) holds we conclude $(C(\lambda) D_\mu \Sigma^+) X \in c_0$ for all $X \in cs$ and $cs \subset c_0(C(\lambda) D_\mu \Sigma^+)$.

Conversely let $X \in c_0(C(\lambda) D_\mu \Sigma^+)$. By elementary calculations we easily get

$$C(\lambda) D_\mu \Sigma^+ = \begin{pmatrix} \sigma_1/\lambda_1 & . & . & . & . & \sigma_1/\lambda_1 & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ \sigma_1/\lambda_n & \sigma_2/\lambda_n & . & \sigma_n/\lambda_n & . & \sigma_n/\lambda_n & . \\ . & . & . & . & . & . & . \end{pmatrix} \tag{3.9}$$

that is

$$\left[C(\lambda) D_\mu \Sigma^+\right]_{nk} = \begin{cases} \sigma_k/\lambda_n \text{ for } k < n, \\ \sigma_n/\lambda_n \text{ for } k \geq n. \end{cases}$$

We deduce

$$\left(C(\lambda) D_\mu \Sigma^+\right) X = \left(\phi_n(X)\right)_{n \geq 1} \in c_0.$$

Then the series $r_n = \sum_{k=n}^{\infty} x_k$ is convergent for all n and $X \in cs$. This shows $c_0\left(C(\lambda) D_\mu \Sigma^+\right) \subset cs$. We conclude $c_0\left(C(\lambda) D_\mu \Sigma^+\right) = cs$. Since

$$\left[C(\lambda) D_\mu \left(\Sigma^+ X\right)\right]_n = \frac{1}{\lambda_n} \sum_{k=1}^{n} \mu_k r_k \text{ for all } n,$$

statement (3.6) comes from identity (3.8).

v) b) is a direct consequence of v) a) where we put $\mu = e$, furthermore condition $\sup_n (n/\lambda_n) < \infty$ trivially implies $\lambda_n \to \infty \ (n \to \infty)$.

vi) can be obtained reasoning as in v) a) by using the characterization of (c_0, c). □

4. $\alpha-$tauberian results

4.1. General case

For given λ, $\mu \in U^+$ the aim of this paper is to determine the set of all sequences $\alpha \in U^+$ such that

$$\frac{1}{\lambda_n} \sum_{k=1}^{n} \mu_k \left(\sum_{j=k}^{\infty} x_j \right) \to l \text{ implies } \frac{x_n}{\alpha_n} \to l' \ \ (n \to \infty) \text{ for all } X \in cs, \tag{4.1}$$

for some l, $l' \in \mathbb{C}$.

Now state a lemma which is a characterization of condition (4.1).

Lemma 4.1. *For λ, μ, $\alpha \in U^+$ condition (4.1) holds if and only if*

$$\Delta^+ D_{1/\mu} \Delta(\lambda) \in \left(c \bigcap C(\lambda) D_\mu c_0, s_\alpha^{(c)}\right). \tag{4.2}$$

Proof. First condition (4.1) means that

$$C(\lambda) D_\mu \left(\Sigma^+ X\right) \in c \text{ implies } X \in s_\alpha^{(c)} \text{ for all } X \in cs. \tag{4.3}$$

Since $\Sigma^+ X \in c_0$ for all $X \in cs$, condition (4.3) is equivalent to the statement

$$Y = C(\lambda) D_\mu \left(\Sigma^+ X\right) \in c \bigcap C(\lambda) D_\mu c_0 \text{ implies } X \in s_\alpha^{(c)}. \tag{4.4}$$

Since $C(\lambda) D_\mu$ is a triangle and $\Sigma^+ \in (cs, c_0)$ by Lemma 3.2 we have

$$C(\lambda) D_\mu \left(\Sigma^+ X\right) = \left(C(\lambda) D_\mu \Sigma^+\right) X \text{ for all } X \in cs.$$

Then by Lemma 3.4 the operator $C(\lambda) D_\mu \Sigma^+ \in \left(cs, C(\lambda) s_\mu^0\right)$ is invertible and

$$\left(C(\lambda) D_\mu \Sigma^+\right)^{-1} = \Delta^+ D_{1/\mu} \Delta(\lambda),$$

we deduce $Y = C(\lambda) D_\mu (\Sigma^+ X)$ if and only if $X = \Delta^+ D_{1/\mu} \Delta(\lambda) Y$ for all $X \in cs$ and condition (4.4) is equivalent to

$$Y \in c \bigcap C(\lambda) D_\mu c_0 \text{ implies } X = \Delta^+ D_{1/\mu} \Delta(\lambda) Y \in s_\alpha^{(c)}$$

and to (4.2). □

To state the next results we need the next lemma.

Lemma 4.2. *Let κ and $\kappa' \in U^{+}$. Then conditions $\kappa + \kappa' \in \ell_\infty$ and $\kappa - \kappa' \in c$ together are equivalent to $\kappa \in \ell_\infty$ and $\kappa - \kappa' \in c$.*

Proof. First we have $\kappa + \kappa' \in \ell_\infty$ if and only if $\kappa, \kappa' \in \ell_\infty$. Then $\kappa - \kappa' \in c$ is equivalent to $\kappa_n = \kappa'_n + L + 0(1)$ $(n \to \infty)$, for some $L \in \mathbb{C}$, which shows that κ is bounded if and only if κ' is bounded. This gives the conclusion. $\square$

In this way it can be easily seen that conditions $\kappa + \kappa' \in \ell_\infty$ and $\kappa - \kappa' \in c$ together are equivalent to $\kappa' \in \ell_\infty$ and $\kappa - \kappa' \in c$.

Now consider the next conditions

$$\frac{1}{\alpha_n}\left(\frac{\lambda_{n-1}}{\mu_n} + \frac{\lambda_{n+1}}{\mu_{n+1}}\right) = O(1) \quad (n \to \infty) \tag{4.5}$$

$$\lim_{n\to\infty}\left\{\frac{1}{\alpha_n}\left[-\frac{\lambda_{n-1}}{\mu_n} + \lambda_n\left(\frac{1}{\mu_n} + \frac{1}{\mu_{n+1}}\right) - \frac{\lambda_{n+1}}{\mu_{n+1}}\right]\right\} = L \text{ for some } L \in \mathbb{C} \tag{4.6}$$

We obtain the following $\alpha-$tauberian theorem.

Theorem 4.3. *Let λ, $\mu \in U^{+}$. Then*

i) condition (4.1) holds if α satisfies one of the conditions a) or b), where

a) $1/\alpha \in \ell_\infty$,

b) conditions (4.5) and (4.6) hold.

ii) If there is $L \in \mathbb{R}^{+*} \bigcup \{+\infty\}$ such that

$$\sigma/\lambda \in \ell_\infty \text{ and } \lambda_n \to L \quad (n \to \infty) \tag{4.7}$$

then condition (4.1) holds if and only if $1/\alpha \in \ell_\infty$.

iii) If $(-\lambda_{n-1} + \lambda_n)/\mu_n \to 0$ $(n \to \infty)$ and there is $K' > 0$ such that

$$\frac{\lambda_{n-1} + \lambda_n}{\mu_n} \le K' \text{ for all } n \ge 1 \tag{4.8}$$

then condition (4.1) holds if and only if (4.5) and (4.6) hold.

Proof. i) First we show that a) implies (4.1). Assume $1/\alpha \in \ell_\infty$. Then the condition

$$\frac{1}{\lambda_n}\sum_{k=1}^{n}\mu_k r_k \to l$$

necessary implies $X \in cs$. Then trivially $X \in c_0$ and $(1/\alpha_n)\, x_n \to 0$ $(n \to \infty)$. So we have shown a) implies (4.1).

Next we show that b) implies (4.1). Since trivially $c \bigcap C(\lambda) D_\mu c_0 \subset c$ we have $\left(c, s_\alpha^{(c)}\right) \subset \left(c \bigcap C(\lambda) D_\mu c_0, s_\alpha^{(c)}\right)$. We show that we have

$$\widetilde{\Delta} = \Delta^{+} D_{1/\mu} \Delta(\lambda) \in \left(c, s_\alpha^{(c)}\right) \tag{4.9}$$

which implies (4.2) and (4.1) by Lemma 4.1. Now the calculations of $D_{1/\mu}\Delta(\lambda)$ and $\widetilde{\Delta}$ successively give

$$D_{1/\mu}\Delta(\lambda) = \begin{pmatrix} \frac{\lambda_1}{\mu_1} & & & & \\ -\frac{\lambda_1}{\mu_2} & \frac{\lambda_2}{\mu_2} & & \mathbf{0} & \\ & \cdot & \cdot & & \\ \mathbf{0} & & -\frac{\lambda_{n-1}}{\mu_n} & \frac{\lambda_n}{\mu_n} & \\ & & & \cdot & \cdot \end{pmatrix} \tag{4.10}$$

and

$$\widetilde{\Delta} = \begin{pmatrix} \lambda_1\left(\frac{1}{\mu_1}+\frac{1}{\mu_2}\right) & -\frac{\lambda_2}{\mu_2} & & & \\ -\frac{\lambda_1}{\mu_2} & \lambda_2\left(\frac{1}{\mu_2}+\frac{1}{\mu_3}\right) & -\frac{\lambda_3}{\mu_3} & \mathbf{0} & \\ & \cdot & \cdot & & \\ \mathbf{0} & & -\frac{\lambda_{n-1}}{\mu_n} & \lambda_n\left(\frac{1}{\mu_n}+\frac{1}{\mu_{n+1}}\right) & -\frac{\lambda_{n+1}}{\mu_{n+1}} \\ & & & \cdot & \cdot \end{pmatrix}. \tag{4.11}$$

Then condition (4.9) means $D_{1/\alpha}\widetilde{\Delta} \in (c,c)$ and from the characterization of (c,c) this condition is equivalent to $\kappa + \kappa' \in \ell_\infty$ and $\kappa - \kappa' \in c$ together where $\kappa = (\kappa_n)_{n\geq 1}$, $\kappa' = (\kappa'_n)_{n\geq 1}$ with

$$\kappa_n = \frac{1}{\alpha_n}\left[\lambda_n\left(\frac{1}{\mu_n}+\frac{1}{\mu_{n+1}}\right)\right] \text{ and } \kappa'_n = \frac{1}{\alpha_n}\left(\frac{\lambda_{n-1}}{\mu_n}+\frac{\lambda_{n+1}}{\mu_{n+1}}\right).$$

Then from Lemma 4.2 condition (4.9) is equivalent to (4.5) and (4.6) and as we have just seen (4.9) implies (4.1). This completes the proof of i).

ii). From Theorem 3.5 vi) we see that (4.1) means that $cs \subset s_\alpha^{(c)}$. Since $cs = c(\Sigma) = \Sigma^{-1}c$ we then have $I \in \left(\Sigma^{-1}c, s_\alpha^{(c)}\right)$ and $D_{1/\alpha}\Sigma^{-1} = D_{1/\alpha}\Delta \in (c,c)$. We have

$$D_{1/\alpha}\Delta = \begin{pmatrix} 1/\alpha_1 & & & \\ \cdot & \cdot & \mathbf{0} & \\ & -1/\alpha_n & 1/\alpha_n & \\ \mathbf{0} & & \cdot & \cdot \end{pmatrix}$$

and from the characterization of (c,c) given in Lemma 3.1 iv) we conclude $D_{1/\alpha}\Delta \in (c,c)$ if and only if $1/\alpha \in \ell_\infty$.

iii) We have $c \subset C(\lambda)D_\mu c_0$. Indeed from the expression of $D_{1/\mu}\Delta(\lambda)$ given by (4.10) it follows that $(C(\lambda)D_\mu)^{-1} = D_{1/\mu}\Delta(\lambda) \in (c,c_0)$ if and only if the hypotheses of iii) hold. Then (4.1) means that $\widetilde{\Delta}Y \in s_\alpha^{(c)}$ for all $Y \in c$ by Lemma 4.1 and $\widetilde{\Delta} \in \left(c, s_\alpha^{(c)}\right)$ that is $D_{1/\alpha}\widetilde{\Delta} \in (c,c)$. Using the characterization of (c,c) given in Lemma 3.1 and Lemma 4.2 we easily conclude that $D_{1/\alpha}\widetilde{\Delta} \in (c,c)$ if and only if (4.5) and (4.6) hold. □

These results lead to the next corollary

Corollary 4.4. *Assume* (4.5) *and* (4.6) *hold. Then condition* (4.1) *holds with* $l' = Ll$.

Proof. This result is a direct consequence of Lemma 3.1 iv) c) and of the proof of i) b) implies (4.1) in Theorem 4.3. □

Example 4.5. If we put $\lambda = e$ in Theorem 4.3 iii), then for given $\mu \in U^+$ with $\sup_n 1/\mu_n < \infty$ we have

$$\sum_{k=1}^{n} \mu_k r_k \to l \text{ implies } \frac{x_n}{\alpha_n} \to l' \ \ (n \to \infty) \ \text{ for all } X \in cs \tag{4.12}$$

if and only if α satisfies

$$\sup_n \left\{ \frac{1}{\alpha_n} \left(\frac{1}{\mu_n} + \frac{1}{\mu_{n+1}} \right) \right\} < \infty.$$

By Corollary 4.4, since $L = 0$ we have $l' = 0$. Particularly if $\mu_n = n$ for all n, (4.12) holds if and only if $1/\alpha_n = O(n)$ $(n \to \infty)$.

In this way we obtain the next result.

Proposition 4.6. *Let* $\lambda \in U^+$ *and assume* $\sup_n (n/\lambda_n) < \infty$. *Then*

i) $c_0 (C(\lambda) \Sigma^+) = cs$.

ii) The condition

$$\frac{1}{\lambda_n} \sum_{k=1}^{n} r_k \to l \text{ implies } \frac{x_n}{\alpha_n} \to l' \ \ (n \to \infty) \ \text{ for all } X \in cs \tag{4.13}$$

is equivalent to $1/\alpha \in \ell_\infty$.

Proof. i) is a direct consequence of Theorem 3.5 v) b) since $\sup_n (n/\lambda_n) < \infty$. ii) is a direct consequence of Theorem 4.3 ii). □

4.2. Case when $\lambda_n = n$ and $\mu_n = n^\xi$ where ξ is a real

Now we consider the case when $\lambda_n = n$ and $\mu_n = n^\xi$ with ξ real in condition (4.1), that is

$$\frac{1}{n} \sum_{k=1}^{n} k^\xi r_k \to l \text{ implies } \frac{x_n}{\alpha_n} \to l' \ \ (n \to \infty) \ \text{ for all } X \in cs \tag{4.14}$$

for some l, $l' \in \mathbb{C}$. As another consequence of Theorem 4.3 we obtain the next corollary.

Corollary 4.7. *i) Let* $\xi \geq 1$. *Then condition* (4.14) *holds if and only if*

$$\sup_n \left(\frac{1}{n^{\xi - 1} \alpha_n} \right) < \infty. \tag{4.15}$$

ii) If $\xi \leq 0$, *condition* (4.14) *holds if and only if* $1/\alpha \in \ell_\infty$.

Proof. i) is a direct consequence of Theorem 4.3 iii). Indeed for $\lambda_n = n$ and $\mu_n = n^\xi$ we have

$$\frac{\lambda_n + \lambda_{n-1}}{\mu_n} = \frac{2n-1}{n^\xi} = O(1) \quad (n \to \infty).$$

We need to verify (4.5). We have

$$\kappa_n = \frac{n-1}{n^\xi} + \frac{n+1}{(n+1)^\xi} = \frac{1}{n^{\xi-1}} - \frac{1}{n^\xi} + \frac{1}{(n+1)^{\xi-1}} \sim \frac{2}{n^{\xi-1}} \quad (n \to \infty).$$

Then

$$\frac{\kappa_n}{\alpha_n} \sim \frac{2}{n^{\xi-1}\alpha_n} \quad (n \to \infty)$$

and so the condition (4.5) is equivalent to (4.15). To show (4.6), put

$$b_n = -\frac{n-1}{n^\xi} + n\left(\frac{1}{n^\xi} + \frac{1}{(n+1)^\xi}\right) - \frac{1}{(n+1)^{\xi-1}}.$$

We immediately get

$$b_n = \frac{1}{n^\xi}\left[1 - \left(\frac{n}{n+1}\right)^\xi\right] \sim \frac{\xi}{n^{\xi+1}} \quad (n \to \infty)$$

and so there is $C > 0$ such that $b_n/\alpha_n \leq C/n^{\xi+1}\alpha_n$ $(n \to \infty)$. Then by (4.15) we have $1/\alpha_n \leq C' n^{\xi-1}$,

$$\frac{b_n}{\alpha_n} \leq CC' \frac{n^{\xi-1}}{n^{\xi+1}} = O\left(\frac{1}{n^2}\right) \quad (n \to \infty)$$

and $b_n/\alpha_n \to 0$ $(n \to \infty)$. We conclude (4.6) holds and the conditions (4.5) and (4.6) together are equivalent to (4.15).

ii) We only have to apply Theorem 4.3 ii). Indeed for $\xi = -1$ we have

$$\frac{\sigma_n}{n} = \frac{1}{n}\sum_{m=1}^{n} \frac{1}{k} = O(1) \quad (n \to \infty).$$

For $\xi \leq 0$ and $\xi \neq -1$ we have

$$\sum_{k=2}^{n} k^\xi \leq \int_1^n x^\xi dx \leq \frac{n^{\xi+1}}{\xi+1}$$

and we conclude

$$\frac{\sigma_n}{n} = \frac{1}{n}\sum_{k=1}^{n} k^\xi = \frac{1}{n} + \frac{n^\xi}{\xi+1} = O(1) \quad (n \to \infty).$$

$\square$

Remark 4.8. As we have seen in the proof of Theorem 4.3 i) for any real ξ the condition $1/\alpha \in \ell_\infty$ trivially implies condition (4.14).

Example 4.9. Taking $\xi = 1$ in Corollary 4.7 we deduce that for every $X \in cs$ we have
$$\frac{1}{n}\sum_{k=1}^{n} kr_k \to l \text{ implies } \frac{x_n}{\alpha_n} \to l' \ \ (n \to \infty) \tag{4.16}$$
for some l, $l' \in \mathbb{C}$ if and only if $1/\alpha \in \ell_\infty$.

4.3. A simplification of the previous results.

In this subsection we will characterize (4.1) and then rewrite Theorem 4.3 in each of the cases $\mu \in \widehat{C_1}$ and $\lambda \in \widehat{\Gamma}$.

Recall the definitions of the sets $\widehat{C_1}$ and $\widehat{\Gamma}$ defined in [3],
$$\widehat{C_1} = \left\{ X \in U^+ : \quad [C(X)X]_n = \frac{1}{x_n}\left(\sum_{k=1}^{n} x_k\right) = O(1) \ \ (n \to \infty) \right\}$$
and
$$\widehat{\Gamma} = \left\{ X \in U^+ : \quad \lim_{n\to\infty}\left(\frac{x_{n-1}}{x_n}\right) < 1 \right\}.$$
It can easily be seen that $\widehat{\Gamma} \subset \widehat{C_1}$ and note that for $a > 1$ we have $(a^n)_{n\geq 1} \in \widehat{\Gamma}$. By [3] if $X \in \widehat{C_1}$ there are $M > 0$ and $\gamma > 1$ such that
$$x_n \geq M\gamma^n \text{ for all } n.$$
From [4, Lemma 11, p. 49] we obtain the next lemma.

Lemma 4.10. *Let $\alpha \in U^+$. Then*

i) $\alpha \in \widehat{C_1}$ if and only if Σ is bijective from s_α^0 to itself,

ii) $\alpha \in \widehat{\Gamma}$ if and only if Σ is bijective from $s_\alpha^{(c)}$ to itself.

Theorem 4.3 can be reduced to the next corollaries.

Corollary 4.11. *Let $\mu \in \widehat{C}_1$.*

i) Let $\lambda \in U^+$ with $\lambda/\mu \in c_0$. Then condition (4.1) holds if and only if (4.5) and (4.6) hold.

ii) Let $\lambda \in U^+$ with $\mu/\lambda \in \ell_\infty$. Then condition (4.1) holds if and only if $1/\alpha \in \ell_\infty$.

Proof. Since $\mu \in \widehat{C}_1$ the operator Σ is bijective from s_μ^0 to itself and
$$C(\lambda)\, s_\mu^0 = D_{1/\lambda}\Sigma s_\mu^0 = D_{1/\lambda} s_\mu^0 = s_{\mu/\lambda}^0.$$
Now show i). We have $c \subset s_{\mu/\lambda}^0$ since $D_{\lambda/\mu} \in (c, c_0)$ which is equivalent to $\lambda/\mu \in c_0$. By Lemma 4.1 for every Y we have
$$Y \in c \bigcap C(\lambda)\, s_\mu^0 = c \text{ implies } \Delta^+ D_{1/\mu}\Delta(\lambda)\, Y \in s_\alpha^{(c)}$$
that is $\Delta^+ D_{1/\mu}\Delta(\lambda) \in \left(c, s_\alpha^{(c)}\right)$. As we have seen in the proof of Theorem 4.3 iii) this means that (4.5) and (4.6) hold.

ii) By Lemma 1 we have $D_{\mu/\lambda} \in (c_0, c)$ if and only if $\mu/\lambda \in \ell_\infty$ and then $s_{\mu/\lambda}^0 \subset c$. Then (4.1) means that $\Delta^+ D_{1/\mu}\Delta(\lambda)\, Y \in s_\alpha^{(c)}$ for all $Y \in c \bigcap C(\lambda) s^0_\mu = s^0_{\mu/\lambda}$, that is $\Delta^+ D_{1/\mu}\Delta(\lambda) \in \left(s^0_{\mu/\lambda}, s^{(c)}_\alpha\right)$ and $D_{1/\alpha}\Delta^+ D_{1/\mu}\Delta(\lambda) D_{\mu/\lambda} \in (c_0, c)$. Now since

$$\Delta^+ D_{1/\mu}\Delta(\lambda) D_{\mu/\lambda} = \Delta^+ D_{1/\mu}\Delta(\mu)$$

we have

$$D_{1/\alpha}\Delta^+ D_{1/\mu}\Delta(\mu) \in (c_0, c). \tag{4.17}$$

Using the calculation of $\widetilde{\Delta}$ explicited in (4.11) with $\lambda = \mu$ we deduce (3.1) is equivalent to

$$\sup_n \left\{ \frac{1}{\alpha_n} \left[\frac{\mu_{n-1}}{\mu_n} + \left(1 + \frac{\mu_n}{\mu_{n+1}}\right) + 1 \right] \right\} < \infty. \tag{4.18}$$

Now since $\mu \in \widehat{C_1}$ implies there is $M > 1$ such that $\mu_n^{-1} \sum_{k=1}^n \mu_k \leq M$ for all $n \geq 1$ and we successively obtain

$$\frac{\mu_{n-1}}{\mu_n} + \left(1 + \frac{\mu_n}{\mu_{n+1}}\right) + 1 \leq \frac{1}{\mu_n} \sum_{k=1}^n \mu_k + \frac{1}{\mu_{n+1}} \sum_{k=1}^{n+1} \mu_k + 1 \leq 2M + 1,$$

and

$$\frac{2}{\alpha_n} \leq \frac{1}{\alpha_n} \left[\frac{\mu_{n-1}}{\mu_n} + \left(1 + \frac{\mu_n}{\mu_{n+1}}\right) + 1 \right] \leq \frac{1}{\alpha_n}(2M+1) \text{ for all } n \geq 1,$$

thus (4.18) is equivalent to $1/\alpha \in \ell_\infty$. This concludes the proof. □

Now consider the following conditions,

$$\sup_n \left\{ \frac{1}{\alpha_n} \left(\frac{\lambda_n}{\mu_n} + \frac{\lambda_{n+1}}{\mu_{n+1}} \right) \right\} < \infty, \tag{4.19}$$

$$\sup_n \frac{1}{\alpha_n} \frac{\lambda_n}{\mu_n} < \infty, \tag{4.20}$$

$$\lim_{n\to\infty} \frac{1}{\alpha_n} \left(\frac{\lambda_n}{\mu_n} - \frac{\lambda_{n+1}}{\mu_{n+1}} \right) = \chi \text{ for some } \chi \in \mathbb{C}. \tag{4.21}$$

We can state the next corollary.

Corollary 4.12. *Let $\lambda \in \widehat{\Gamma}$, $\mu \in U^+$ and assume conditions of Theorem* 4.3 *iii) hold.*
Then condition (4.1) *holds with $l' = l(1-a)\chi$, ($a = \lim_{n\to\infty} \lambda_{n-1}/\lambda_n < 1$) if and only if α satisfies* (4.20) *and* (4.21)*.*

Proof. By conditions of Theorem 4.3 iii) we have $D_{1/\mu}\Delta(\lambda) \in (c, c_0)$ and $\Delta(\lambda) c \subset s^0_\mu$ and since $C(\lambda) = \Delta(\lambda)^{-1}$ we have $c \subset C(\lambda) s^0_\mu$. So (4.1) means that

$$X = \Delta^+ D_{1/\mu}\Delta(\lambda) Y \in s^{(c)}_\alpha \text{ for all } Y \in c,$$

that is $\Delta^+ D_{1/\mu}\Delta(\lambda) c \subset s^{(c)}_\alpha$. Now by Lemma 4.10 ii) $\lambda \in \widehat{\Gamma}$ implies $\Delta s^{(c)}_\lambda = s^{(c)}_\lambda$ and

$$\Delta^+ D_{1/\mu}\Delta(\lambda) c = \Delta^+ D_{1/\mu}\Delta s^{(c)}_\lambda = \Delta^+ D_{1/\mu} s^{(c)}_\lambda = \Delta^+ s^{(c)}_{\lambda/\mu}.$$

Then (4.1) is equivalent to $\Delta^+ \in \left(s^{(c)}_{\lambda/\mu}, s^{(c)}_{\alpha}\right)$ and to (4.19) and (4.21). By Lemma 4.2 where $\kappa_n = \lambda_n/\alpha_n\mu_n$ and $\kappa'_n = \lambda_{n+1}/\alpha_n\mu_{n+1}$ we deduce that $\Delta^+ \in \left(s^{(c)}_{\lambda/\mu}, s^{(c)}_{\alpha}\right)$ is equivalent to (4.20) and (4.21).

Now show $l' = l\,(1-a)\,\chi$. If $X \in cs$ and

$$\begin{aligned} X &= \Delta^+ D_{1/\mu}\Delta(\lambda)\, Y = \Delta^+ D_{1/\mu}\Delta D_\lambda Y \\ &= \Delta^+ D_{1/\mu} D_\lambda \left(D_{1/\lambda}\Delta D_\lambda\right) Y = \Delta^+ D_{\lambda/\mu}\left(D_{1/\lambda}\Delta D_\lambda\right) Y, \end{aligned}$$

and letting $\widehat{Y} = (\widehat{y}_n)_{n\geq 1} = \left(D_{1/\lambda}\Delta D_\lambda\right) Y$, we have

$$\widehat{y}_n = -\frac{\lambda_{n-1}}{\lambda_n} y_{n-1} + y_n.$$

Thus in particular if $Y = e$, then

$$\lim_{n\to\infty} \widehat{y}_n = \lim_{n\to\infty}\left(1 - \frac{\lambda_{n-1}}{\lambda_n}\right) = 1 - a.$$

And if $Y \in c_0$ then, clearly, $\widehat{Y} = \left(D_{1/\lambda}\Delta D_\lambda\right) Y \in c_0$. Consequently, if $Y \in c$ with $l = \lim_{n\to\infty} y_n$, then $\widehat{y}_n - l \to -al + l - l = -al$ $(n \to \infty)$. Then by (4.21), we obtain

$$\begin{aligned} \frac{x_n}{\alpha_n} &= \left(D_{1/\alpha}\Delta^+ D_{\lambda/\mu}\right)_n \left(le + \left(\widehat{Y} - le\right)\right) \\ &= \frac{1}{\alpha_n}\left(\frac{\lambda_n}{\mu_n} - \frac{\lambda_{n+1}}{\mu_{n+1}}\right) l + \frac{1}{\alpha_n}\left(\frac{\lambda_n}{\mu_n} - \frac{\lambda_{n+1}}{\mu_{n+1}}\right)(\widehat{y}_n - l) \to \chi l - \chi a l \ \ (n \to \infty). \end{aligned}$$

This concludes the proof. □

Example 4.13. As a direct application of the preceding we have

$$\frac{1}{(n-1)!}\sum_{k=1}^{n} k! r_k \to 0 \text{ implies } \frac{x_n}{\alpha_n} \to l' \ \ (n \to \infty) \ \text{ for all } X \in cs \tag{4.22}$$

if and only if there is $C > 0$ such that $\alpha_n \geq C/n$ for all n. Indeed conditions (4.20) and (4.21) mean that $\sup_n \{1/(n\alpha_n)\} < \infty$ and $\lim_{n\to\infty} 1/\left(n^2\alpha_n\right) = \chi$ for some scalar χ. It can easily be seen that $\sup_n \{1/(n\alpha_n)\} < \infty$ implies $\lim_{n\to\infty} 1/\left(n^2\alpha_n\right) = 0$. Since $\chi = 0$ we have $l' = 0$. This concludes the proof.

Example 4.14. In the same way it can easily be shown that for $1 < \mathbf{a} < \mathbf{b}$ and $\lim_{n\to\infty} \mathbf{a}^n/\mathbf{b}^n\alpha_n = L$, we then have

$$\mathbf{a}^{-n}\sum_{k=1}^{n} \mathbf{b}^k r_k \to l \text{ implies } \frac{x_n}{\alpha_n} \to l\left(1 - \frac{1}{\mathbf{a}}\right)\left(1 - \frac{\mathbf{a}}{\mathbf{b}}\right) L \ \ (n \to \infty)$$

for all $X \in cs$ if and only if $\left(\mathbf{a}^n/\left(\mathbf{b}^n\alpha_n\right)\right)_{n\geq 1} \in c$.

4.4. Study of the converse of tauberian results

For given $\alpha \in U^+$ we will determine the set of all λ, $\mu \in U^+$ such that

$$\frac{x_n}{\alpha_n} \to l \text{ implies } \frac{1}{\lambda_n}\sum_{k=1}^{n} \mu_k r_k \to l' \ (n \to \infty) \text{ for all } X \in cs \tag{4.23}$$

and give a characterization of (4.23).

We get the following theorem

Theorem 4.15. *Let λ, μ, $\alpha \in U^+$. Suppose $\alpha \in cs$. Then the sequences λ and μ satisfy condition (4.23) if and only if $1/\lambda \in c$ and*

$$\lim_{n\to\infty} \phi_n(\alpha) = L \text{ for some } L \in \mathbb{C}. \tag{4.24}$$

Proof. First we note that $\alpha \in cs$ if and only if $s_\alpha^{(c)} \subset cs$. Now condition (4.23) means that

$$X \in s_\alpha^{(c)} \bigcap cs = s_\alpha^{(c)} \text{ implies } \left(C(\lambda) D_\mu \Sigma^+\right) X = C(\lambda) D_\mu \left(\Sigma^+ X\right) \in c$$

by Lemma 3.2 which is equivalent to

$$C(\lambda) D_\mu \Sigma^+ D_\alpha \in (c, c). \tag{4.25}$$

We deduce from the proof of Theorem 3.5 (iv) that if we put $C(\lambda) D_\mu \Sigma^+ D_\alpha = (c_{nk})_{n,k\geq 1}$, then

$$c_{nk} = \begin{cases} \dfrac{\sigma_k}{\lambda_n}\alpha_k \text{ for } k < n, \\ \dfrac{\sigma_n}{\lambda_n}\alpha_k \text{ for } k \geq n. \end{cases}$$

So condition (4.25) is equivalent to $1/\lambda \in c$, (4.24) and

$$\sup_n \{\phi_n(\alpha)\} < \infty. \tag{4.26}$$

We conclude the proof since condition (4.24) implies condition (4.26). $\square$

Now to state the next result recall the following result due to Kizmaz.

Lemma 4.16. *([7]) Let $p = (p_n)_{n\geq 1}$ be a strictly increasing sequence. If $pX \in cs$ then $(p_n r_{n+1})_{n\geq 1} \in c_0$.*

Corollary 4.17. *Let $\xi > 0$ be a real, $\alpha \in U^+$ and assume $\left(n^{\xi+1}\alpha_n\right)_{n\geq 1} \in c$ and $\left(n^{\xi}\alpha_n\right)_{n\geq 1} \in cs$. Then*

$$\frac{x_n}{\alpha_n} \to l \textit{ implies } \frac{1}{n}\sum_{k=1}^{n} k^{\xi} r_k \to l' \ (n \to \infty).$$

for all $X \in cs$ and for some scalars l, l'.

Proof. We only have to apply Theorem 4.15. For this it suffices to show that

$$\frac{1}{n}\sum_{k=1}^{n} \sigma_k \alpha_k \to l_1 \text{ and } \frac{1}{n}\sigma_n \sum_{k=n+1}^{\infty} \alpha_k \to l_2$$

for some l_1, $l_2 \geq 0$ with $\sigma_n = \sum_{k=1}^{n} k^{\xi}$. First we have

$$\frac{n^{\xi+1}}{\xi+1} \leq \sigma_n \leq \frac{(n+1)^{\xi+1}-1}{\xi+1} \text{ for all } n$$

and then $\sigma_n \sim n^{\xi+1}/(\xi+1)$ $(n \to \infty)$. Since $n^{\xi+1}\alpha_n \to L$ $(n \to \infty)$ we deduce $(\sigma_n \alpha_n)_{n\geq 1} \in c$ and $\left(n^{-1}\sum_{k=1}^{n} \sigma_k \alpha_k\right)_{n\geq 1} \in c$. Then putting $p_n = \sigma_n/n$ we get

$$p_n \sim \frac{1}{n}\frac{n^{\xi+1}}{\xi+1} = \frac{n^{\xi}}{\xi+1} \ \ (n \to \infty)$$

and by Lemma 4.16 condition $\sum_{n=1}^{\infty} n^{\xi}\alpha_n < \infty$ implies

$$\frac{1}{n}\sigma_n \sum_{k=n+1}^{\infty} \alpha_k \to 0 \ \ (n \to \infty).$$

This concludes the proof. □

Example 4.18. Let $\gamma > 2$, then $n^{\gamma}x_n \to l$ implies $n^{-1}\sum_{k=1}^{n} kr_k \to l'$ $(n \to \infty)$ for all $X \in cs$.

Indeed it is enough to put $\xi = 1$ and $\alpha_n = n^{-\gamma}$.

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