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The double Orlicz sequence spaces $\chi_M^2(p)$ and $\Lambda_M^2(p)$

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Abstract. In this paper, we introduce two general double sequence spaces $\chi_M^2(p)$ and $\Lambda_M^2(p)$ using Orlicz functions. We establish some inclusion relations, topological results and we characterize the duals of these double sequence spaces.

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1. Introduction

Throughout w, χ and Λ denote the classes of gai and analytic scalar valued single sequences, respectively.

We write w^2 for the set of all complex sequences (x_{mn}), where $m, n \in \mathbb{N}$, the set of positive integers. Then, w^2 is a linear space under the coordinate wise addition and scalar multiplication.

Some initial works on double sequence spaces are due to Bromwich [4]. Later on, the double sequence spaces were studied by Hardy [8], Moricz [12], Moricz and Rhoades [13], Basarir and Solankan [2], Tripathy [20], Colak and Turkmenoglu [6], Turkmenoglu [22], and many others.

Let us define the following sets of double sequences:

$$\mathcal{M}_u(t) := \left\{ (x_{mn}) \in w^2 : \sup_{m,n \in N} |x_{mn}|^{p_{mn}} < \infty \right\},$$

$$\mathcal{C}_p(t) := \left\{ (x_{mn}) \in w^2 : p - \lim_{m,n \to \infty} |x_{mn} - \ell|^{p_{mn}} = 1 \text{ for some } \ell \in \mathbb{C} \right\},$$

$$\mathcal{C}_{0p}(t) := \left\{ (x_{mn}) \in w^2 : p - \lim_{m,n \to \infty} |x_{mn}|^{p_{mn}} = 1 \right\},$$

$$\mathcal{L}_u(t) := \left\{ (x_{mn}) \in w^2 : \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |x_{mn}|^{p_{mn}} < \infty \right\},$$

$$\mathcal{C}_{bp}(t) := \mathcal{C}_p(t) \bigcap \mathcal{M}_u(t) \text{ and } \mathcal{C}_{0bp}(t) = \mathcal{C}_{0p}(t) \bigcap \mathcal{M}_u(t),$$

where $p = (p_{mn})$ is the sequence of strictly positive reals p_{mn} for all $m, n \in \mathbb{N}$ and $p - \lim_{m,n\to\infty}$ denotes the limit in the Pringsheim's sense. In the case $p_{mn} = 1$ for all $m, n \in \mathbb{N}; \mathcal{M}_u(t), \mathcal{C}_p(t), \mathcal{C}_{0p}(t), \mathcal{L}_u(t), \mathcal{C}_{bp}(t)$ and $\mathcal{C}_{0bp}(t)$ reduce to the sets $\mathcal{M}_u, \mathcal{C}_p, \mathcal{C}_{0p}, \mathcal{L}_u, \mathcal{C}_{bp}$ and $\mathcal{C}_{0bp}$, respectively. Now, we may summarize the knowledge given in some document related to the double sequence spaces. Gökhan and Colak [27, 28] have proved that $\mathcal{M}_u(t)$ and $\mathcal{C}_p(t), \mathcal{C}_{bp}(t)$ are complete paranormed spaces of double sequences and gave the $\alpha-, \beta-, \gamma-$ duals of the spaces $\mathcal{M}_u(t)$ and $\mathcal{C}_{bp}(t)$. Quite recently, in her PhD thesis, Zeltser [29] has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [30] have recently introduced the statistical convergence and Cauchy for double sequences and given the relation between statistical convergent and strongly Cesàro summable double sequences. Next, Mursaleen [31] and Mursaleen and Edely [32] have defined the almost strong regularity of matrices for double sequences and applied these matrices to establish a core theorem and introduced the $M-$core for double sequences and determined those four dimensional matrices transforming every bounded double sequences $x = (x_{jk})$ into one whose core is a subset of the $M-$core of x. More recently, Altay and Basar [33] have defined the spaces $\mathcal{BS}, \mathcal{BS}(t), \mathcal{CS}_p, \mathcal{CS}_{bp}, \mathcal{CS}_r$ and $\mathcal{BV}$ of double sequences consisting of all double series whose sequence of partial sums are in the spaces $\mathcal{M}_u, \mathcal{M}_u(t), \mathcal{C}_p, \mathcal{C}_{bp}, \mathcal{C}_r$ and $\mathcal{L}_u$, respectively, and also examined some properties of those sequence spaces and determined the $\alpha-$ duals of the spaces $\mathcal{BS}, \mathcal{BV}, \mathcal{CS}_{bp}$ and the $\beta(\vartheta)-$ duals of the spaces $\mathcal{CS}_{bp}$ and $\mathcal{CS}_r$ of double series. Also Basar and Sever [34] have introduced the Banach space $\mathcal{L}_q$ of double sequences corresponding to the well-known space ℓ_q of single sequences and examined some properties of the space $\mathcal{L}_q$. Quite recently Subramanian and Misra [35] have studied the space $\chi^2_M(p, q, u)$ of double sequences and gave some inclusion relations.

We need the following inequality in the sequel of the paper. For $a, b, \geq 0$ and $0 < p < 1$, we have

$$(a+b)^p \leq a^p + b^p. \tag{1.1}$$

The double series $\sum_{m,n=1}^{\infty} x_{mn}$ is called convergent if and only if the double sequence (s_{mn}) is convergent, where $s_{mn} = \sum_{i,j=1}^{m,n} x_{ij} (m, n \in \mathbb{N})$ (see [1]).

A sequence $x = (x_{mn})$ is said to be double analytic if

$$\sup_{mn} |x_{mn}|^{1/m+n} < \infty.$$

The vector space of all double analytic sequences will be denoted by Λ^2. A sequence $x = (x_{mn})$ is called double gai sequence if $((m+n)! |x_{mn}|)^{1/m+n} \to 0$ as $m, n \to \infty$. The double gai sequences will be denoted by χ^2. By ϕ, we denote the set of all finite sequences.

Consider a double sequence $x = (x_{ij})$. The $(m,n)^{th}$ section $x^{[m,n]}$ of the sequence is defined by $x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{ij}\Im_{ij}$ for all $m,n \in \mathbb{N}$; where $\Im_{ij}$ denotes the double sequence whose only nonzero term is $\frac{1}{(i+j)!}$ in the $(i,j)^{th}$ place for each $i,j \in \mathbb{N}$.

An FK-space (or a metric space) X is said to have AK property if $(\Im_{mn})$ is a Schauder basis for X. Or equivalently $x^{[m,n]} \to x$.

An FDK-space is a double sequence space endowed with a complete metric; locally convex topology under which the coordinate mappings $x = (x_k) \to (x_{mn})(m,n \in \mathbb{N})$ are also continuous.

Orlicz [16] used the idea of Orlicz function to construct the space $\left(L^M\right)$. Lindenstrauss and Tzafriri [10] investigated Orlicz sequence spaces in more details, and they proved that every Orlicz sequence space ℓ_M contains a subspace isomorphic to $\ell_p\,(1 \le p < \infty)$. Subsequently, different classes of sequence spaces were defined by Parashar and Choudhary [17], Mursaleen et al. [14], Bektas and Altin [3], Tripathy et al. [21], Rao and Subramanian [18], and many others. The Orlicz sequence spaces are the special cases of Orlicz spaces studied in [9].

Recalling [16] and [9], an Orlicz function is a function $M : [0,\infty) \to [0,\infty)$ which is continuous, non-decreasing and convex, with $M(0) = 0$, $M(x) > 0$, for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. If convexity of Orlicz function M is replaced by subadditivity of M, then this function is called modulus function, defined by Nakano [15] and further discussed by Ruckle [19] and Maddox [11], and many others.

An Orlicz function M is said to satisfy the Δ_2- condition for all values of u if there exists a constant $K > 0$ such that $M(2u) \le KM(u)\,(u \ge 0)$. The Δ_2- condition is equivalent to $M(\ell u) \le K\ell M(u)$, for all values of u and for $\ell > 1$.

Lindenstrauss and Tzafriri [10] used the idea of Orlicz function to construct Orlicz sequence space

$$\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\},$$

The space ℓ_M with the norm

$$\|x\| = \inf\left\{\rho > 0 : \textstyle\sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \le 1\right\},$$

becomes a Banach space which is called an Orlicz sequence space. For $M(t) = t^p\,(1 \le p < \infty)$, the spaces ℓ_M coincide with the classical sequence space ℓ_p.

If X is a sequence space, we give the following definitions:

(i) $X^{'}$= the continuous dual of X;

(ii) $X^{\alpha} = \left\{a = (a_{mn}) : \sum_{m,n=1}^{\infty} |a_{mn}x_{mn}| < \infty, \text{ for each } x \in X\right\}$;

(iii) $X^{\beta} = \left\{a = (a_{mn}) : \sum_{m,n=1}^{\infty} a_{mn}x_{mn} \text{ is convegent, for each } x \in X\right\}$;

(iv) $X^{\gamma} = \left\{a = (a_{mn}) : \sup_{MN} \ge 1 \left|\sum_{m,n=1}^{M,N} a_{mn}x_{mn}\right| < \infty, \text{ for each } x \in X\right\}$;

(v) let X be a FK-space $\supset \phi$; then $X^f = \left\{f(\Im_{mn}) : f \in X^{'}\right\}$;

(vi) $X^{\delta}=\left\{a=(a_{mn}):\sup_{mn}|a_{mn}x_{mn}|^{1/m+n}<\infty,\text{ for each } x\in X\right\}$;
$X^{\alpha},X^{\beta},X^{\gamma}$ and X^{δ} are called $\alpha-$ (or Köthe-Toeplitz) dual of X, $\beta-$ (or generalized-Köthe-Toeplitz) dual of X, $\gamma-$ dual of X, $\delta-$ dual of X respectively. X^{α} is defined by Gupta and Kamptan [24]. It is clear that $X^{\alpha}\subset X^{\beta}$ and $X^{\alpha}\subset X^{\gamma}$, but $X^{\alpha}\subset X^{\gamma}$ does not hold, since the sequence of partial sums of a double convergent series need not to be bounded.

The notion of difference spaces of single sequences was introduced by Kizmaz [36] as follows

$$Z(\Delta)=\{x=(x_k)\in w:(\Delta x_k)\in Z\}$$

for $Z=c,c_0$ and ℓ_{∞}, where $\Delta x_k=x_k-x_{k+1}$ for all $k\in\mathbb{N}$. Here w,c,c_0 and ℓ_{∞} denote the classes of all, convergent, null and bounded scalar valued single sequences respectively. The above difference spaces are Banach spaces normed by

$$\|x\|=|x_1|+\sup_{k\geq 1}|\Delta x_k|.$$

Later on the notion was further investigated by many others. We now introduce the following difference double sequence spaces defined by

$$Z(\Delta)=\left\{x=(x_{mn})\in w^2:(\Delta x_{mn})\in Z\right\},$$

where $Z=\Lambda^2,\chi^2$ and $\Delta x_{mn}=(x_{mn}-x_{mn+1})-(x_{m+1n}-x_{m+1n+1})=x_{mn}-x_{mn+1}-x_{m+1n}+x_{m+1n+1}$ for all $m,n\in\mathbb{N}$.

This paper deals with various duals namely α,β,γ, complete paranormed space of $\Lambda^2_M(p)$ and paranormed space of $\chi^2_M(p)$ using Orlicz fuctions.

2. Definitions and preliminaries

Throughout the paper w^2 denotes the spaces of all sequences. $\chi^2_M(p)$ and $\Lambda^2_M(p)$ denote the Pringscheim's sense of double Orlicz space of gai sequences and Pringscheim's sense of double Orlicz space of bounded sequences respectively.

Let w^2 denote the set of all complex double sequences $x=(x_{mn})_{m,n=1}^{\infty}$ and $M:[0,\infty)\rightarrow[0,\infty)$ be an Orlicz function, or a modulus function.

Given a double sequence, $x\in w^2$. If $p=(p_{mn})$ is a double sequence of strictly positive real numbers p_{mn} then we write

$$\chi^2_M(p)=\left\{x\in w^2:\left(M\left(\frac{((m+n)!\,|x_{mn}|)^{1/m+n}}{\rho}\right)^{p_{mn}}\right)\rightarrow 0\right.$$
$$\left.\text{as } m,n\rightarrow\infty\text{ for some }\rho>0\right\}$$

and

$$\Lambda^2_M(p)=\left\{x\in w^2:\sup_{m,n\geq 1}\left(M\left(\frac{|x_{mn}|^{1/m+n}}{\rho}\right)^{p_{mn}}\right)<\infty\text{ for some }\rho>0\right\}.$$

The space $\Lambda^2_M(p)$ is a metric space with the metric

$$d(x,y)=\inf\left\{\rho>0:\sup_{m,n\geq 1}\left(M\left(\frac{|x_{mn}-y_{mn}|}{\rho}\right)\right)^{p_{mn}/m+n}\leq 1\right\}.$$

The space $\chi^2_M(p)$ is a metric space with the metric

$$d(x,y)=\inf\left\{\rho>0:\sup_{m,n\geq 1}\left(M\left(\frac{(m+n)!\,|x_{mn}-y_{mn}|}{\rho}\right)\right)^{p_{mn}/m+n}\leq 1\right\}.$$

Throughout the paper we write $\inf p_{mn}$, $\sup p_{mn}$ and $\sum_{mn}$ instead of $\inf_{m,n\geq 1}$, $\sup_{m,n\geq 1}$ and $\sum_{m,n=1}^{\infty}$ respectively.

3. Main results

Theorem 3.1. *For every $p=(p_{mn})$,*

$$\left[\Lambda^2_M(p)\right]^{\beta}=\left[\Lambda^2_M(p)\right]^{\alpha}=\left[\Lambda^2_M(p)\right]^{\gamma}=\eta^2_M(p),$$

where $\eta^2_M(p)=\bigcap_{N\in N-\{1\}}\left\{x=x_{mn}:\sum_{m,n}\left(M\left(\frac{|x_{mn}|N^{m+n/p_{mn}}}{\rho}\right)\right)<\infty\right\}.$

Proof. (1) First we show that $\eta^2_M(p)\subset\left[\Lambda^2_M(p)\right]^{\beta}$.

Let $x\in\eta^2_M(p)$ and $y\in\Lambda^2_M(p)$. Then we can find a positive integer N such that $\left(|y_{mn}|^{1/m+n}\right)^{p_{mn}}<max\left(1,\sup_{m,n\geq 1}\left(|y_{mn}|^{1/m+n}\right)^{p_{mn}}\right)<N$, for all m,n.

Hence we may write

$$\left|\sum_{m,n}x_{mn}y_{mn}\right|\leq\sum_{m,n}|x_{mn}y_{mn}|\leq\sum_{mn}\left(M\left(\frac{|x_{mn}y_{mn}|}{\rho}\right)\right)$$
$$\leq\sum_{m,n}\left(M\left(\frac{|x_{mn}|\,N^{m+n/p_{mn}}}{\rho}\right)\right).$$

Since $x\in\eta^2_M(p)$ the series on the right side of the above inequality is convergent, whence $x\in\left[\Lambda^2_M(p)\right]^{\beta}$. Hence $\eta^2_M(p)\subset\left[\Lambda^2_M(p)\right]^{\beta}$.

Now we show that $\left[\Lambda^2_M(p)\right]^{\beta}\subset\eta^2_M(p)$.

For this, let $x\in\left[\Lambda^2_M(p)\right]^{\beta}$, and suppose that $x\notin\Lambda^2_M(p)$. Then there exists a positive integer $N>1$ such that $\sum_{m,n}\left(M\left(\frac{|x_{mn}|N^{m+n/p_{mn}}}{\rho}\right)\right)=\infty$.

If we define $y_{mn}=N^{m+n/p_{mn}}Sgnx_{mn}\,m,n=1,2,\cdots$, then $y\in\Lambda^2_M(p)$.

But, since

$$\left|\sum_{m,n}x_{mn}y_{mn}\right|=\sum_{mn}\left(M\left(\frac{|x_{mn}y_{mn}|}{\rho}\right)\right)$$
$$=\sum_{m,n}\left(M\left(\frac{|x_{mn}|\,N^{m+n/p_{mn}}}{\rho}\right)\right)=\infty,$$

we get $x \notin \left[\Lambda^2_M(p)\right]^\beta$, which contradicts to the assumption $x \in \left[\Lambda^2_M(p)\right]^\beta$. Therefore $x \in \eta^2_M(p)$. Hence $\left[\Lambda^2_M(p)\right]^\beta = \eta^2_M(p)$.

(ii) and (iii) can be shown in a similar way with (i). □

Theorem 3.2. *Let $p = (p_{mn})$ be an analytic double sequence of strictly positive real numbers p_{mn}. Then*

(i) $\Lambda^2_M(p)$ is a paranormed space with

$$g(x) = \sup_{m,n\geq 1}\left(M\left(\frac{|x_{mn}|^{1/m+n}}{\rho}\right)^{p_{mn}/M}\right)$$

if and only if $h = \inf p_{mn} > 0$, where $M = max(1, H)$ and $H = \sup p_{mn}$.

(ii) $\Lambda^2_M(p)$ is a complete paranormed linear metric space if the condition p in (i) is satisfied.

Proof. (i) **Sufficiency.** Let $h > 0$. It is trivial that $g(\theta) = 0$ and $g(-x) = g(x)$. The inequality $g(x+y) \leq g(x) + g(y)$ follows from the inequality (1.1), since $p_{mn}/M \leq 1$ for all positive integers m, n. We also may write $g(\lambda x) \leq max\left(|\lambda|, |\lambda|^{h/M}\right) g(x)$, since $|\lambda|^{p_{mn}} \leq max\left(|\lambda|^h, |\lambda|^M\right)$ for all positive integers m, n and for any $\lambda \in C$, the set of complex numbers. Using this inequality, it can be proved that $\lambda x \to \theta$, when x is fixed and $\lambda \to 0$, or $\lambda \to 0$ and $x \to \theta$, or λ is fixed and $x \to \theta$.

Necessity. Let $\Lambda^2_M(p)$ be a paranormed space with the paranorm

$$g(x) = \sup_{m,n\geq 1}\left(M\left(\frac{|x_{mn}|^{1/m+n}}{\rho}\right)^{p_{mn}/M}\right)$$

and suppose that $h = 0$. Since $|\lambda|^{p_{mn}/M} \leq |\lambda|^{h/M} = 1$ for all positive integers m, n and $\lambda \in C$ such that $0 < |\lambda| \leq 1$, we have

$$\sup_{m,n\geq 1}\left(M\left(\frac{|\lambda|^{p_{mn}/M}}{\rho}\right)\right) = 1.$$

Hence it follows that $g(\lambda x) = \sup_{m,n\geq 1}\left(M\left(\frac{|\lambda|^{p_{mn}/M}}{\rho}\right)\right) = 1$ for $x = (\alpha) \in \Lambda^2_M(p)$ as $\lambda \to 0$. But this contradicts the assumption $\Lambda^2_M(p)$ is a paranormed space with $g(x)$.

(ii) The proof is clear. □

Corollary 3.3. *$\Lambda^2_M(p)$ is a complete paranormed space with the natural paranorm if and only if $\Lambda^2_M(p) = \Lambda^2_M$.*

Theorem 3.4. *Let*

$$N_1 = min\left\{n_0 : \sup_{m,n\geq n_0}\left(M\left(\frac{((m+n)!\,|x_{mn}|)^{1/m+n}}{\rho}\right)^{p_{mn}}\right) < \infty\right\},$$

$$N_2 = min\left\{n_0 : \sup_{m,n\geq n_0} p_{mn} < \infty\right\} \text{ and } N = max(N_1, N_2).$$

(i) $\chi_M^2(p)$ *is a paranormed space with*

$$g(x) = \lim_{N\to\infty} \sup_{m,n\geq N} \left(M \left(\frac{((m+n)!\,|x_{mn}|)^{1/m+n}}{\rho} \right)^{p_{mn}/M} \right) \tag{3.1}$$

if and only if $\mu > 0$, *where*

$$\mu = \lim_{N\to\infty} \inf_{m,n\geq N} p_{mn} \text{ and } M = max\left(1, \sup_{m,n\geq N} p_{mn}\right).$$

(ii) $\chi_M^2(p)$ *is complete with the paranorm (3.1).*

Proof. (i) **Necessity.** Let $\chi_M^2(p)$ be a paranormed space with (3.1) and suppose that $\mu = 0$.
Then $\alpha = \inf_{m,n\geq N} p_{mn} = 0$ for all $N \in \mathbb{N}$, and hence we obtain $g(\lambda x) = \lim_{N\to\infty} \sup_{m,n\geq N} |\lambda|^{p_{mn}/M} = 1$ for all $\lambda \in (0,1]$, where $x = (\alpha) \in \chi_M^2(p)$. Whence $\lambda \to 0$ does not imply $\lambda x \to \theta$, when x is fixed. But this contradicts (3.1) to be a paranorm.

Sufficiency. Let $\mu > 0$. It is trivial that $g(\theta) = 0, g(-x) = g(x)$ and $g(x+y) \leq g(x) + g(y)$. Since $\mu > 0$ there exists a positive number β such that $p_{mn} > \beta$ for sufficiently large positive integer m, n. Hence for any $\lambda \in \mathbb{C}$, we may write $|\lambda|^{p_{mn}} \leq max\left(|\lambda|^M, |\lambda|^\beta\right)$ for sufficiently large positive integers $m, n \geq N$. Therefore, we obtain that $g(\lambda x) \leq max\left(|\lambda|, |\lambda|^{\beta/M}\right) g(x)$ using this, one can prove that $\lambda x \to \theta$, whenever x is fixed and $\lambda \to 0$, or $\lambda \to 0$ and $x \to \theta$, or λ is fixed and $x \to \theta$.

(ii) Let $\left(x^{k\ell}\right)$ be a Cauchy sequence in $\chi_M^2(p)$, where

$$x^{k\ell} = \left(x_{mn}^{k\ell}\right)_{mn\in N}.$$

Then for every $\epsilon > 0\,(0 < \epsilon < 1)$ there exists a positive integer s_0 such that

$$g\left(x^{k\ell} - x^{rt}\right) = \lim_{N\to\infty} \sup_{m,n\geq N} \left(M \left(\frac{\left((m+n)!\,\left|x_{mn}^{k\ell} - x_{mn}^{rt}\right|\right)^{1/m+n}}{\rho} \right)^{p_{mn}/M} \right) < \epsilon/2 \text{ for all } k, \ell, r, t > s_0. \tag{3.2}$$

By (3.2) there exists a positive integer n_0 such that

$$\sup_{m,n\geq N} \left(M \left(\frac{\left((m+n)!\,\left|x_{mn}^{k\ell} - x_{mn}^{rt}\right|\right)^{1/m+n}}{\rho} \right)^{p_{mn}/M} \right) < \epsilon/2$$

for all $k, \ell, r, t > s_0$ and for $N > n_0$. Hence we obtain

$$\left(M \left(\frac{\left((m+n)!\,\left|x_{mn}^{k\ell} - x_{mn}^{rt}\right|\right)^{1/m+n}}{\rho} \right)^{p_{mn}/M} \right) < \epsilon/2 < 1 \tag{3.3}$$

so that

$$\left(M\left(\frac{\left((m+n)!\left|x_{mn}^{k\ell}-x_{mn}^{rt}\right|\right)^{1/m+n}}{\rho}\right)\right)$$

$$<\left(M\left(\frac{\left((m+n)!\left|x_{mn}^{k\ell}-x_{mn}^{rt}\right|\right)^{1/m+n}}{\rho}\right)^{p_{mn}/M}\right)<\epsilon/2$$

for all $k,\ell,r,t,>s_0$. This implies that $\left(x_{mn}^{k\ell}\right)_{k\ell\in N}$ is a Cauchy sequence in C for each fixed $m,n>n_0$. Hence the sequence $\left(x_{mn}^{k\ell}\right)_{k\ell\in N}$ is convergent to x_{mn} say,

$$\lim_{k,\ell\to\infty} x_{mn}^{k\ell}=x_{mn} \text{ for each fixed } m,n>n_0 \tag{3.4}$$

Getting x_{mn}, we define $x=(x_{mn})$. From (3.3) we obtain

$$g\left(x^{k\ell}-x\right)$$

$$=\lim_{N\to\infty}\sup_{m,n\geq N}\left(M\left(\frac{\left((m+n)!\left|x_{mn}^{k\ell}-x_{mn}\right|\right)^{1/m+n}}{\rho}\right)^{p_{mn}/M}\right)<\epsilon/2 \tag{3.5}$$

as $r,t\to\infty$, for $k,\ell>s_0$ by (3.5). This implies that $\lim_{k\ell\to\infty} x^{k\ell}=x$.

Now we show that $x=(x_{mn})\in\chi_M^2(p)$. Since $x^{k\ell}\in\chi_M^2(p)$ for each $(k,1)\in N\times N$ for every $\epsilon>0\,(0<\epsilon<1)$ there exists a positive integer $n_1\in N$ such that

$$\left(M\left(\frac{\left((m+n)!\left|x_{mn}^{k\ell}\right|\right)^{1/m+n}}{\rho}\right)^{p_{mn}/M}\right)<\epsilon/2 \text{ for every } m,n>n_1. \tag{3.6}$$

By (3.5) and (3.6) and (3.1) we obtain

$$\left(M\left(\frac{\left((m+n)!\left|x_{mn}\right|\right)^{1/m+n}}{\rho}\right)^{p_{mn}/M}\right)$$

$$\leq\left(M\left(\frac{\left((m+n)!\left|x_{mn}^{k\ell}\right|\right)^{1/m+n}}{\rho}\right)^{p_{mn}/M}\right)$$

$$+\left(M\left(\frac{\left((m+n)!\left|x_{mn}^{k\ell}-x_{mn}\right|\right)^{1/m+n}}{\rho}\right)^{p_{mn}/M}\right)$$

$$\leq\epsilon/2+\epsilon/2=\epsilon$$

for all $k,\ell>max\,(s_0,s_1)$ and $m,n>max\,(n_0,n_1)$. This implies that $x\in\chi_M^2(p)$. This completes the proof. □

Theorem 3.5. *For every* $p=(p_{mn})$, *then* $\eta_M^2(p)\subset\left[\chi_M^2(p)\right]^\beta \overset{\subset}{\neq}\Lambda^2$.

Proof. **Case 1.** First we show that $\eta^2_M(p) \subset \left[\chi^2_M(p)\right]^\beta$.
We know that $\chi^2(p) \subset \Lambda^2_M(p)$, $\left[\Lambda^2_M(p)\right]^\beta \subset \left[\chi^2_M(p)\right]^\beta$.
But $\left[\Lambda^2_M(p)\right]^\beta = \eta^2_M(p)$, by Theorem 3.1.
Therefore

$$\eta^2_M(p) \subset [\chi^2_M(p)]^\beta. \tag{3.7}$$

Case 2. Now we show that $\left[\chi^2_M(p)\right]^\beta \underset{\neq}{\subset} \Lambda^2$.
Let $y = \{y_{mn}\}$ be an arbitrary point in $\left(\chi^2_M(p)\right)^\beta$. If y is not in Λ^2, then for each natural number q, we can find an index $m_q n_q$ such that

$$\left(M\left(\frac{\left((m_q+n_q)!\left|y_{m_q n_q}\right|\right)^{1/m_q+n_q}}{\rho}\right)\right)^{p_{mn}} > q, (1, 2, 3, \cdots).$$

Define $x = \{x_{mn}\}$ by $\left(M\left(\frac{(m+n)!x_{mn}}{\rho}\right)\right)^{p_{mn}} = \frac{1}{q^{m+n}}$ for $(m,n) = (m_q, n_q)$ for some $q \in \mathbb{N}$; and $\left(M\left(\frac{(m+n)!x_{mn}}{\rho}\right)\right)^{p_{mn}} = 0$ otherwise.
Then x is in $\chi^2_M(p)$, but for infinitely mn,

$$\left(M\left(\frac{(m+n)!\left|y_{mn}x_{mn}\right|}{\rho}\right)\right)^{p_{mn}} > 1. \tag{3.8}$$

Consider the sequence $z = \{z_{mn}\}$, where

$$\left(M\left(\frac{2!z_{11}}{\rho}\right)\right)^{p_{mn}} = \left(M\left(\frac{2!x_{11}}{\rho}\right)\right)^{p_{mn}} - s$$

with

$$s = \sum\left(M\left(\frac{(m+n)!x_{mn}}{\rho}\right)\right)^{p_{mn}};$$

and

$$\left(M\left(\frac{(m+n)!z_{mn}}{\rho}\right)\right)^{p_{mn}} = \left(M\left(\frac{(m+n)!x_{mn}}{\rho}\right)\right)^{p_{mn}} (m,n = 1, 2, 3, \cdots).$$

Then z is a point of $\chi^2_M(p)$. Also $\sum\left(M\left(\frac{(m+n)!z_{mn}}{\rho}\right)\right)^{p_{mn}} = 0$. Hence z is in $\chi^2_M(p)$.
But, by the equation (3.8), $\sum\left(M\left(\frac{(m+n)!z_{mn}y_{mn}}{\rho}\right)\right)^{p_{mn}}$ does not converge.
$\Rightarrow \sum(m+n)!x_{mn}y_{mn}$ diverges.
Thus the sequence y would not be in $\left(\chi^2_M(p)\right)^\beta$. This contradiction proves that

$$\left(\chi^2_M(p)\right)^\beta \subset \Lambda^2. \tag{3.9}$$

If we now choose $p = (p_{mn})$ constant, $M = id$, where id is the identity and $(1+n)!y_{1n} = (1+n)!x_{1n} = 1$ and $(m+n)!y_{mn} = (m+n)!x_{mn} = 0$ $(m > 1)$ for all n, then obviously $x \in \chi^2_M(p)$ and $y \in \Lambda^2$, but

$$\sum_{m,n=1}^{\infty}(m+n)!x_{mn}y_{mn} = \infty,$$

hence

$$y \notin \left(\chi^2_M(p)\right)^{\beta}. \tag{3.10}$$

From (3.9) and (3.10) we are granted

$$\left(\chi^2_M(p)\right)^{\beta} \overset{\subset}{\neq} \Lambda^2. \tag{3.11}$$

Hence (3.7) and (3.11) we are granted $\eta^2_M(p) \subset \left[\chi^2_M(p)\right]^{\beta} \overset{\subset}{\neq} \Lambda^2$. This completes the proof. □

Theorem 3.6. *Let M be an Orlicz function or modulus function which satisfies the Δ_2-condition. Then $\chi^2(p) \subset \chi^2_M(p)$.*

Proof. Let

$$x \in \chi^2(p). \tag{3.12}$$

Then $\left(((m+n)!\,|x_{mn}|)^{1/m+n}\right)^{p_{mn}} \leq \epsilon$ for sufficiently large m, n and every $\epsilon > 0$.
But then by taking $\rho \geq 1/2$,

$$\left(M\left(\frac{((m+n)!\,|x_{mn}|)^{1/m+n}}{\rho}\right)\right)^{p_{mn}} \leq \left(M\left(\frac{\epsilon}{\rho}\right)\right)$$

(because M is non-decreasing)

$$\leq (M(2\epsilon))$$

$$\Rightarrow \left(M\left(\frac{((m+n)!\,|x_{mn}|)^{1/m+n}}{\rho}\right)\right)^{p_{mn}} \leq KM(\epsilon)$$

(by the Δ_2- condition, for some $k > 0$)

$$\leq \epsilon$$

(by defining $M(\epsilon) < \epsilon/K$)

$$\left(M\left(\frac{((m+n)!\,|x_{mn}|)^{1/m+n}}{\rho}\right)\right)^{p_{mn}} \to 0 \text{ as } m, n \to \infty. \tag{3.13}$$

Hence

$$x \in \chi^2_M(p). \tag{3.14}$$

From (3.12) and (3.14) we get $\chi^2(p) \subset \chi^2_M(p)$. This completes the proof. □

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