Stud. Univ. Babeş-Bolyai Math. 56(2011), No. 4, 71–80

A class of uniformly convex functions involving a differential operator

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Abstract. The main purpose of this paper is to introduce a new class $\mathcal{UH}(\alpha,\ \beta, \gamma,\ \ \lambda,\ k)$, of functions which are analytic in the open disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. We obtain various results including characterization, coefficients estimates, distortion and covering theorems, radii of close-to-convexity, starlikeness and convexity for functions belonging to the class $\mathcal{UH}(\alpha,\ \beta, \gamma,\ \ \lambda,\ k)$.

Mathematics Subject Classification (2010): 30C45.

Keywords: Analytic function, starlike function, convex function, uniformly convex function, convolution product, Cho-Srivastava operator.

1. Introduction and motivations

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

that are analytic in the open unit disc $\Delta := \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{S}$ be a subclass of $\mathcal{A}$ consisting of univalent functions in Δ. By $\mathcal{K}(\beta)$, and $\mathcal{S}^*(\beta)$ respectively, we mean the classes of analytic functions that satisfy the analytic conditions

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \beta \quad \text{and} \quad \operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \beta, \quad z \in \Delta$$

for $0 \leqq \beta < 1$. In particular, $\mathcal{K} = \mathcal{K}(0)$ and $\mathcal{S}^* = \mathcal{S}^*(0)$ respectively, are the well-known standard class of convex and starlike functions.

The function $f \in \mathcal{A}$ is said to be close-to-convex of order β, $\beta \geqq 0$, with respect to a starlike function g and $\phi \in \mathbb{R}$ if

$$\left|\arg e^{i\phi}\frac{f(z)}{g(z)}\right| \leqq \beta\frac{\pi}{2}, \quad z \in \Delta.$$

Let $\mathcal{CC}(\beta)$ denote the union of all such close-to-convex functions of order β.

Let $\mathcal{T}$ denote the subclass of $\mathcal{S}$ of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geqq 0, \tag{1.1}$$

that are analytic in the open unit disk Δ. This class was introduced and studied in [9]. Analogous to the subclasses $\mathcal{S}^*(\beta)$ and $\mathcal{K}(\beta)$ of $\mathcal{S}$ respectively, the subclasses of $\mathcal{T}$ denoted by $\mathcal{T}^*(\beta)$ and $\mathcal{C}(\beta)$, $0 \leqq \beta < 1$, were also investigated in [9].

The main class which we investigate in this present paper uses the operator known as the Cho-Srivastava operator. In fact, One important concept that is useful in discussing this operator is the convolution or Hadamard product. Here by convolution we mean the following: For f, g analytic with $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots$ and $g(z) = b_0 + b_1 z + b_2 z^2 + \cdots$, the (Hadamard) convolution of f and g is defined by $(f * g)(z) = a_0 b_0 + a_1 b_1 z + a_2 b_2 z^2 + \cdots$. It is natural to use the notation $f(z) * g(z)$ for $(f * g)(z)$ and vice versa frequently.

For functions $f \in \mathcal{A}$, we recall the multiplier transformation $I(\lambda, k)$ introduced by Cho and Srivastava [3] defined as

$$I(\lambda, k)f(z) = z + \sum_{n=2}^{\infty} \Psi_n a_n z^n \quad (\lambda \geqq 0;\ k \in \mathbb{Z}) \tag{1.2}$$

where

$$\Psi_n := \left(\frac{n+\lambda}{1+\lambda}\right)^k \tag{1.3}$$

so that, obviously,

$$I(\lambda, k)\left(I(\lambda, m)\ f(z)\right) = I(\lambda, k+m)\ f(z) \quad (k, m \in \mathbb{Z})\,. \tag{1.4}$$

For $\lambda = 1$, the operators $I(\lambda, k)$ were studied by Uralegaddi and Somanatha [12]. The operators $I(\lambda, k)$ are closely related to the multiplier transformations studied by Flett [4] and also to the differential and integral operators investigated by Sălăgean [7]. For a detailed analysis of various convolution operators, which are related to the multiplier transformations of Flett [4], refer the work of Li and Srivastava [5] (as well as the references cited by them). Now we define an unified class of analytic function based on this operator.

Definition 1.1. *For* $0 \leqq \gamma \leqq 1,\ \ 0 \leqq \beta < 1,\ \ \alpha \geqq 0$, *and for all* $z \in \Delta$, *we let the class* $\mathcal{UH}(\alpha,\ \beta, \gamma,\ \lambda,\ k)$, *consists of functions* $f \in \mathcal{T}$ *is said to be in the class satisfying the condition*

$$\operatorname{Re}\left\{\frac{zF'(z)}{F(z)}\right\} > \alpha \left|\frac{zF'(z)}{F(z)} - 1\right| + \beta, \tag{1.5}$$

with,

$$F(z) := \gamma(1+\lambda)I(\lambda, k+1)f(z) + (1 - \gamma(1+\lambda))I(\lambda, k)f(z), \tag{1.6}$$

where $I(\lambda, k)f(z)$ *is the Cho-Srivastava operator as defined by* (1.2)

The family $\mathcal{UH}(\alpha,\ \beta, \gamma,\ \ \lambda,\ k)$, unifies various well known classes of analytic univalent functions. We list a few of them. The class $\mathcal{UH}(2,\ 1,\ \lambda,\ \beta,\ 0)$ studied in [1]. Many classes including $\mathcal{UH}(2,\ 1,\ 0,\ \beta,\ 0)$ and $\mathcal{UH}(2,\ 1,\ 1,\ \beta,\ 0)$ given in [11], are particular cases of this class. Further that, the class $\mathcal{UH}(2, 1,\ \lambda,\ 0,\ \beta,\ k)$ is the class of $k-$uniformly convex of order β, was introduced and studied in [10] (also see [2]).

In this present paper, we obtain a characterization, coefficients estimates, distortion theorem and covering theorem, extreme points and radii of close-to-convexity, starlikeness and convexity for functions belonging to the class $\mathcal{UH}(\alpha,\ \beta, \gamma,\ \ \lambda,\ k)$,.

2. Characterization and coefficient estimates

Theorem 2.1. *Let* $f \in \mathcal{T}$. *Then* $f \in \mathcal{UH}(\alpha,\ \beta, \gamma,\ \lambda,\ k)$, $\quad 0 \leqq \gamma \leqq 1,\ \ 0 \leqq \beta < 1$ *and* $\alpha \geqq 0$,

$$\sum_{n=2}^{\infty} \left[n(\alpha+1) - (\alpha+\beta)\right] \left(\gamma(n-1)+1\right) \Psi_n |a_n| \leqq 1-\beta. \tag{2.1}$$

This result is sharp for the function

$$f(z) = z - \frac{1-\beta}{[n(\alpha+1)-(\alpha+\beta)][\gamma(n-1)+1]\Psi_n}\, z^n \ \ n \geqq 2. \tag{2.2}$$

Proof. We employ the technique adopted by [2]. We have

$$f \in \mathcal{UH}(\alpha,\ \beta,\ \gamma,\ \lambda,\ k),$$

if and only if the condition (1.5) is satisfied, which is equivalent to

$$\operatorname{Re}\left\{\frac{zF'(z)(1+ke^{i\theta}) - F(z)ke^{i\theta}}{F(z)}\right\} > \beta, \quad -\pi \leqq \theta < \pi. \tag{2.3}$$

Now, letting $G(z) = zF'(z)(1+ke^{i\theta}) - F(z)ke^{i\theta}$, equation (2.3) is equivalent to

$$|G(z) + (1-\beta)F(z)| > |G(z) - (1+\beta)F(z)|,\ 0 \leqq \beta < 1.$$

where $F(z)$ is as defined in (1.6). Now a simple computation gives

$$\begin{aligned} &|G(z) + (1-\beta)F(z)| \\ &\quad \geqq \ (2-\beta)|z| - \sum_{n=2}^{\infty} \Big(n(\alpha+1) - (\alpha+\beta) + 1\Big) \Big(\gamma(n-1)+1\Big) \Psi_n a_n |z|^n \end{aligned}$$

and similarly,

$$\begin{aligned} &|G(z) - (1+\beta)F(z)| \\ &\quad \leqq \beta|z| + \sum_{n=2}^{\infty} \Big((n(\alpha+1) - (\alpha+\beta) - 1)\Big) \Big(\gamma(n-1)+1\Big) \Psi_n a_n |z|^n. \end{aligned}$$

Therefore,

$$|G(z) + (1-\beta)F(z)| - |G(z) - (1+\beta)F(z)|$$

$$\geqq 2(1-\beta)|z| -2\sum_{n=2}^{\infty}\Big((n(\alpha+1)-(\alpha+\beta))\Big)\,(\gamma(n-1)+1)\,\Psi_n a_n|z|^n \geqq 0,$$

which is equivalent to the result (2.1).

On the other hand, for all $-\pi \leqq \theta < \pi$, we must have

$$\operatorname{Re}\left\{\frac{zF'(z)}{F(z)}(1+ke^{i\theta}) - ke^{i\theta}\right\} > \beta.$$

Now, choosing the values of z on the positive real axis, where $0 \leqq |z| = r < 1$, and using $\operatorname{Re}\{-e^{i\theta}\} \geqq -|e^{i\theta}| = -1$, the above inequality can be written as

$$\operatorname{Re}\left\{\frac{(1-\beta) - \sum_{n=2}^{\infty}\Big(n(\alpha+1)-(\alpha+\beta)\Big)\Big(\gamma(n-1)+1\Big)\,\Psi_n a_n r^{n-1}}{1-\sum_{n=2}^{\infty}\Big(\gamma(n-1)+1\Big)\,\Psi_n a_n r^{n-1}}\right\} \geqq 0.$$

Setting $r \to 1^{-}$, we get the desired result. $\square$

Many known results can be obtained as particular cases of Theorem 2.1. For details, we refer to [6, 8].

By taking $\alpha = 0, \gamma = 1, \lambda = 0$ and $k = 1$ in Theorem 2.1, we get the following interesting result given in [9].

Corollary 2.2. [9] *If $f \in \mathcal{T}$, then $f \in \mathcal{C}(\beta)$ if and only if*

$$\sum_{n=2}^{\infty} n(n-\beta)a_n \leqq 1-\beta.$$

Indeed, since $f \in \mathcal{UH}(\alpha,\ \beta,\ \gamma,\ \lambda,\ k)$, (2.1), we have

$$\sum_{n=2}^{\infty}\Big(n(\alpha+1)-(\alpha+\beta)\Big)\Big(\gamma(n-1)+1\Big)\,\Psi_n a_n \leqq 1-\beta.$$

Hence for all $n \geqq 2$, we have

$$a_n \leqq \frac{1-\beta}{\Big(n(\alpha+1)-(\alpha+\beta)\Big)\Big(\gamma(n-1)+1\Big)\,\Psi_n},$$

whenever $0 \leqq \gamma \leqq 1,\ \ 0 \leqq \beta < 1$ and $\alpha \geqq 0$. Hence we state this important observation as a separate theorem.

Theorem 2.3. *If $f \in \mathcal{UH}(q, s, \lambda, \beta, k)$, then*

$$a_n \leqq \frac{1-\beta}{\Big(n(\alpha+1)-(\alpha+\beta)\Big)\Big(\gamma(n-1)+1\Big)\,\Psi_n},\ \ n \geqq 2, \tag{2.4}$$

where $0 \leqq \gamma \leqq 1,\ \ 0 \leqq \beta < 1$ and $\alpha \geqq 0$. Equality in (2.4) holds for the function

$$f(z) = z - \frac{1-\beta}{\Big(n(\alpha+1)-(\alpha+\beta)\Big)\Big(\gamma(n-1)+1\Big)\,\Psi_n}. \tag{2.5}$$

This theorem also contains many known results for the special values of the parameters. For example, see [6, 8].

3. Distortion and covering theorems

Theorem 3.1. *If* $f \in \mathcal{UH}(\alpha, \beta, \gamma, \lambda, k)$, *then* $f \in \mathcal{T}^*(\delta)$, *where*

$$\delta = 1 - \frac{1-\beta}{\big(2(\alpha+1) - (\alpha+\beta)\big)\big(\gamma+1\big)\Psi_2 - (1-\beta)}.$$

This result is sharp with the extremal function being

$$f(z) = z - \frac{1-\beta}{\big(2(\alpha+1) - (\alpha+\beta)\big)\big(\gamma+1\big)\Psi_2} z^2.$$

Proof. It is sufficient to show that (2.1) implies $\sum_{n=2}^{\infty}(n-\delta)a_n \leqq 1-\delta$ [9], that is,

$$\frac{n-\delta}{1-\delta} \leqq \frac{\big(n(\alpha+1) - (\alpha+\beta)\big)\big(\gamma(n-1)+1\big)\Psi_n}{1-\beta}, \quad n \geqq 2. \tag{3.1}$$

Since, for $n \geqq 2$, (3.1) is equivalent to

$$\delta \leqq 1 - \frac{(n-1)(1-\beta)}{\big(n(\alpha+1) - (\alpha+\beta)\big)\big(\gamma(n-1)+1\big)\Psi_n - (1-\beta)} = \Phi(n),$$

and $\Phi(n) \leqq \Phi(2)$, (3.1) holds true for any $0 \leqq \gamma \leqq 1$, $0 \leqq \beta < 1$ and $\alpha \geqq 0$. This completes the proof of the Theorem 3.1. □

As in the previous cases we note this result has many special cases. If we take $\alpha_1 = 1, \alpha_2 = 1, \beta_1 = 1, q = 2, s = 1$, $\lambda = 1$ and $k = 0$ in Theorem 3.1, then we have the following result of [9].

Corollary 3.2. [9] *If* $f \in \mathcal{C}(\beta)$, *then* $f \in \mathcal{T}^*\left(\frac{2}{3-\beta}\right)$. *The result is sharp for the extremal function*

$$f(z) = z - \frac{1-\beta}{2(2-\beta)} z^2.$$

Remark. Since distortion theorem and covering theorem are available for the class $\mathcal{T}^*(\beta)$ [9], we can also obtain the corresponding results for the class $\mathcal{UH}(\alpha, \beta, \gamma, \lambda, k)$, from the respective results of $\mathcal{T}^*(\beta)$ by using Theorem 3.1, and we state them without proof.

Theorem 3.3. *Let* Ψ_n *be defined as in* (1.3). *Then, for* $f \in \mathcal{UH}(\alpha, \beta, \gamma, \lambda, k)$, *with* $z = re^{i\theta} \in \Delta$, *we have*

$$r - B(\alpha, \beta, \gamma, \lambda)r^2 \leqq |f(z)| \leqq r + B(\alpha, \beta, \gamma, \lambda)r^2, \tag{3.2}$$

where,

$$B(\alpha, \beta, \gamma, \lambda) := \frac{1-\beta}{\Big(2(\alpha+1) - (\alpha+\beta)\Big)\Big(\gamma+1\Big)\Psi_2}.$$

Theorem 3.4. *If* $f \in \mathcal{UH}(\alpha, \beta, \gamma, \lambda, k)$, *then for* $|z| = r < 1$

$$1 - B(\alpha, \beta, \gamma, \lambda)r \leqq |f'(z)| \leqq 1 + B(\alpha, \beta, \gamma, \lambda)r, \tag{3.3}$$

where $B(\alpha, \beta, \gamma, \lambda)$ *as in Theorem* 3.3.

Note that in Theorem 3.3 and Theorem 3.4 equality holds for the function

$$f(z) = z - \frac{1-\beta}{\Big(2(\alpha+1)-(\alpha+\beta)\Big)\Big(\gamma+1\Big)\Psi_2} z^2.$$

4. Extreme points of the class $\mathcal{UH}(\alpha,\ \beta, \gamma,\ \lambda,\ k)$,

Theorem 4.1. *Let* $f_1(z) = z$ *and*

$$f_n(z) = z - \frac{1-\beta}{\big(n(\alpha+1)-(\alpha+\beta)\big)\big(\gamma(n-1)+1\big)\Psi_n} z^n, \quad n \geqq 2$$

and Ψ_n *be as defined in* (1.3). *Then* $f \in \mathcal{UH}(\alpha,\ \beta,\ \gamma,\ \lambda,\ k)$, *if and only if it can be represented in the form*

$$f(z) = \sum_{n=1}^{\infty} \mu_n f_n(z), \quad \mu_n \geqq 0, \quad \sum_{n=1}^{\infty} \mu_n = 1. \tag{4.1}$$

Proof. Suppose $f(z)$ can be written as in (4.1). Then

$$f(z) \quad = \quad z - \sum_{n=2}^{\infty} \mu_n \left\{ \frac{1-\beta}{\big(n(\alpha+1)-(\alpha+\beta)\big)\big(\gamma(n-1)+1\big)\Psi_n} \right\} z^n.$$

Now,

$$\sum_{n=2}^{\infty} \mu_n \frac{\big(n(\alpha+1)-(\alpha+\beta)\big)\big(\gamma(n-1)+1\big)\Psi_n(1-\beta)}{(1-\beta)\big(n(\alpha+1)-(\alpha+\beta)\big)\big(\gamma(n-1)+1\big)\Psi_n} = \sum_{n=2}^{\infty} \mu_n = 1-\mu_1 \leqq 1.$$

Thus $f \in \mathcal{UH}(\alpha,\ \beta,\ \gamma,\ \lambda,\ k)$. Conversely, let us have $f \in \mathcal{UH}(\alpha,\ \beta,\ \gamma,\ \lambda,\ k)$. Then by using (2.4), we may write

$$\mu_n = \frac{\big(n(\alpha+1)-(\alpha+\beta)\big)\big(\gamma(n-1)+1\big)\Psi_n}{1-\beta} a_n, \quad n \geqq 2,$$

and $\mu_1 = 1 - \sum_{n=2}^{\infty} \mu_n$. Then $f(z) = \sum_{n=1}^{\infty} \mu_n f_n(z)$, with $f_n(z)$ is as in the Theorem. □

Corollary 4.2. *The extreme points of* $f \in \mathcal{UH}(\alpha,\ \beta,\ \gamma,\ \lambda,\ k)$, *are the functions* $f_1(z) = z$ *and*

$$f_n(z) = z - \frac{1-\beta}{\big(n(\alpha+1)-(\alpha+\beta)\big)\big(\gamma(n-1)+1\big)\Psi_n} z^n, \quad n \geqq 2.$$

Remark. As in earlier theorems, we can deduce known results for various other classes and we omit details.

Theorem 4.3. *The class* $\mathcal{UH}(\alpha,\ \beta,\ \gamma,\ \lambda,\ k)$ *is a convex set.*

Proof. Let the function

$$f_j(z) = \sum_{n=2}^{\infty} a_{n,\,j} z^n, \quad a_{n,\,j} \geqq 0, \quad j = 1, 2\,, \tag{4.2}$$

be the class $\mathcal{UH}(\alpha,\ \beta,\ \gamma,\ \lambda,\ k)$. It sufficient to show that the function $g(z)$ defined by

$$g(z) = \mu f_1(z) + (1-\mu) f_2(z), \quad 0 \leqq \mu \leqq 1,$$

is in the class $\mathcal{UH}(\alpha,\ \beta,\ \gamma,\ \lambda,\ k)$. Since

$$g(z) = z - \sum_{n=2}^{\infty} [\mu a_{n,1} + (1-\mu) a_{n,2}] z^n,$$

an easy computation with the aid of Theorem 2.1 gives,

$$\sum_{n=2}^{\infty} \Big(n(\alpha+1) - (\alpha+\beta) \Big) \Big(\gamma(n-1) + 1 \Big) \Psi_n [\mu a_{n,1} + (1-\mu) a_{n,2}]$$

$$+(1-\mu) \sum_{n=2}^{\infty} (\, n(\alpha+1) - (\alpha+\beta))(\, \gamma(n-1) + 1) \Psi_n$$

$$\leqq \mu(1-\beta) + (1-\mu)(1-\beta) \leqq 1-\beta,$$

which implies that $g \in \mathcal{UH}(\alpha,\ \beta,\ \gamma,\ \lambda,\ k)$. Hence $\mathcal{UH}(\alpha,\ \beta,\ \gamma,\ \lambda,\ k)$ is convex. □

5. Modified Hadamard products

For functions of the form (4.2), we define the modified Hadamard product as

$$(f_1 * f_2)(z) = z - \sum_{n=2}^{\infty} a_{n,\,1}\, a_{n,\,2} z^n. \tag{5.1}$$

Theorem 5.1. *If* $f_j(z) \in \mathcal{UH}(q, s, \lambda, \beta, k)$, $j = 1,\ 2,$ *then*

$$(f_1 * f_2)(z) \in \mathcal{UH}(q, s, \lambda, \beta, k,\ \xi),$$

where

$$\xi = \frac{(2-\beta)\, \big(\, 2(\alpha+1) - (\alpha+\beta) \big)\, \big(\, \gamma+1 \big)\, \Psi_2 - 2(1-\beta)^2}{(2-\beta)\, \big(\, 2(\alpha+1) - (\alpha+\beta) \big)\, \big(\, \gamma+1 \big)\, \Psi_2 - (1-\beta)^2},$$

with Ψ_n *be defined as in* (1.3).

Proof. Since $f_j(z) \in \mathcal{UH}(q, s, \lambda, \beta, k)$, $j = 1,\ 2$, we have

$$\sum_{n=2}^{\infty} \Big(n(\alpha+1) - (\alpha+\beta) \Big) \Big(\gamma(n-1) + 1 \Big) \Psi_n a_{n,\,j} \leqq 1-\beta, \quad j = 1,\ 2.$$

The Cauchy-Schwartz inequality leads to

$$\sum_{n=2}^{\infty} \frac{\big(\, n(\alpha+1) - (\alpha+\beta)\big)\, \big(\, \gamma(n-1) + 1 \big)\, \Psi_n a_{n,\,j}}{1-\beta} \sqrt{a_{n,\,1}\, a_{n,\,2}} \leqq 1. \tag{5.2}$$

Note that we need to find the largest ξ such that

$$\sum_{n=2}^{\infty} \frac{\big(n(k+1)-(k+\xi)\big)\big(\gamma(n-1)+1\big)\Psi_n a_{n,j}}{1-\xi} a_{n,1}\, a_{n,2} \leqq 1. \tag{5.3}$$

Therefore, in view of (5.2) and (5.3), whenever

$$\frac{n-\xi}{1-\xi}\sqrt{a_{n,1}\, a_{n,2}} \leqq \frac{n-\beta}{1-\beta},\ n \geqq 2$$

holds, then (5.3) is satisfied. We have, from (5.2),

$$\sqrt{a_{n,1}\, a_{n,2}} \leqq \frac{1-\beta}{\big(n(\alpha+1)-(\alpha+\beta)\big)\big(\gamma(n-1)+1\big)\Psi_n},\ n \geqq 2. \tag{5.4}$$

Thus, if

$$\left(\frac{n-\xi}{1-\xi}\right)\left[\frac{1-\beta}{\big(n(\alpha+1)-(\alpha+\beta)\big)\big(\gamma(n-1)+1\big)\Psi_n}\right] \leqq \frac{n-\beta}{1-\beta},\ n \geqq 2,$$

or, if

$$\xi \leqq \frac{(n-\beta)\big(n(\alpha+1)-(\alpha+\beta)\big)\big(\gamma(n-1)+1\big)\Psi_n - n(1-\beta)^2}{(n-\beta)\big(n(\alpha+1)-(\alpha+\beta)\big)\big(\gamma(n-1)+1\big)\Psi_n - (1-\beta)^2},\ n \geqq 2,$$

then (5.2) is satisfied. Note that the right hand side of the above expression is an increasing function on n. Hence, setting $n = 2$ in the above inequality gives the required result. Finally, by taking the function

$$f(z) = z - \frac{1-\beta}{(2-\beta)\big(2(\alpha+1)-(\alpha+\beta)\big)\big(\gamma+1\big)\Psi_2} z^2,$$

we see that the result is sharp. □

6. Radii of close-to-convexity, starlikeness and convexity

Theorem 6.1. *Let the function $f \in \mathcal{T}$ be in the class $\mathcal{UH}(q, s, \lambda, \beta, k)$. Then $f(z)$ is close-to-convex of order ρ, $0 \leqq \rho < 1$ in $|z| < r_1(\alpha, \beta, \gamma, \rho)$, where*

$$r_1(\alpha, \beta, \gamma, \rho) = \inf_n \left[\frac{(1-\rho)\big(n(\alpha+1)-(\alpha+\beta)\big)\big(\gamma(n-1)+1\big)\Psi_n}{n(1-\beta)}\right]^{\frac{1}{n-1}},$$

$n \geqq 2$, with Ψ_n be defined as in (1.3)*. This result is sharp for the function $f(z)$ given by* (2.2)*.*

Proof. It is sufficient to show that $|f'(z) - 1| \leqq 1 - \rho$, $0 \leqq \rho < 1$, for $|z| < r_1(\alpha, \beta, \gamma, \rho)$, or equivalently

$$\sum_{n=2}^{\infty} \left(\frac{n}{1-\rho}\right) a_n |z|^{n-1} \leqq 1. \tag{6.1}$$

By Theorem 2.1, (6.1) will be true if

$$\left(\frac{n}{1-\rho}\right)|z|^{n-1} \leqq \frac{\big(n(\alpha+1)-(\alpha+\beta)\big)\big(\gamma(n-1)+1\big)\Psi_n}{1-\beta}$$

or, if

$$|z| \leqq \left[\frac{(1-\rho)\left(n(\alpha+1)-(\alpha+\beta)\right)\left(\gamma(n-1)+1\right)\Psi_n}{n(1-\beta)}\right]^{\frac{1}{n-1}}. \qquad (6.2)$$

The theorem follows easily from (6.2). □

Theorem 6.2. *Let the function* $f(z)$ *defined by* (1.1) *be in the class* $\mathcal{UH}(\alpha,\ \beta,\gamma,\ \lambda,\ k)$. *Then* $f(z)$ *is starlike of order* ρ, $0 \leqq \rho < 1$ *in* $|z| < r_2(\alpha,\ \beta,\gamma,\rho)$, *where*

$$r_2(\alpha,\ \beta,\gamma,\rho) = \inf_n \left[\frac{(1-\rho)\left(n(\alpha+1)-(\alpha+\beta)\right)\left(\gamma(n-1)+1\right)\Psi_n}{(n-\rho)(1-\beta)}\right]^{\frac{1}{n-1}},$$

$n \geqq 2$, *with* Ψ_n *be defined as in* (1.3). *This result is sharp for the function* $f(z)$ *given by* (2.2).

Proof. It is sufficient to show that

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leqq 1-\rho, \text{ or equivalently } \sum_{n=2}^{\infty}\left(\frac{n-\rho}{1-\rho}\right) a_n|z|^{n-1} \leqq 1, \qquad (6.3)$$

for $0 \leqq \rho < 1$, and $|z| < r_2(\alpha,\ \beta,\gamma,\rho)$. Proceeding as in Theorem 6.1, with the use of Theorem 2.1, we get the required result. □

Theorem 6.3. *Let the function* $f(z)$ *defined by* (1.1) *be in the class* $\mathcal{UH}(\alpha,\ \beta,\gamma,\lambda,k)$. *Then* $f(z)$ *is convex of order* ρ, $0 \leqq \rho < 1$ *in* $|z| < r_3(\alpha,\ \beta,\gamma,\rho)$, *where*

$$r_3(\alpha,\ \beta,\gamma,\rho) = \inf_n \left[\frac{(1-\rho)\left(n(\alpha+1)-(\alpha+\beta)\right)\left(\gamma(n-1)+1\right)\Psi_n}{n(n-\rho)(1-\beta)}\right]^{\frac{1}{n-1}},$$

$n \geqq 2$, *with* Ψ_n *be defined as in* (1.3). *This result is sharp for the function* $f(z)$ *given by* (2.2).

Proof. It is sufficient to show that

$$\left|\frac{zf''(z)}{f'(z)}\right| \leqq 1-\rho \quad \text{or equivalently} \quad \sum_{n=2}^{\infty}\left(\frac{n(n-\rho)}{1-\rho}\right) a_n|z|^{n-1} \leqq 1, \qquad (6.4)$$

for $0 \leqq \rho < 1$ and $|z| < r_3(\alpha,\ \beta,\gamma,\rho)$. Proceeding as in Theorem 6.1, we get the required result. □

References

[1] Altintas, O., *On a subclass of certain starlike functions with negative coefficients*, Math. Japon., **36**(1991), no. 3, 1-7.

[2] Aqlan, E., Jahangiri, J.M., Kulkarni, S.R., *Classes of k− uniformly convex and starlike functions*, Tamkang J. Math., **35**(2004), no. 3, 1-7.

[3] Cho, N.E., Srivastava, H.M., *Argument estimates of certain analytic functions defined by a class of multiplier transformations*, Math. Comput. Modelling, **37**(2003), no. 1-2, 39-49.

[4] Flett, T.M., *The dual of an inequality of Hardy and Littlewood and some related inequalities*, J. Math. Anal. Appl., **38**(1972), 746-765.

[5] Li, J.-L., Srivastava, H.M., *Some inclusion properties of the class* $\mathcal{P}_\alpha(\beta)$, Integral Transform. Spec. Funct., **8**(1999), no. 1-2, 57-64.

[6] Gangadharan, A., Shanmugam, T.N., Srivastava, H.M., *Generalized hypergeometric functions associated with k-uniformly convex functions*, Comput. Math. Appl., **44**(2002), 1515-1526.

[7] Sălăgean, G.S., *Subclasses of univalent functions*, in Complex analysis - fifth Romanian-Finnish seminar, Part 1 (Bucharest, 1981), 362-372, Lecture Notes in Math., 1013, Springer, Berlin.

[8] Shanmugam, T.N., Sivasubramanian, S., Kamali, M., *On the unified class of* $k-$*uniformly convex functions associated with Sălăgean derivative*, J. Approx. Theory and Appl., **1(2)**(2005), 141-155.

[9] Silverman, H., *Univalent functions with negative coefficients*, Proc. Amer. Math. Soc., **51**(1975), 109-116.

[10] Srivastava, H.M., Owa, S., Chatterjea, S.K., *A note on certain classes of starlike functions*, Rend. Sem. Mat. Univ Padova, **77**(1987), 115-124.

[11] Srivastava, H.M., Saigo, M., Owa, S., *A class of distortion theorems involving certain operator of fractional calculus*, J. Math. Anal. Appl., **131**(1988), 412-420.

[12] Uralegaddi, B.A., Somanatha, C., Certain classes of univalent functions, in Current topics in analytic function theory, (Edited by H.M. Srivastava and S.Owa), 371–374, World Scientific, Singapore, 1992.

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