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Coefficient bounds for certain classes of multivalent functions

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Abstract. In this paper, sharp upper bounds for $\left|a_{p+2}-\eta a_{p+1}^2\right|$ and $|a_{p+3}|$ are derived for a class of Mocanu α-convex p-valent functions defined by an extended linear multiplier differential operator (LMDO) $\mathcal{J}_p^{\delta}(\lambda,\mu,l)$.

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1. Introduction

Let $\mathcal{A}_p$ be the class of all functions of the form

$$f(z)=z^p+\sum_{n=p+1}^{\infty}a_nz^n \tag{1.1}$$

which are analytic in the open unit disk $\mathcal{U}:=\{z:|z|<1\}$ and let $\mathcal{A}:\ =\mathcal{A}_1$. For $f(z)$ given by (1.1) and $g(z)$ given by $g(z)=z^p+\sum\limits_{n=p+1}^{\infty}b_nz^n$, their convolution (or Hadamard product), denoted by $f*g$, is defined by

$$(f*g)(z):\ =z^p+\sum_{n=p+1}^{\infty}a_nb_nz^n.$$

The function $f(z)$ is subordinate to the function $g(z)$, written $f(z)\prec g(z)$, provided there exists analytic function $w(z)$ defined on $\mathcal{U}$ with $w(0)=0$ and $|w(z)|<1$ such that $f(z)=g(w(z))$. Let φ be an analytic function with positive real part in the unit disk $\mathcal{U}$ with $\varphi(0)=1$ and $\varphi'(0)>0$ that maps $\mathcal{U}$ onto a region which is starlike with respect to 1 and symmetric with respect to the real axis. R. M. Ali *et al.* [1] defined and studied the class $\mathcal{S}^*_{b,p}(\varphi)$ consisting of functions in $f \in \mathcal{A}_p$ for which

$$1+\frac{1}{b}\left(\frac{1}{p}\frac{zf'(z)}{f(z)}-1\right) \prec \varphi(z) \qquad (z\in\mathcal{U},\ \ b\in\mathbb{C}\setminus\{0\}), \tag{1.2}$$

and the class $\mathcal{C}_{b,p}(\varphi)$ of all functions in $f \in \mathcal{A}_p$ for which

$$1-\frac{1}{b}+\frac{1}{bp}\left(1+\frac{zf''(z)}{f'(z)}\right) \prec \varphi(z) \qquad (z\in\mathcal{U},\ \ b\in\mathbb{C}\setminus\{0\}). \tag{1.3}$$

R. M. Ali *et al.* [1] also defined and studied the class $\mathcal{R}_{b,p}(\varphi)$ to be the class of all functions in $f \in \mathcal{A}_p$ for which

$$1+\frac{1}{b}\left(\frac{f'(z)}{pz^{p-1}}-1\right) \prec \varphi(z) \qquad (z\in\mathcal{U},\ \ b\in\mathbb{C}\setminus\{0\}). \tag{1.4}$$

Note that $\mathcal{S}^*_{1,1}(\varphi) = \mathcal{S}^*(\varphi)$ and $\mathcal{C}_{1,1}(\varphi) = \mathcal{C}(\varphi)$, the classes introduced and studied by Ma and Minda [8]. The familiar class $\mathcal{S}^*(\alpha)$ of starlike functions of order α and the class $\mathcal{C}(\alpha)$ of convex functions of order α, $0 \le \alpha < 1$ are the special case of $\mathcal{S}^*_{1,1}(\varphi)$ and $\mathcal{C}_{1,1}(\varphi)$, respectively, when $\varphi(z) = (1+(1-2\alpha)z)/(1-z)$.

Owa [9] introduced and studied the class $\mathcal{H}_p(A,B,\alpha,\beta)$ of all functions in $f \in \mathcal{A}_p$ satisfying

$$(1-\beta)\left(\frac{f(z)}{z^p}\right)^{\alpha}+\beta\frac{zf'(z)}{pf(z)}\left(\frac{f(z)}{z^p}\right)^{\alpha} \prec \frac{1+Az}{1+Bz} \tag{1.5}$$

where $z\in\mathcal{U}$, $-1 \le B < A \le 1$, $0 \le \beta \le 1$, $\alpha \ge 0$.

We note that $\mathcal{H}_1(A,B,\alpha,\beta)$ is a subclass of Bazilevic functions [4].

A function $f \in \mathcal{A}_p$ is said to be in the class $\mathcal{R}_{(b,p,\alpha,\beta)}(\varphi)$ if

$$1+\frac{1}{b}\left\{(1-\beta)\left(\frac{f(z)}{z^p}\right)^{\alpha}+\beta\frac{zf'(z)}{pf(z)}\left(\frac{f(z)}{z^p}\right)^{\alpha}-1\right\} \prec \varphi(z) \tag{1.6}$$

$(0 \le \beta \le 1, \alpha \ge 0)$. The class $\mathcal{R}_{(b,p,\alpha,\beta)}(\varphi)$ was defined and studied by Ramachandran *et al.* [12].

A class of functions which unifies the classes $\mathcal{S}^*_{b,p}(\varphi)$ and $\mathcal{C}_{b,p}(\varphi)$ was introduced by T. N. Shanmugam, S. Owa, C. Ramachandran, S. Sivasubramanian and Y. Nakamura in [14]. They defined this class in the following way.

Let $\varphi(z)$ be a univalent starlike function with respect to 1 which maps the open unit disk $\mathcal{U}$ onto a region in the right half plane and is symmetric with respect to real axis, $\varphi(0) = 1$ and $\varphi'(0) > 0$. A function $f \in \mathcal{A}_p$ is in the class $\mathcal{M}_{(b,p,\alpha,\lambda)}(\varphi)$ if

$$1+\frac{1}{b}\left[\frac{1}{p}\left((1-\alpha)\frac{zF'(z)}{F(z)}+\alpha\left(1+\frac{zF''(z)}{F'(z)}\right)\right)-1\right] \prec \varphi(z) \tag{1.7}$$

$(0 \le \alpha \le 1)$, where

$$F(z):=(1-\lambda)f(z)+\lambda zf'(z).$$

T. N. Shangumugam *et al.* [14] obtained certain coefficient inequalities for function $f \in \mathcal{A}_p$ which are in the class $\mathcal{M}_{(b,p,\alpha,\lambda)}(\varphi)$.

For a function f in $\mathcal{A}_p$, the *linear multiplier differential operator (LMDO)* $\mathcal{J}_p^\delta(\lambda,\mu,l)f : \mathcal{A}_p \to \mathcal{A}_p$ was defined by the authors in [5] in the following way.

Definition 1.1. *Let $f \in \mathcal{A}_p$. For the parameters $\delta, \lambda, \mu, l \in \mathbb{R}$; $\lambda \geq \mu \geq 0$ and $\delta, l \geq 0$ the LMDO $\mathcal{J}_p^\delta(\lambda,\mu,l)$ on $\mathcal{A}_p$ is defined by*

$$\mathcal{J}_p^0(\lambda,\mu,l)f(z) = f(z) \tag{1.8}$$

$$(p+l)\mathcal{J}_p^1(\lambda,\mu,l)f(z)$$
$$= \lambda\mu z^2 f''(z) + (\lambda - \mu + (1-p)\lambda\mu)\, zf'(z) + (p(1-\lambda+\mu)+l)\, f(z)$$

$$(p+l)\mathcal{J}_p^2(\lambda,\mu,l)f(z)$$
$$= \lambda\mu z^2[\mathcal{J}_p^1(\lambda,\mu,l)f(z)]'' + (\lambda-\mu+(1-p)\lambda\mu)\, z[\mathcal{J}_p^1(\lambda,\mu,l)f(z)]'$$
$$+ (p(1-\lambda+\mu)+l)\, \mathcal{J}_p^1(\lambda,\mu,l)f(z)$$

$$\mathcal{J}_p^{\delta_1}(\lambda,\mu,l)(\mathcal{J}_p^{\delta_2}(\lambda,\mu,l)f(z)) = \mathcal{J}_p^{\delta_2}(\lambda,\mu,l)(\mathcal{J}_p^{\delta_1}(\lambda,\mu,l)f(z)), \quad \delta_1,\delta_2 \geq 0$$

for $z \in \mathcal{U}$ and $p \in \mathbb{N} := \{1,2,...\}$.

If f is given by (1.1) then from the definition of the *LMDO* $\mathcal{J}_p^\delta(\lambda,\mu,l)$, we can easily see that

$$\mathcal{J}_p^\delta(\lambda,\mu,l)f(z) = z^p + \sum_{k=p+1}^{\infty} \Phi_p^k(\delta,\lambda,\mu,l)a_k z^k$$

where

$$\Phi_p^k(\delta,\lambda,\mu,l) = \left[\frac{(k-p)(\lambda\mu k+\lambda-\mu)+p+l}{p+l}\right]^\delta.$$

When $p=1$, $l=0$ and $\delta = m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ we get Deniz-Orhan [6] (Also for earlier $0 \leq \mu \leq \lambda \leq 1$ Raducanu-Orhan [11]) differential operator, when $p=1$, $l=0=\mu$ and $\delta = m \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}$ we obtain the differential operator defined by Al-Oboudi [2] and when $p=1$, $l=0=\mu$, $\lambda=1$ and $\delta = m \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}$ we obtain the differential operator defined by Sălăgean [13]. We note that by specializing the parameters δ, λ, μ, l and p, the *LMDO* $\mathcal{J}_p^\delta(\lambda,\mu,l)$ reduces to other several well-known operators of analytic functions. Detailed information can be found in [5].

Now, by making use of the operator $\mathcal{J}_p^\delta(\lambda,\mu,l)$, we define a new subclass of functions belonging to the class $\mathcal{A}_p$.

Definition 1.2. *Let $\varphi(z)$ be a univalent starlike function with respect to 1 which maps the open unit disk $\mathcal{U}$ onto a region in the right half plane and is symmetric with respect to real axis, $\varphi(0)=1$ and $\varphi'(0)>0$. A function $f \in \mathcal{A}_p$ is in the class $\mathcal{M}^\delta_{(b,p,\alpha,\lambda,\mu,l,\nu)}(\varphi)$ if*

$$1+\frac{1}{b}\left[\frac{1}{p}\left((1-\alpha)\frac{z(F_{\nu,\delta}(z))'}{F_{\nu,\delta}(z)} + \alpha\left(1+\frac{z(F_{\nu,\delta}(z))''}{(F_{\nu,\delta}(z))'}\right)\right)-1\right] \prec \varphi(z) \tag{1.9}$$

where $0 \leq \alpha \leq 1$; $\delta, \lambda, \mu, l \in \mathbb{R}$; $\delta, l \geq 0$; $0 \leq \mu \leq \lambda$; $p \in \mathbb{N}$, *and*

$$F_{\nu,\delta}(z) = (1-\nu)J_p^{\delta}(\lambda,\mu,l)f(z) + \nu J_p^{\delta+1}(\lambda,\mu,l)f(z) \qquad (0 \leq \nu \leq 1).$$

Note that the class $\mathcal{M}^{\delta}_{(b,p,\alpha,\lambda,\mu,l,\nu)}(\varphi)$ reduces to the classes

$$\mathcal{M}^{1}_{(1,1,1,0,0,0,0)}(\varphi) \equiv \mathcal{C}(\varphi),$$

$$\mathcal{M}^{1}_{(1,1,0,0,0,0,0)}(\varphi) \equiv \mathcal{S}^{*}(\varphi)$$

which were introduced and studied by Ma and Minda [8]. Also,

$$\mathcal{M}^{1}_{(1,p,0,0,0,0,0)}(\varphi) \equiv \mathcal{S}^{*}_{p}(\varphi),$$

$$\mathcal{M}^{1}_{(1,p,1,0,0,0,0)}(\varphi) \equiv \mathcal{C}_{p}(\varphi),$$

$$\mathcal{M}^{1}_{(b,p,0,0,0,0,0)}(\varphi) \equiv \mathcal{S}^{*}_{b,p}(\varphi)$$

and $\mathcal{M}^{1}_{(b,p,1,0,0,0,0)}(\varphi) \equiv \mathcal{C}_{b,p}(\varphi)$ were introduced and studied by R. M. Ali *et al.* [1]. Also recently for $\delta \in \mathbb{N}_0$ Altuntaş and Kamali [3] were introduced and studied the class $\mathcal{M}^{\delta}_{(b,p,\alpha,1,0,0,\nu)}(\varphi) = \mathcal{M}_{(b,p,\alpha,\nu,\delta)}(\varphi)$

In this paper, we obtain Fekete-Szegö like inequalities and bounds for the coefficient a_{p+3} for the class $\mathcal{M}^{\delta}_{(1,p,\alpha,\lambda,\mu,l,\nu)}(\varphi)$. These results can be extended to other classes defined earlier.

Let Ω be the class of analytic functions of the form

$$w(z) = w_1 z + w_2 z^2 + w_3 z^3 + ...$$

in the unit disk $\mathcal{U}$ satisfying the condition $|w(z)| < 1$.

We need the following lemmas to prove our main results.

Lemma 1.3. [1] *If* $w \in \Omega$, *then*

$$\left|w_2 - t w_1^2\right| \leq \begin{cases} -t & \text{if} \quad t \leq -1, \\ 1 & \text{if} \quad -1 \leq t \leq 1, \\ t & \text{if} \quad t \geq 1. \end{cases} \tag{1.10}$$

When $t < -1$ *or* $t > 1$, *the equality holds if and only if* $w(z) = z$ *or one of its rotations.*

If $-1 < t < 1$, *then equality holds if and only if* $w(z) = z^2$ *or one of its rotations.*

Equality holds for $t = -1$ *if and only if* $w(z) = \frac{z(z+\lambda)}{1+\lambda z}$ $(0 \leq \lambda \leq 1)$ *or one of its rotations, while for* $t = 1$ *the equality holds if and only if* $w(z) = -\frac{z(z+\lambda)}{1+\lambda z}$ $(0 \leq \lambda \leq 1)$ *or one of its rotations.*

Although the above upper bound is sharp, it can be improved as follows when $-1 < t < 1$:

$$\left|w_2 - t w_1^2\right| + (1+t)\left|w_1\right|^2 \leq 1 \quad (-1 < t \leq 0) \tag{1.11}$$

and

$$\left|w_2 - t w_1^2\right| + (1-t)\left|w_1\right|^2 \leq 1 \quad (0 < t < 1). \tag{1.12}$$

Lemma 1.4. [7] *If $w \in \Omega$, then for any complex number t*

$$\left|w_2 - tw_1^2\right| \leq \max\{1; |t|\}. \tag{1.13}$$

The result is sharp for the functions $w(z) = z$ or $w(z) = z^2$.

Lemma 1.5. [10] *If $w \in \Omega$, then for any real numbers q_1 and q_2 the following sharp estimate holds:*

$$\left|w_3 + q_1 w_1 w_2 + q_2 w_1^3\right| \leq H(q_1, q_2) \tag{1.14}$$

where

$$H(q_1,q_2) =$$

$$= \begin{cases} 1 & \text{for} \quad (q_1,q_2) \in D_1 \cup D_2, \\ |q_2| & \text{for} \quad (q_1,q_2) \in \bigcup_{k=3}^{7} D_k, \\ \frac{2}{3}(|q_1|+1)\left(\frac{|q_1|+1}{3(|q_1|+1+q_2)}\right)^{\frac{1}{2}} & \text{for} \quad (q_1,q_2) \in D_8 \cup D_9, \\ \frac{q_2}{3}\left(\frac{q_1^2-4}{q_1^2-4q_2}\right)\left(\frac{q_1^2-4}{3(q_2-1)}\right)^{\frac{1}{2}} & \text{for} \quad (q_1,q_2) \in D_{10} \cup D_{11} \setminus \{\pm 2, 1\}, \\ \frac{2}{3}(|q_1|-1)\left(\frac{|q_1|-1}{3(|q_1|-1-q_2)}\right)^{\frac{1}{2}} & \text{for} \quad (q_1,q_2) \in D_{12}. \end{cases}$$

The extremal functions, up to rotations, are of the form

$$w(z) = z^3,\ w(z) = z,\ w(z) = w_0(z) = \frac{(z\left[(1-\lambda)\varepsilon_2 + \lambda\varepsilon_1\right] - \varepsilon_1\varepsilon_2 z)}{1 - \left[(1-\lambda)\varepsilon_1 + \lambda\varepsilon_2\right] z},$$

$$w(z) = w_1(z) = \frac{z(t_1 - z)}{1 - t_1 z},\ w(z) = w_2(z) = \frac{z(t_2 + z)}{1 + t_2 z},$$

$$|\varepsilon_1| = |\varepsilon_2| = 1,\ \varepsilon_1 = t_0 - e^{\frac{-i\theta_0}{2}}(a \pm b),\ \varepsilon_2 = -e^{\frac{-i\theta_0}{2}}(ia \pm b),$$

$$a = t_0 \cos\frac{\theta_0}{2},\ b = \sqrt{1 - t_0^2 \sin^2\frac{\theta_0}{2}},\ \lambda = \frac{b \pm a}{2b},$$

$$t_0 = \left[\frac{2q_2(q_1^2+2) - 3q_1^2}{3(q_2-1)(q_1^2+4q_2)}\right],\ t_1 = \left(\frac{|q_1|+1}{3(|q_1|+1+q_2)}\right)^{\frac{1}{2}},$$

$$t_2 = \left(\frac{|q_1|-1}{3(|q_1|-1-q_2)}\right)^{\frac{1}{2}},\ \cos\frac{\theta_0}{2} = \frac{q_1}{2}\left[\frac{q_2(q_1^2+8) - 2(q_1^2+2)}{2q_2(q_1^2+2) - 3q_1^2}\right]$$

The sets D_k, $k = 1, 2, ..., 12$, are defined as follows:

$$D_1 = \left\{(q_1,q_2) : |q_1| \leq \frac{1}{2},\quad |q_2| \leq 1\right\},$$

$$D_2 = \left\{(q_1,q_2) : \frac{1}{2} \leq |q_1| \leq 2,\quad \frac{4}{27} \leq (|q_1|+1)^3 - (|q_1|+1) \leq q_2 \leq 1\right\},$$

$$D_3 = \left\{(q_1,q_2) : |q_1| \leq \frac{1}{2},\quad q_2 \leq -1\right\},$$

$$D_4 = \left\{(q_1,q_2) : |q_1| \geq \frac{1}{2},\quad q_2 \leq -\frac{2}{3}(|q_1|+1)\right\},$$

$$D_5 = \{(q_1,q_2) : |q_1| \leq 2,\quad q_2 \geq 1\},$$

$$D_6=\left\{(q_1,q_2): 2\le |q_1|\le 4,\quad q_2\ge \frac{1}{12}(q_1^2+8)\right\},$$

$$D_7=\left\{(q_1,q_2): |q_1|\ge 4,\quad q_2\ge \frac{2}{3}(|q_1|-1)\right\},$$

$$D_8=\left\{(q_1,q_2): \frac{1}{2}\le |q_1|\le 2, -\frac{2}{3}(|q_1|+1)\le q_2\le \frac{4}{27}(|q_1|+1)^3-(|q_1|+1)\right\},$$

$$D_9=\left\{(q_1,q_2): |q_1|\ge 2,\quad -\frac{2}{3}(|q_1|+1)\le q_2\le \frac{2|q_1|(|q_1|+1)}{q_1^2+2|q_1|+4}\right\},$$

$$D_{10}=\left\{(q_1,q_2): 2\le |q_1|\le 4,\quad \frac{2|q_1|(|q_1|+1)}{q_1^2+2|q_1|+4}\le q_2\le \frac{1}{12}(q_1^2+8)\right\},$$

$$D_{11}=\left\{(q_1,q_2): |q_1|\ge 4,\quad \frac{2|q_1|(|q_1|+1)}{q_1^2+2|q_1|+4}\le q_2\le \frac{2|q_1|(|q_1|-1)}{q_1^2-2|q_1|+4}\right\},$$

$$D_{12}=\left\{(q_1,q_2): |q_1|\ge 4,\quad \frac{2|q_1|(|q_1|-1)}{q_1^2-2|q_1|+4}\le q_2\le \frac{2}{3}(|q_1|-1)\right\}.$$

2. Coefficient Bounds

By making use of Lemmas 1.3-1.5, we obtain the following results.

Theorem 2.1. *Let $\varphi(z)=1+B_1z+B_2z^2+B_3z^3+\dots$, where B_n are real with $B_1>0$ and $B_2\ge 0$. Let $0\le\alpha\le 1$; $\delta,\lambda,\mu,l\in\mathbb{R}$; $\delta,l\ge 0$; $0\le\mu\le\lambda$; $p\in\mathbb{N}; 0\le\nu\le 1$.*

If $f(z)$ given by (1.1) belongs to $\mathcal{M}^{\delta}_{(1,p,\alpha,\lambda,\mu,l,\nu)}(\varphi)$, then

$$\left|a_{p+2}-\eta a_{p+1}^2\right|\le$$

$$\begin{cases} \frac{p^2}{2(p+2\alpha)}\frac{(p+l)^{\delta+1}}{M_2N_2^{\delta}}\left\{B_2+pB_1^2\phi(\alpha,p,\mu,\eta,\lambda,\delta,l,\nu)\right\} & \text{if} \quad \eta\le\psi_1, \\ \frac{p^2B_1}{2(p+2\alpha)}\frac{(p+l)^{\delta+1}}{M_2N_2^{\delta}} & \text{if} \quad \psi_1\le\eta\le\psi_2, \\ -\frac{p^2}{2(p+2\alpha)}\frac{(p+l)^{\delta+1}}{M_2N_2^{\delta}}\left\{B_2+pB_1^2\phi(\alpha,p,\mu,\eta,\lambda,\delta,l,\nu)\right\} & \text{if} \quad \eta\ge\psi_2. \end{cases} \tag{2.1}$$

Further, if $\psi_1\le\eta\le\psi_3$, then

$$\begin{aligned} &\left|a_{p+2}-\eta a_{p+1}^2\right|+\frac{M_1^2N_1^{2\delta}}{M_2N_2^{\delta}}\frac{(p+\alpha)^2}{2p^2B_1^2(p+2\alpha)(p+l)^{\delta+1}} \\ &\times\left(B_1-B_2-pB_1^2\phi(\alpha,p,\mu,\eta,\lambda,\delta,l,v)\right)\left|a_{p+1}\right|^2 \\ \le\; &\frac{p^2B_1}{2(p+2\alpha)}\frac{(p+l)^{\delta+1}}{M_2N_2^{\delta}}. \end{aligned} \tag{2.2}$$

If $\psi_3 \le \eta \le \psi_2$, *then*

$$\begin{aligned} & \left|a_{p+2} - \eta a_{p+1}^2\right| + \frac{M_1^2 N_1^{2\delta}}{M_2 N_2^\delta} \frac{(p+\alpha)^2}{2p^2 B_1^2 (p+2\alpha)(p+l)^{\delta+1}} \\ & \times \left(B_1 + B_2 + pB_1^2 \phi(\alpha, p, \mu, \eta, \lambda, \delta, l, v)\right) \left|a_{p+1}\right|^2 \\ \le \;& \frac{p^2 B_1}{2(p+2\alpha)} \frac{(p+l)^{\delta+1}}{M_2 N_2^\delta}. \end{aligned} \tag{2.3}$$

For any complex number η,

$$\left|a_{p+2} - \eta a_{p+1}^2\right| \le \frac{p^2 B_1}{2(p+2\alpha)} \frac{(p+l)^{\delta+1}}{M_2 N_2^\delta}$$

$$\times \max\left\{1; \left|\frac{B_2}{B_1} + pB_1 \phi(\alpha, p, \mu, \eta, \lambda, \delta, l, \nu)\right|\right\}$$

where

$$\psi_1 = \frac{M_1^2 N_1^{2\delta}}{M_2 N_2^\delta} \frac{\left\{(B_2 - B_1)(p+\alpha)^2 + pB_1^2(p^2 + 2\alpha p + \alpha)\right\}}{2p^2 B_1^2 (p+2\alpha)(p+l)^{\delta+1}}$$

$$\psi_2 = \frac{M_1^2 N_1^{2\delta}}{M_2 N_2^\delta} \frac{\left\{(B_2 + B_1)(p+\alpha)^2 + pB_1^2(p^2 + 2\alpha p + \alpha)\right\}}{2p^2 B_1^2 (p+2\alpha)(p+l)^{\delta+1}}$$

$$\psi_3 = \frac{M_1^2 N_1^{2\delta}}{M_2 N_2^\delta} \frac{\left\{B_2(p+\alpha)^2 + pB_1^2(p^2 + 2\alpha p + \alpha)\right\}}{2p^2 B_1^2 (p+2\alpha)(p+l)^{\delta+1}}$$

$$\phi(\alpha, p, \mu, \eta, \lambda, \delta, l, \nu) = \frac{p^2 + 2\alpha p + \alpha}{(p+\alpha)^2} - 2\eta p \frac{M_2 N_2^\delta}{M_1^2 N_1^{2\delta}} \frac{(p+2\alpha)(p+l)^{\delta+1}}{(p+\alpha)^2}$$

and $M_c = [p + c\nu(\lambda\mu(p+c) + \lambda - \mu)]$, $N_c = [c(\lambda\mu(p+c) + \lambda - \mu) + p + l]$, $M_c^d = (M_c)^d$, $N_c^d = (N_c)^d$, $c \in \mathbb{N} = \{1, 2, 3, ...\}$.

Further,

$$|a_{p+3}| \le \frac{p^2 B_1}{3(p+3\alpha)} \frac{(p+l)^{\delta+1}}{M_3 N_3^\delta} H(q_1, q_2) \tag{2.4}$$

where $H(q_1, q_2)$ *is defined as in Lemma* 1.5, *with*

$$\begin{aligned} q_1 &= \frac{2B_2}{B_1} + \frac{3}{2} \frac{(p^2 + 3\alpha p + 2\alpha)}{(p+\alpha)(p+2\alpha)} pB_1, \\ q_2 &= \frac{B_3}{B_1} + \frac{3}{2} \frac{\left(p^2 + 3\alpha p + 2\alpha\right)}{(p+\alpha)(p+2\alpha)} pB_1 \left[\frac{B_2}{B_1} + pB_1 \frac{\left(p^2 + 2\alpha p + \alpha\right)}{(p+\alpha)^2}\right] \\ & \quad - \frac{\left(p^3 + 3\alpha p^2 + 3\alpha p + \alpha\right)}{(p+\alpha)^3} p^2 B_1^2. \end{aligned}$$

These results are sharp.

Proof. If $f(z) \in \mathcal{M}^\delta_{(1,p,\alpha,\lambda,\mu,l,\nu)}(\varphi)$, then there is a Schwarz function

$$w(z) = w_1 z + w_2 z^2 + ... \in \Omega$$

such that

$$\frac{1}{p}\left\{(1-\alpha)\frac{z(F_{\nu,\delta}(z))'}{F_{\nu,\delta}(z)}+\alpha\left(1+\frac{z(F_{\nu,\delta}(z))''}{(F_{\nu,\delta}(z))'}\right)\right\}=\varphi(w(z)) \tag{2.5}$$

where $0\leq\alpha\leq 1$; $\delta,\lambda,\mu,l\in\mathbb{R}$; $\delta,l\geq 0$; $0\leq\mu\leq\lambda$; $p\in\mathbb{N}$; $0\leq\nu\leq 1$; $F_{\nu,\delta}(z)=(1-\nu)J_p^{\delta}(\lambda,\mu,l)f(z)+\nu J_p^{\delta+1}(\lambda,\mu,l)f(z)$ and

$$J_p^{\delta}(\lambda,\mu,l)f(z)=z^p+\sum_{k=p+1}^{\infty}\left[\frac{(k-p)(\lambda\mu k+\lambda-\mu)+p+l}{p+l}\right]^{\delta}a_kz^k.$$

By definition of $J_p^{\delta}(\lambda,\mu,l)f(z)$ and $F_{\nu,\delta}(z)$, we can write

$$\begin{aligned}F_{\nu,\delta}(z)&=z^p+\frac{M_1N_1^{\delta}}{(p+l)^{\delta+1}}a_{p+1}z^{p+1}+\frac{M_2N_2^{\delta}}{(p+l)^{\delta+1}}a_{p+2}z^{p+2}\\&\quad+\frac{M_3N_3^{\delta}}{(p+l)^{\delta+1}}a_{p+3}z^{p+3}+...\end{aligned} \tag{2.6}$$

where

$$M_c=\left[p+c\nu(\lambda\mu(p+c)+\lambda-\mu)\right],$$
$$N_c=\left[c(\lambda\mu(p+c)+\lambda-\mu)+p+l\right],$$

$c\in\mathbb{N}=\{1,2,3,...\}$.

Let

$$T_{p+c}=\frac{M_cN_c^{\delta}}{(p+l)^{\delta+1}}a_{p+c};\quad c\in\mathbb{N}=\{1,2,3,...\}.$$

Then, we have

$$F_{\nu,\delta}(z)=z^p+T_{p+1}z^{p+1}+T_{p+2}z^{p+2}+T_{p+3}z^{p+3}+... \tag{2.7}$$

and differentiating both sides of the (2.7), we obtain the following equality

$$(F_{\nu,\delta}(z))'=pz^{p-1}+(p+1)T_{p+1}z^p+(p+2)T_{p+2}z^{p+1}+(p+3)T_{p+3}z^{p+2}+.... \tag{2.8}$$

From (2.7) and (2.8), we deduce

$$\frac{z\left(F_{\nu,\delta}(z)\right)'}{F_{\nu,\delta}(z)}=p+T_{p+1}z+(2T_{p+2}-T_{p+1}^2)z^2+(3T_{p+3}-3T_{p+2}T_{p+1}+T_{p+1}^3)z^3+.... \tag{2.9}$$

Similarly, if we take $U_{p+c}=(p+c)T_{p+c}$, we have

$$\begin{aligned}\frac{z\left(F_{\nu,\delta}(z)\right)''}{(F_{\nu,\delta}(z))'}&=p-1+\frac{1}{p}U_{p+1}z+\frac{1}{p}(2U_{p+2}-\frac{1}{p}U_{p+1}^2)z^2\\&\quad+\frac{1}{p}(3U_{p+3}-\frac{3}{p}U_{p+2}U_{p+1}+\frac{1}{p^2}U_{p+1}^3)z^3+....\end{aligned} \tag{2.10}$$

Since

$$\begin{aligned}&\frac{1}{p}\left\{(1-\alpha)\frac{z(F_{\nu,\delta}(z))'}{F_{\nu,\delta}(z)}+\alpha\left(1+\frac{z(F_{\nu,\delta}(z))''}{(F_{\nu,\delta}(z))'}\right)\right\}=\frac{1}{p}\left\{(1-\alpha)\left[p+T_{p+1}z\right.\right.\\&\left.+(2T_{p+2}-T_{p+1}^2)z^2+(3T_{p+3}-3T_{p+2}T_{p+1}+T_{p+1}^3)z^3+...\right]\end{aligned} \tag{2.11}$$

$$+\alpha\left[1+p-1+\frac{1}{p}U_{p+1}z+\frac{1}{p}(2U_{p+2}-\frac{1}{p}U_{p+1}^2)z^2\right.$$
$$\left.+\frac{1}{p}(3U_{p+3}-\frac{3}{p}U_{p+2}U_{p+1}+\frac{1}{p^2}U_{p+1}^3)z^3+...\right]\Bigg\}$$
$$=1+\frac{1}{p}(\frac{p+\alpha}{p})T_{p+1}z+\frac{1}{p}\left(\frac{2\,(p+2\alpha)}{p}T_{p+2}-\frac{p^2+2\alpha p+\alpha}{p^2}T_{p+1}^2\right)z^2$$
$$+\frac{1}{p}\left(\frac{3}{p}(p+3\alpha)T_{p+3}-\frac{3}{p^2}(p^2+3\alpha p+2\alpha)T_{p+2}T_{p+1}\right.$$
$$\left.+\frac{1}{p^3}(p^3+3\alpha p^2+3\alpha p+\alpha)T_{p+1}^3\right)z^3+...$$

and

$$\begin{aligned}\varphi(w(z)) &= 1+B_1w_1z+(B_1w_2+B_2w_1^2)z^2 \\ &\quad+(B_1w_3+2B_2w_1w_2+B_3w_1^3)z^3+...,\end{aligned} \tag{2.12}$$

by using equality (2.5), we have the equalities that follow.

Firstly, from

$$B_1w_1=\frac{1}{p}(\frac{p+\alpha}{p})\frac{M_1N_1^\delta}{(p+l)^{\delta+1}}a_{p+1}$$

we can write

$$a_{p+1}=\frac{p^2B_1w_1}{(p+\alpha)}\frac{(p+l)^{\delta+1}}{M_1N_1^\delta}. \tag{2.13}$$

Secondly, from

$$B_1w_2+B_2w_1^2=$$
$$=\frac{1}{p}\left(\frac{2(p+2\alpha)}{p}\frac{M_2N_2^\delta}{(p+l)^{\delta+1}}a_{p+2}-\frac{p^2+2\alpha p+\alpha}{p^2}\frac{M_1^2N_1^{2\delta}}{(p+l)^{2(\delta+1)}}a_{p+1}^2\right)$$

we can write

$$a_{p+2}=\frac{p^2B_1}{2(p+2\alpha)}\frac{(p+l)^{\delta+1}}{M_2N_2^\delta}\left\{w_2-w_1^2\left[-\frac{B_2}{B_1}-\frac{pB_1\left(p^2+2\alpha p+\alpha\right)}{(p+\alpha)^2}\right]\right\}. \tag{2.14}$$

Thus, by using (2.13) and (2.14), we can write

$$a_{p+2}-\eta a_{p+1}^2=\frac{p^2B_1}{2(p+2\alpha)}\frac{(p+l)^{\delta+1}}{M_2N_2^\delta}\left\{w_2-w_1^2\left[-\frac{B_2}{B_1}-\frac{pB_1\left(p^2+2\alpha p+\alpha\right)}{(p+\alpha)^2}\right.\right.$$
$$\left.\left.+\;2\eta p^2\frac{M_2N_2^\delta}{M_1^2N_1^{2\delta}}\frac{B_1(p+2\alpha)\,(p+l)^{\delta+1}}{(p+\alpha)^2}\right]\right\}.$$

Let

$$t=-\frac{B_2}{B_1}-pB_1\frac{\left(p^2+2\alpha p+\alpha\right)}{(p+\alpha)^2}+2\eta p^2\frac{M_2N_2^\delta}{M_1^2N_1^{2\delta}}\frac{B_1(p+2\alpha)\,(p+l)^{\delta+1}}{(p+\alpha)^2}.$$

Therefore, we have

$$a_{p+2}-\eta a_{p+1}^2=\frac{p^2B_1}{2(p+2\alpha)}\frac{(p+l)^{\delta+1}}{M_2N_2^{\delta}}\{w_2-tw_1^2\}. \tag{2.15}$$

By using Lemma 1.3, we can write for $\eta\leq\psi_1$

$$\begin{aligned}\left|a_{p+2}-\eta a_{p+1}^2\right| &\leq \frac{p^2}{2(p+2\alpha)}\frac{(p+l)^{\delta+1}}{M_2N_2^{\delta}}\left\{B_2+pB_1^2\left[\frac{\left(p^2+2\alpha p+\alpha\right)}{(p+\alpha)^2}\right.\right.\\ &\left.\left.-2\eta p\frac{M_2N_2^{\delta}}{M_1^2N_1^{2\delta}}\frac{(p+2\alpha)(p+l)^{\delta+1}}{(p+\alpha)^2}\right]\right\}\\ &= \frac{p^2}{2(p+2\alpha)}\frac{(p+l)^{\delta+1}}{M_2N_2^{\delta}}\left\{B_2+pB_1^2\phi(\alpha,p,\mu,\eta,\lambda,\delta,l,\nu)\right\}\end{aligned}$$

for $\eta\geq\psi_2$

$$\begin{aligned}\left|a_{p+2}-\eta a_{p+1}^2\right| &\leq -\frac{p^2}{2(p+2\alpha)}\frac{(p+l)^{\delta+1}}{M_2N_2^{\delta}}\left\{B_2+pB_1^2\left[\frac{\left(p^2+2\alpha p+\alpha\right)}{(p+\alpha)^2}\right.\right.\\ &\left.\left.-2\eta p\frac{M_2N_2^{\delta}}{M_1^2N_1^{2\delta}}\frac{(p+2\alpha)(p+l)^{\delta+1}}{(p+\alpha)^2}\right]\right\}\\ &= -\frac{p^2}{2(p+2\alpha)}\frac{(p+l)^{\delta+1}}{M_2N_2^{\delta}}\left\{B_2+pB_1^2\phi(\alpha,p,\mu,\eta,\lambda,\delta,l,\nu)\right\}\end{aligned}$$

and for $\psi_1\leq\eta\leq\psi_2$

$$\left|a_{p+2}-\eta a_{p+1}^2\right|\leq\frac{p^2}{2(p+2\alpha)}\frac{(p+l)^{\delta+1}}{M_2N_2^{\delta}}$$

where

$$\begin{aligned}\psi_1 &= \frac{M_1^2N_1^{2\delta}}{M_2N_2^{\delta}}\frac{\left\{(B_2-B_1)(p+\alpha)^2+pB_1^2(p^2+2\alpha p+\alpha)\right\}}{2p^2B_1^2(p+2\alpha)(p+l)^{\delta+1}}\\ \psi_2 &= \frac{M_1^2N_1^{2\delta}}{M_2N_2^{\delta}}\frac{\left\{(B_2+B_1)(p+\alpha)^2+pB_1^2(p^2+2\alpha p+\alpha)\right\}}{2p^2B_1^2(p+2\alpha)(p+l)^{\delta+1}}\\ \psi_3 &= \frac{M_1^2N_1^{2\delta}}{M_2N_2^{\delta}}\frac{\left\{B_2(p+\alpha)^2+pB_1^2(p^2+2\alpha p+\alpha)\right\}}{2p^2B_1^2(p+2\alpha)(p+l)^{\delta+1}}\end{aligned}$$

and

$$\phi(\alpha,p,\mu,\eta,\lambda,\delta,l,\nu)=\frac{p^2+2\alpha p+\alpha}{(p+\alpha)^2}-2\eta p\frac{M_2N_2^{\delta}}{M_1^2N_1^{2\delta}}\frac{(p+2\alpha)(p+l)^{\delta+1}}{(p+\alpha)^2}.$$

Further, if $\psi_1\leq\eta\leq\psi_3$, then

$$\left|a_{p+2}-\eta a_{p+1}^2\right|+\frac{M_1^2N_1^{2\delta}}{M_2N_2^{\delta}}\frac{(p+\alpha)^2}{2p^2B_1^2(p+2\alpha)(p+l)^{\delta+1}}$$

$$\times\left(B_1-B_2-pB_1^2\phi(\alpha,p,\mu,\eta,\lambda,\delta,l,v)\right)\left|a_{p+1}\right|^2$$

$$\leq \frac{p^2 B_1}{2(p+2\alpha)} \frac{(p+l)^{\delta+1}}{M_2 N_2^{\delta}}.$$

If $\psi_3 \leq \eta \leq \psi_2$, then

$$\begin{aligned} & \left|a_{p+2} - \eta a_{p+1}^2\right| + \frac{M_1^2 N_1^{2\delta}}{M_2 N_2^{\delta}} \frac{(p+\alpha)^2}{2p^2 B_1^2 (p+2\alpha)(p+l)^{\delta+1}} \\ & \times \left(B_1 + B_2 + pB_1^2 \phi(\alpha, p, \mu, \eta, \lambda, \delta, l, v)\right) |a_{p+1}|^2 \\ \leq \ & \frac{p^2 B_1}{2(p+2\alpha)} \frac{(p+l)^{\delta+1}}{M_2 N_2^{\delta}}. \end{aligned}$$

By using Lemma 1.4, we can write

$$|a_{p+2} - \eta a_{p+1}^2| \leq \frac{p^2 B_1}{2(p+2\alpha)} \frac{(p+l)^{\delta+1}}{M_2 N_2^{\delta}} \max\left\{1; \left|\frac{B_2}{B_1} + pB_1 \phi(\alpha, p, \mu, \eta, \lambda, \delta, l, \nu)\right|\right\}$$

for any complex number η.

By using equalities (2.11) and (2.12)

$$\begin{aligned} & \frac{1}{p}\left\{\frac{3}{p}(p+3\alpha) \frac{M_3 N_3^{\delta}}{(p+l)^{\delta+1}} a_{p+3} - \frac{3}{p^2}\left(p^2 + 3\alpha p + 2\alpha\right)\right. \\ & \left. \times \frac{M_2 N_2^{\delta}}{(p+l)^{\delta+1}} \frac{M_1 N_1^{\delta}}{(p+l)^{\delta+1}} a_{p+2} a_{p+1} + \frac{1}{p^3}\left(p^3 + 3\alpha p^2 + 3\alpha p + \alpha\right) \frac{M_1^3 N_1^{3\delta}}{(p+l)^{3(\delta+1)}} a_{p+1}^3 \right\} \\ & = B_1 w_3 + 2B_2 w_1 w_2 + B_3 w_1^3. \end{aligned}$$

Thus, we obtain

$$\begin{aligned} a_{p+3} = & \frac{p^2 B_1}{3(p+3\alpha)} \frac{(p+l)^{\delta+1}}{M_3 N_3^{\delta}} \left\{ w_3 + \left(\frac{2B_2}{B_1} + \frac{3}{2} \frac{\left(p^2 + 3\alpha p + 2\alpha\right)}{(p+\alpha)(p+2\alpha)} pB_1 \right) w_1 w_2 \right. \\ & + \left(\frac{B_3}{B_1} + \frac{3}{2} \frac{\left(p^2 + 3\alpha p + 2\alpha\right)}{(p+\alpha)(p+2\alpha)} pB_1 \left[\frac{B_2}{B_1} + pB_1 \frac{\left(p^2 + 2\alpha p + \alpha\right)}{(p+\alpha)^2} \right] \right. \\ & \left. \left. - \frac{\left(p^3 + 3\alpha p^2 + 3\alpha p + \alpha\right)}{(p+\alpha)^3} p^2 B_1^2 \right) w_1^3 \right\}. \end{aligned} \tag{2.16}$$

Let

$$\begin{aligned} q_1 &= \frac{2B_2}{B_1} + \frac{3}{2} \frac{\left(p^2 + 3\alpha p + 2\alpha\right)}{(p+\alpha)(p+2\alpha)} pB_1, \\ q_2 &= \frac{B_3}{B_1} + \frac{3}{2} \frac{\left(p^2 + 3\alpha p + 2\alpha\right)}{(p+\alpha)(p+2\alpha)} pB_1 \left[\frac{B_2}{B_1} + pB_1 \frac{\left(p^2 + 2\alpha p + \alpha\right)}{(p+\alpha)^2} \right] \\ &\quad - \frac{\left(p^3 + 3\alpha p^2 + 3\alpha p + \alpha\right)}{(p+\alpha)^3} p^2 B_1^2. \end{aligned}$$

Then, from equality (2.16), we obtain

$$a_{p+3} = \frac{p^2 B_1}{3(p+3\alpha)} \frac{(p+l)^{\delta+1}}{M_3 N_3^{\delta}} \left\{ w_3 + q_1 w_1 w_2 + q_2 w_1^3 \right\}.$$

Thus, we can write

$$|a_{p+3}| \leq \frac{p^2 B_1}{3(p+3\alpha)} \frac{(p+l)^{\delta+1}}{M_3 N_3^{\delta}} H(q_1, q_2) \tag{2.17}$$

where $H(q_1, q_2)$ is defined as in Lemma 1.5.

To show that the bounds in (2.1)-(2.3) are sharp, we define the functions $K_{\varphi,n}$ $(n = 2, 3, ...)$ by

$$\frac{1}{p}\left\{(1-\alpha)\frac{z(K_{\varphi,n})'(z)}{(K_{\varphi,n})(z)} + \alpha\left(1 + \frac{z(K_{\varphi,n})''(z)}{(K_{\varphi,n})'}\right)\right\} = \varphi(z^{n-1}) \tag{2.18}$$

$$[K_{\varphi,n}](0) = 0 = [K_{\varphi,n}]'(0) - 1,$$

and the function $F_{\lambda,m}$ and $G_{\lambda,m}$ $(0 \leq \lambda \leq 1,\ m \in \mathbb{N}_0)$ by

$$\frac{1}{p}\left\{(1-\alpha)\frac{z(F_{\lambda,m})'(z)}{(F_{\lambda,m})(z)} + \alpha\left(1 + \frac{z(F_{\lambda,m})''(z)}{(F_{\lambda,m})'}\right)\right\} = \varphi\left(z\frac{z+\lambda}{1+\lambda z}\right) \tag{2.19}$$

$$[F_{\lambda,m}](0) = 0 = [F_{\lambda,m}]'(0) - 1,$$

and

$$\frac{1}{p}\left\{(1-\alpha)\frac{z(G_{\lambda,m})'(z)}{(G_{\lambda,m})(z)} + \alpha\left(1 + \frac{z(G_{\lambda,m})''(z)}{(G_{\lambda,m})'}\right)\right\} = \varphi\left(-z\frac{z+\lambda}{1+\lambda z}\right) \tag{2.20}$$

$$(G_{\lambda,m})(0) = 0 = [G_{\lambda,m}]'(0) - 1.$$

Clearly the functions $K_{\varphi,n}$, $F_{\lambda,m}$, $G_{\lambda,m} \in \mathcal{M}^{\delta}_{(1,p,\alpha,\lambda,\mu,l,\nu)}(\varphi)$. Also we write $K_{\varphi} = K_{\varphi,2}$. If $\eta < \psi_1$ or $\eta > \psi_2$, then the equality holds if and only if f is K_{φ} or one of its rotations. When $\psi_1 < \eta < \psi_2$, then the equality holds if and only if f is $K_{\varphi,3}$ or one of its rotations. If $\eta = \psi_1$, then the equality holds if and only if f is $F_{\lambda,m}$ or one of its rotations. If $\eta = \psi_2$, then the equality holds if and only if f is $G_{\lambda,m}$ or one of its rotations. □

Remark 2.2.

1. For $l = \mu = 0$ and $\lambda = 1$ in Theorem 2.1,we get the result obtained by Altuntaş and Kamali [3].
2. For $l = \delta = \mu = \alpha = \nu = 0$ and $\lambda = 1$ in Theorem 2.11, we obtain the result obtained by R. M. Ali et al. [1].
3. For $l = \delta = \mu = \alpha = \nu = 0$ and $p = \lambda = 1$ in Theorem 2.1, we obtain the result obtained by Ma and Minda et al. [8].
4. For $l = \alpha = 0$ and $p = b = 1$ in Theorem 2.1, we obtain the result obtained by Deniz and Orhan et al. [6].

3. Applications to functions defined by convolution

We define $\mathcal{M}^{\delta}_{(b,p,\alpha,\lambda,\mu,l,\nu,g)}(\varphi)$ to be the class of all functions $f \in \mathcal{A}_p$ for which $f * g \in \mathcal{M}^{\delta}_{(b,p,\alpha,\lambda,\mu,l,\nu)}(\varphi)$, where g is a fixed function with positive coefficients and the class $\mathcal{M}^{\delta}_{(b,p,\alpha,\lambda,\mu,l,\nu)}(\varphi)$ is as in Definition 1.2. In Theorem

2.1 we obtained the coefficient estimate for the class $\mathcal{M}^{\delta}_{(1,p,\alpha,\lambda,\mu,l,\nu)}(\varphi)$. Now we obtain the coefficient estimates for the class $\mathcal{M}^{\delta}_{(1,p,\alpha,\lambda,\mu,l,\nu,g)}(\varphi)$.

Theorem 3.1. *Let $\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + ...,$ where B_n are real with $B_1 > 0$ and $B_2 \geq 0$. Let $0 \leq \alpha \leq 1$; $\delta, \lambda, \mu, l \in \mathbb{R}$; $\delta, l \geq 0$; $0 \leq \mu \leq \lambda$; $p \in \mathbb{N}; 0 \leq \nu \leq 1$.*

If $f(z)$ given by (1.1) belongs to $\mathcal{M}^{\delta}_{(1,p,\alpha,\lambda,\mu,l,\nu,g)}(\varphi)$, then

$$\left|a_{p+2} - \eta a_{p+1}^2\right| \leq$$

$$\begin{cases} \frac{p^2}{2(p+2\alpha)g_{p+2}} \frac{(p+l)^{\delta+1}}{M_2 N_2^{\delta}} \left\{B_2 + pB_1^2 \phi(\alpha,p,\mu,\eta,\lambda,\delta,l,\nu)\right\} & if \quad \eta \leq \psi_1, \\ \frac{p^2 B_1}{2(p+2\alpha)g_{p+2}} \frac{(p+l)^{\delta+1}}{M_2 N_2^{\delta}} & if \quad \psi_1 \leq \eta \leq \psi_2, \\ -\frac{p^2}{2(p+2\alpha)g_{p+2}} \frac{(p+l)^{\delta+1}}{M_2 N_2^{\delta}} \left\{B_2 + pB_1^2 \phi(\alpha,p,\mu,\eta,\lambda,\delta,l,\nu)\right\} & if \quad \eta \geq \psi_2. \end{cases}$$

Further, if $\psi_1 \leq \eta \leq \psi_3$, then

$$\begin{aligned} & \left|a_{p+2} - \eta a_{p+1}^2\right| + \frac{g_{p+1}^2}{g_{p+2}} \frac{M_1^2 N_1^{2\delta}}{M_2 N_2^{\delta}} \frac{(p+\alpha)^2}{2p^2 B_1^2 (p+2\alpha)(p+l)^{\delta+1}} \\ & \times \left(B_1 - B_2 - pB_1^2 \phi(\alpha,p,\mu,\eta,\lambda,\delta,l,v,g)\right) \left|a_{p+1}\right|^2 \\ \leq \quad & \frac{p^2 B_1}{2g_{p+2}(p+2\alpha)} \frac{(p+l)^{\delta+1}}{M_2 N_2^{\delta}}. \end{aligned}$$

If $\psi_3 \leq \eta \leq \psi_2$, then

$$\begin{aligned} & \left|a_{p+2} - \eta a_{p+1}^2\right| + \frac{M_1^2 N_1^{2\delta}}{M_2 N_2^{\delta}} \frac{(p+\alpha)^2}{2p^2 B_1^2 (p+2\alpha)(p+l)^{\delta+1}} \\ & \times \left(B_1 + B_2 + pB_1^2 \phi(\alpha,p,\mu,\eta,\lambda,\delta,l,v,g)\right) \left|a_{p+1}\right|^2 \\ \leq \quad & \frac{p^2 B_1}{2g_{p+2}(p+2\alpha)} \frac{(p+l)^{\delta+1}}{M_2 N_2^{\delta}}. \end{aligned}$$

For any complex number η,

$$\begin{aligned} \left|a_{p+2} - \eta a_{p+1}^2\right| \leq & \frac{g_{p+1}^2}{g_{p+2}} \frac{p^2 B_1}{2g_{p+2}(p+2\alpha)} \frac{(p+l)^{\delta+1}}{M_2 N_2^{\delta}} \\ & \times \max\left\{1; \left|\frac{B_2}{B_1} + pB_1 \phi(\alpha,p,\mu,\eta,\lambda,\delta,l,\nu,g)\right|\right\} \end{aligned}$$

where

$$\begin{aligned} \psi_1 &= \frac{g_{p+1}^2}{g_{p+2}} \frac{M_1^2 N_1^{2\delta}}{M_2 N_2^{\delta}} \frac{\left\{(B_2 - B_1)(p+\alpha)^2 + pB_1^2(p^2 + 2\alpha p + \alpha)\right\}}{2p^2 B_1^2 (p+2\alpha)(p+l)^{\delta+1}} \\ \psi_2 &= \frac{g_{p+1}^2}{g_{p+2}} \frac{M_1^2 N_1^{2\delta}}{M_2 N_2^{\delta}} \frac{\left\{(B_2 + B_1)(p+\alpha)^2 + pB_1^2(p^2 + 2\alpha p + \alpha)\right\}}{2p^2 B_1^2 (p+2\alpha)(p+l)^{\delta+1}} \\ \psi_3 &= \frac{g_{p+1}^2}{g_{p+2}} \frac{M_1^2 N_1^{2\delta}}{M_2 N_2^{\delta}} \frac{\left\{B_2(p+\alpha)^2 + pB_1^2(p^2 + 2\alpha p + \alpha)\right\}}{2p^2 B_1^2 (p+2\alpha)(p+l)^{\delta+1}} \end{aligned}$$

$$\phi(\alpha,p,\mu,\eta,\lambda,\delta,l,\nu,g) = \frac{p^2 + 2\alpha p + \alpha}{(p+\alpha)^2} - 2\eta p \frac{g_{p+2}}{g_{p+1}^2} \frac{M_2 N_2^{\delta}}{M_1^2 N_1^{2\delta}} \frac{(p+2\alpha)(p+l)^{\delta+1}}{(p+\alpha)^2}.$$

and $M_c = [p + c\nu(\lambda\mu(p+c) + \lambda - \mu)]$, $N_c = [c(\lambda\mu(p+c) + \lambda - \mu) + p + l]$, $M_c^d = (M_c)^d$, $N_c^d = (N_c)^d$, $c \in \mathbb{N} = \{1, 2, 3, ...\}$.

Further,

$$|a_{p+3}| \leq \frac{p^2 B_1}{3g_{p+3}(p+3\alpha)} \frac{(p+l)^{\delta+1}}{M_3 N_3^{\delta}} H(q_1, q_2)$$

where $H(q_1, q_2)$ *is defined as in Lemma* 1.5*,*

$$q_1 = \frac{2B_2}{B_1} + \frac{3}{2}\frac{(p^2+3\alpha p+2\alpha)}{(p+\alpha)(p+2\alpha)} pB_1,$$

$$q_2 = \frac{B_3}{B_1} + \frac{3}{2}\frac{(p^2+3\alpha p+2\alpha)}{(p+\alpha)(p+2\alpha)} pB_2$$

$$+ \left(\frac{3}{2}\frac{(p^2+3\alpha p+2\alpha)(p^2+2\alpha p+\alpha)}{(p+2\alpha)(p+\alpha)^3} - \frac{(p^3+3\alpha p^2+3\alpha p+\alpha)}{(p+\alpha)^3}\right) p^2 B_1^2.$$

These results are sharp.

Proof. The proof is similar to the proof of Theorem 2.1 □

Remark 3.2.

1. For $l = \delta = \mu = \alpha = \nu = 0$ and $\lambda = 1$ in Theorem 3.1, we obtain the result obtained by Ali et al. [1].

2. For $l = \delta = \mu = \alpha = \nu = 0$ and $p = \lambda = 1$ in Theorem 3.1, we obtain the result obtained by Ma and Minda et al. [8].

3. For $l = \alpha = 0$ and $p = b = 1$ in Theorem 2.1, we obtain the result obtained by Deniz and Orhan et al. [6].

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